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THOMASSENOVA HYPOTÉZA A SOUVISEJÍCÍ PROBLÉMY

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THOMASSEN'S CONJECTURE AND RELATED PROBLEMS

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Prohlášení

Prohlašuji, že jsem tuto práci vypracoval samostatně s použitím odborné literatury a pramenů, jejichž přehled je její součástí.

V Plzni dne 31.října 2012 ....

podpis

Poděkování

Děkuji svému školiteli Prof. RNDr. Zdeňku Ryjáčkovi, DrSc. za cenné připomínky a rady, které mi v průběhu psaní této práce poskytl a všem spoluautorům přiložených článků za spolupráci na nich.

Věnováno Žanetě

Anotace

V této disertační práci zkoumáme různé verze Thomassenovy hypotézy, která říká, že každý 4-souvislý hranový graf je hamiltonovský. Ukazujeme známé pozitivní výsledky dávající částečná řešení Thomassenovy hypotézy i známá vyvrácení silnějších hypotéz.

Klíčová slova

Hamiltonovská kružnice, hamiltonovská souvislost, hranový graf, hranový graf multigrafu.

Obsah

Kapitola 1

Úvod

Hamiltonovské vlastnosti jsou jednou ze základních otázek teorie grafů. Nacházejí široké uplatnění v praxi například při plánování výroby, v počítačových sítích nebo dopravních problémech. Thomassenova hypotéza patří k důležitým a intenzivně zkoumaným problémům této oblasti. Dokladem této skutečnosti je například i to, že přehledový článek (který je součástí této práce) obsahuje 67 referencí. Tématika, která je v práci zkoumána, zahrnuje široké spektrum otázek. Za všechny jmenujme konstrukce snarků, grafy na plochách, hyperkostry, uzávěrové operace a charakterizace grafových tříd. Rozsah těchto problémů a důkazových technik se neustále rozšiřuje v naději, že nový úhel pohledu přinese nějaký posun - ať už důkaz, nebo nalezení protipříkladu.

Práce je zpracována formou souboru sedmi prací, které jsou publikovány či zaslány k publikaci. V komentáři jsou výsledky, které jsou předmětem disertace, označeny hvězdičkou.

Kapitola 2

Základní terminologie

Značení použité v této práci vychází z knihy [3].

Graf je uspořádaná dvojice $G = (V(G), E(G))$, kde $V(G)$ je konečná množina a $E(G)$ je podmnožina množiny všech dvojic vzájemně různých prvků z $V(G)$. Povolíme-li více (konečný počet) různých hran mezi stejnou dvojicí uzlů, říkáme, že $G = (V(G), E(G))$ je *multigraf*. Množinu hran mezi jednou dvojicí uzlů nazveme *multihrana*. Počet hran v multihraně e nazveme *multiplicitou* multihrany e.

Okolí uzlu x v grafu G budeme značit $N_G(x)$ a dále zavedeme $N_G[x] = N_G(x) \cup \{x\}$. Označíme $d_G(x) = |N_G(x)|$ stupeň uzlu $x \in V(G)$ a definujeme množinu $V_k(G) = \{x \in V(G) | d_G(x) = k\}$. *Volná hrana* je hrana, která má jeden uzel stupně 1. *Klika* je úplný podgraf nikoliv nutně maximální. *Indukovaný podgraf* F grafu G je podgraf, pro který každá hrana grafu G mezi uzly podgrafu F je současně hranou podgrafu F. Podgraf indukovaný množinou uzlů M značíme $\langle M \rangle$. *Grafem bez* $K_{1,3}$ rozumíme graf, který neobsahuje $K_{1,3}$ jako indukovaný podgraf. Stejnou vazbu budeme používat i pro další podgrafy. Graf, který se skládá z kružnice C_k s k uzly a uzlu sousedícího se všemi uzly kružnice C_k (nazveme ho střed), nazveme *k-kolo* a označíme W_k. *Sledem* nazveme posloupnost uzlů takovou, že každé 2 po sobě jdoucí uzly jsou spojené hranou. *Tahem* nazveme sled, v němž se žádná hrana nepoužívá dvakrát. *Hranový graf* H grafu (multigrafu) G, značíme $H = L(G)$, je graf s množinou uzlů $V(H) = E(G)$, v němž jsou uzly spojené hranou právě tehdy, když odpovídající hrany v G mají společný uzel. Graf G nazveme *hranovým grafem* (*hranovým grafem multigrafu*), jestliže existuje graf (multigraf) H tak, že $G = L(H)$. Řekneme, že uzel u je *simpliciální uzel* v G, jestliže jeho okolí $N_G(u)$ indukuje úplný graf. Řekneme, že graf G je *cyklicky hranově k-souvislý*, jestliže neobsahuje hranový řez R takový, že $|R| < k$ a současně alespoň 2 komponenty $G - R$ obsahují kružnici. Řekneme, že graf G je *esenciálně hranově k-souvislý*, jestliže neobsahuje hranový řez R takový, že $|R| < k$ a současně alespoň 2 komponenty $G-R$ obsahují hranu. Jestliže $\{x, y\} \subset V(G)$ je uzlový řez grafu G a K_1, K_2 jsou komponenty $G-\{x, y\}$, pak podgrafy $\langle V(K_1)\cup\{x, y\}\rangle_G$ a $\langle V(K_2) \cup \{x, y\}\rangle_G$ nazveme *bikomponenty* grafu G určené $\{x, y\}$.

Kružnici (cestu) v grafu G nazveme *hamiltonovskou*, jestliže obsahuje všechny uzly grafu G. Graf nazveme *hamiltonovským*, jestliže má hamiltonovskou kružnici. Graf G nazveme *hamiltonovsky souvislým*, jestliže pro každou dvojici uzlů $x, y \in V(G)$ existuje hamiltonovská cesta s koncovými uzly x, y. Graf G nazveme *k-hamiltonovsky souvislým*, jestliže $G - X$ je hamiltonovsky souvislý pro každou množinu uzlů $X \subset V(G)$ s $|X| = k$. Graf G je *2-hranově hamiltonovsky souvislý*, jestliže graf $G + X$ má hamiltonovskou kružnici obsahující všechny hrany z X pro každou $X \subset E^+(G) = \{xy|x, y \in V(G)\}$ s $1 \leq |X| \leq 2$. Řekneme, že uzavřený tah (kružnice) T v grafu G je *dominantní*, jestliže každá hrana v G má alespoň jeden uzel v T. Jestliže kružnice (resp. tah) obsahuje alespoň jeden uzel dané hrany e, řekneme, že hranu e *dominuje*. Jestliže tah, který dostaneme odebráním první a poslední hrany tahu T, dominuje všechny hrany grafu G, řekneme, že tah T je *vnitřně dominantní* (značíme IDT z anglického internally dominating trail). Jestliže tah, který dostaneme odebráním první hrany tahu T, dominuje všechny hrany grafu G, označíme ho HIDT (z anglického half internally dominating trail).

Pokud graf G má pro každý uzel x hamiltonovskou cestu P takovou, že x je její koncový uzel, nazveme graf G *P-souvislým*. Pro podmnožiny X a Y množiny $V(G)$ takové, že $X \cap Y = \emptyset$, označíme $E_G(X, Y)$ množinu hran mezi X a Y, a dále $e_G(X, Y) = |E_G(X, Y)|$. Kružnici C grafu G nazveme *Tutteovou kružnicí grafu G*, jestliže (a) C je hamiltonovská nebo (b) $|V(C)| \geq 4$ a každá komponenta $G - C$ má nejvýše tři sousední uzly na C. Graf je k-regulární, jestliže všechny uzly grafu mají stupeň k. Graf, který je 3-regulární, označíme také jako *kubický*. Graf je *hranově k-obarvitelný*, jestliže hrany lze obarvit k barvami tak, že žádné dvě hrany se společným uzlem nemají stejnou barvu. *Druhou mocninu* grafu G dostaneme spojením všech uzlů ve vzdálenosti 2 hranami a budeme ji značit G^2. Buď $\mathcal{K}$ konečný systém množin. *Průnikový graf* (*průnikový multigraf*) systému $\mathcal{K}$ nazveme graf (multigraf), ve kterém každou množinu z $\mathcal{K}$ reprezentuje jeden uzel a uzly jsou spojené hranou (multihranou s multiplicitou i) právě tehdy, když množiny korespondující uzlům mají neprázdný průnik (s i prvky).

Kapitola 3

Charakterizace podtříd grafů bez $K_{1,3}$

3.1 Charakterizace hranových grafů multigrafů

Krausz [38](1943) dokázal následující charakterizaci hranových grafů multigrafů.

Krauszovým pokrytím grafu G nazveme pokrytí grafu G klikami, pro které platí:

(i) každá hrana je v alespoň jedné klice,

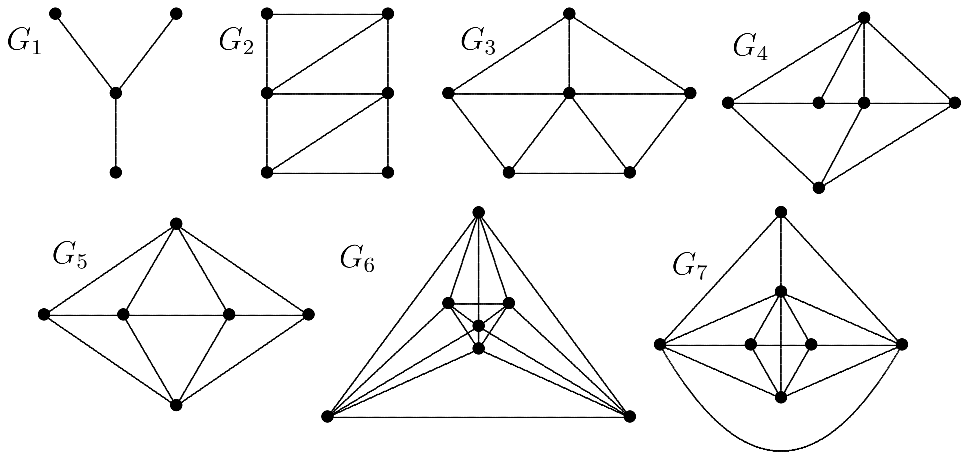
(ii) každý uzel je právě ve dvou klikách.

Věta 3.1 [38]. *Neprázdný graf G je hranový graf multigrafu právě tehdy, když existuje Krauszovo pokrytí grafu G.*

Jestliže G je hranový graf multigrafu a $\mathcal{K} = \{K_1, ..., K_m\}$ je Krauszovo pokrytí G, pak multigraf H takový, že $G = L(H)$, může být získán z $\mathcal{K}$ jako průnikový multigraf systému množin $\{V(K_1), ..., V(K_m)\}$.

Bermond a Meyer [6](1973) dokázali charakterizaci hranových grafů multigrafů přes zakázané podgrafy.

Věta 3.2 [6]. *Nechť G je graf. Pak následující podmínky jsou ekvivalentní:*



Obrázek 3.1

(a) *Graf G je hranový graf multigrafu.*

(b) *Graf G neobsahuje jako indukovaný podgraf žádný z grafů na obrázku 3.1.*

(c) *Z grafu G dostaneme hranový graf, jestliže nahradíme uzlem každou maximální množinu M, pro kterou platí $x, y \in M \Leftrightarrow (N_G[x] = N_G[y])$.*

Bermond a Meyer ve stejném textu uvedli ještě jednu charakterizaci.

Kliku C grafu G nazveme *normalizovanou*, pokud každý uzel $x \in V(G) \setminus V(C)$ buď leží mimo $N_G(C)$, nebo existuje rozdělení C na 3 disjunktní podkliky K_1, K_2, K_3 takové, že $C - \langle N_G(x) \rangle = K_1$, a pro uzly y, z z různých klik K_i platí, že $N_G[y] \neq N_G[z]$.

Věta 3.3 [6]. *Graf G je hranový graf multigrafu právě tehdy, když neobsahuje jako indukovaný podgraf $K_{1,3}$ ani G_5 na obrázku 3.1 a navíc množina maximálních klik, které nejsou normalizované, pokrývá každý uzel grafu G nejvýše dvakrát (každý uzel G je v nejvýše dvou takových klikách).*

Vzor hranového grafu multigrafu není určen jednoznačně. Zverovič [63] (1997) dokázal následující větu. Obdobnou větu pro hranové grafy dokázal Whitney již v roce 1932. *Základní graf* multigrafu (podle basic graph - Zverovič [63]) je graf, který vznikne nahrazením každé jeho multihrany hranou. *Volná* *multihrana* je multihrana, které v základním grafu odpovídá volná hrana. *Multihvězdou* nazveme multigraf, jehož základní graf je izomorfní s grafem $K_{1,i}, i \geq 2$. Střed multihvězdy S nazveme uzel x, který je spojen hranami se všemi ostatními uzly S. Ostatní uzly multihvězdy S nazveme listy. Pokud střed x je jediný uzel S, pro který platí $d_G(x) \neq d_S(x)$, řekneme, že S je *volná multihvězda* a uzel x nazveme kořenem. Multigraf, jehož základní graf je izomorfní s C_3 se nazývá *multitrojúhelník*. Pokud pro právě jeden uzel x v multitrojúhelníku H v grafu G platí $d_G(x) \neq d_H(x)$, nazveme H *volný multitrojúhelník* a uzel x nazveme *kořenem*.

Teď zavedeme transformaci libovolně zvoleného multigrafu G ([63]).

Operace A. V libovolně zvolené maximální (vzhledem k inkluzi) volné multihvězdě sloučíme všechny listy do jednoho uzlu.

Operace B. Pro libovolně zvolený volný multitrojúhelník H s kořenem v a množinou uzlů $V(H) = v, x, y$ odstraníme všechny hrany mezi uzly v, x. Následně doplníme hrany mezi uzly v, y tak, že stupeň v se při transformaci nezmění.

Multigraf, který dostaneme z G opakovaným použitím operací A, B v libovolném pořadí, dokud je to možné, označíme $AB(G)$.

Věta 3.4 [63]. *Buďte H a H' souvislé multigrafy, jejichž hranové grafy jsou izomorfní. Pak multigrafy $AB(H)$ a $AB(H')$ jsou vždy izomorfní, ledaže jeden z H, H' je multitrojúhelník a druhý neizomorfní multitrojúhelník nebo multihvězda.*

Zajímavý důsledek Zverovičovy věty je ukázán v [54]*. Přidání velmi přirozené podmínky, že simpliciální uzel v hranovém grafu odpovídá volné hraně v multigrafu, dává jednoznačný vzor pro hranové grafy multigrafů.

Věta 3.5 [54]*. *Buď G souvislý hranový graf multigrafu. Pak existuje až na izomorfismus jednoznačně určený multigraf H takový, že $G = L(H)$ a uzel $e \in V(G)$ je simpliciální v G právě tehdy, když korespondující hrana $e \in E(H)$ je volná hrana v H.*

Tento vzor je velmi důležitý pro důkaz věty o uzávěru grafů bez $K_{1,3}$ zachovávajícím 1-hamiltonovskou souvislost, budeme ho značit $H = L_M^{-1}(G)$ a pro vzor H hranového multigrafu G jednoznačně určený podle věty 3.5 a hranu $\overline{a} \in E(H)$ odpovídající uzlu $a \in V(G)$ označíme $\overline{a} = L_{M(G)}^{-1}(a)$.

Důležitou třídou z hlediska využitelnosti uzávěrů jsou hranové grafy multigrafů bez trojúhelníků. Charakterizacemi této třídy se zabývají Kloks, Kratsch a Müller [34]. Používají značně nestandardní terminologii. Hranové grafy multigrafů bez trojúhelníků například nazývají domina. Definují je jako třídu, kde každý uzel grafu se nachází nejvýše ve dvou maximálních klikách. Z Krauzsovy charakterizace lze snadno dokázat následující lemma.

Lemma 3.6. *Graf G je domino právě tehdy, když existuje multigraf H bez trojúhelníků takový, že $L(H) = G$.*

Důkaz. Nejprve ukážeme, že každé domino je hranový graf multigrafu bez trojúhelníků. Označme systém klik v grafu G naplňující definici domina $\mathcal{K} = \{K_i, i = 1 \dots m\}$. Systém klik $\mathcal{K}$ pokrývá všechny hrany v G, protože jinak nepokrytá hrana je v maximální klice, která dává spor s definicí. Buď K' systém klik, který dostaneme z K_i doplněním jednouzlových klik pro všechny uzly, které jsou v právě jedné maximální klice (jsou to simpliciální uzly grafu G). Systém klik K' je Krauzsovo pokrytí grafu G. Pokud by vzor odpovídající K' obsahoval trojúhelník T, každá jeho hrana je v G ve dvou klikách $\mathcal{K}$. Obraz T je v G maximální klika a dává spor s definicí $\mathcal{K}$.

Nechť naopak G je hranový graf multigrafu H bez trojúhelníků. Protože v grafu H nejsou trojúhelníky, z definice hranového grafu odpovídají všechny kliky $L(H)$ (tedy i maximální) multihvězdám v H. Z toho vyplývá, že maximální kliky $L(H)$ jsou podmožiny Krauzsových klik $L(H)$ korespondujících s H. Graf $L(H)$ splňuje definici domina. ■

Kloks a kol. charakterizují domina přes zakázané podgrafy.

Věta 3.7 [34]. *Graf je domino právě tehdy, když je bez $K_{1,3}$, bez 4-kola a bez druhé mocniny cesty s pěti uzly.*

3.2 Charakterizace hranových grafů

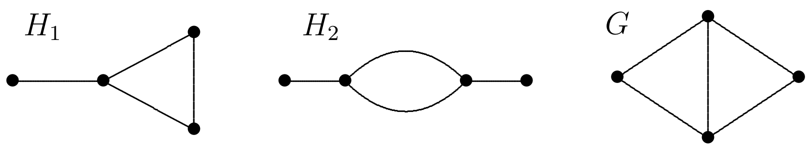
Pokud existuje Krauzsovo pokrytí hranového grafu G multigrafu H takové, že korespondující vzor H' je graf, graf G je podle definice hranový graf a můžeme takovým pokrytím hranové grafy grafů dokonce charakterizovat. Jedná se o nejstarší charakterizaci hranových grafů vyslovenou Krauszem [38] již v roce 1943.

Věta 3.8 [38]. *Neprázdný graf G je hranový právě tehdy, když existuje klikové pokrytí K grafu G takové, že každá hrana je v právě jedné klice v K a každý uzel je právě ve dvou klikách v K.*

Následující věta (známá jako Whitneyho věta) ukazuje jeden z rozdílů mezi hranovými grafy a hranovými grafy multigrafů.

Věta 3.9 [60]. *Buďte G, G' souvislé grafy, jejichž hranové grafy jsou izomorfní. Pak grafy G, G' jsou vždy izomorfní s výjimkou případu, kdy jeden je C_3 a druhý $K_{1,3}$.*

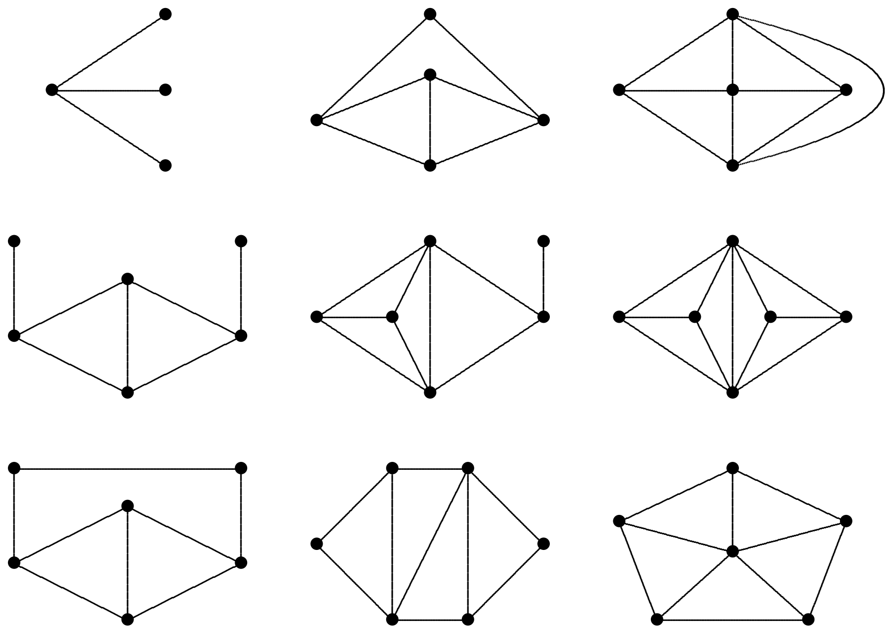
Poznamenejme, že dříve uvedená podmínka pro jednoznačný vzor hranových grafů multigrafů není rozšířením vzoru z Whitneyho věty na hranové grafy multigrafů a existuje nekonečná třída 2-souvislých hranových grafů, pro které se oba vzory liší. Obrázek 3.2 ukazuje příklad takových grafů. Graf G je hranový graf a grafy H_1, H_2 dva různé vzory G. Graf H_1 je jediný vzor G podle věty 3.9, zatímco H_2 je jediný vzor G podle věty 3.5. Nekonečnou třídu snadno dostaneme pokud jeden z trojúhelníků grafu G příslušně nahradíme libovolně velkou klikou.



Obrázek 3.2

Charakterizaci hranových grafů pomocí zakázaných podgrafů našel Beineke [4](1970).

Věta 3.10 [4]. *Graf G je hranový právě tehdy, když neobsahuje indukovaný podgraf izomorfní s některým grafem na obrázku 3.3.*



Obrázek 3.3

Doplníme ještě charakterizaci hranových grafů grafů bez trojúhelníků. Bod (*iii*) pochází od A.R. Rao, zbytek dokázali Beineke a Hemminger. Vše je v [5] (1978).

Věta 3.11 [5]. *Následující tvrzení jsou ekvivalentní pro souvislé grafy H s alespoň čtyřmi uzly:*

(*i*) *H je hranový graf grafu bez trojúhelníků,*

(*ii*) *dvě různé maximální kliky grafu H mají společný nejvýše jeden uzel a průnikový graf maximálních klik v H je bez trojúhelníků,*

(*iii*) *všechny uzly sousední k oběma uzlům libovolné hrany indukují v H kliku a okolí každého uzlu lze pokrýt dvěma klikami,*

(*iv*) *H neobsahuje indukované podgrafy $K_{1,3}$ a druhou mocninu cesty se čtyřmi uzly.*

Kapitola 4

Uzávěrové operace na grafech bez $K_{1,3}$

4.1 Obecný úvod

Uzávěrové operace na grafech jsou velmi důležité, protože umožňují rozšiřovat platnost vět z hranových grafů (multigrafů) na širší grafové třídy. Nás zajímají především hamiltonovské vlastnosti, ale tyto techniky lze využít také v jiných oblastech (např. párování). Hranové grafy multigrafů jsou výhodné zejména možností přejít ke vzoru. Často je možné pomocí uzávěru vzor omezit například na grafy bez trojúhelníků. Uvedeme známé korespondence pro hamiltonovskost, hamiltonovskou souvislost, P-souvislost a Tutteovy kružnice.

Souvislost mezi dominantními tahy a hamiltonovskými kružnicemi dokázali Harary a Nash-Williams [28] (1965).

Věta 4.1 [28]. *Buď G graf s alespoň třemi hranami. Pak $L(G)$ je hamiltonovský právě tehdy, když G má uzavřený dominantní tah.*

Podobný argument dává analogii pro hamiltonovskou souvislost (viz např. [46]).

Věta 4.2 [46]. *Buď G graf s alespoň třemi hranami. Pak $L(G)$ je hamiltonovský právě tehdy, když když G má IDT s koncovými hranami e_1, e_2 pro každý pár hran $e_1, e_2 \in E(G)$.*

Jen s velmi malou změnou lze ukázat obdobu pro P-souvislost.

Věta 4.3. *Buď G graf s alespoň třemi hranami. Následující tři podmínky jsou ekvivalentní:*

(i) *Graf $L(G)$ je P-souvislý.*

(ii) *Graf G má IDT s první hranou e pro každou hranu $e \in E(G)$.*

(iii) *Graf G má HIDT s první hranou e pro každou hranu $e \in E(G)$.*

Čada a kol. [19]* dokázali následující korespondenci. Uzavřený tah T v grafu H nazveme *slabý Tutteův uzavřený tah* grafu H jestliže (a) $E_H(T) = E(H)$, nebo (b) $|E_H(T)| \geq 4$ a $e_H(F,T) \leq 3$ pro všechny $F \in \mathcal{F}_H(T) = \{F|F$ je komponenta $H - T$ pro kterou $|V(F)| \geq 2\}$. Kružnici C v grafu bez $K_{1,3}$ nazveme *kružnicí s $(*)$-vlastností*, jestliže pro každý simpliciální uzel x na kružnici C kružnice C obsahuje všechny uzly $N_G(x)$.

Věta 4.4 [19]*. *Buď H graf bez trojúhelníků. Jestliže H má slabý Tutteův hranově maximální uzavřený tah, pak $L(H)$ má Tutteovu maximální kružnici s $(*)$-vlastností.*

Z hamiltonovských vlastností nelze uzávěrové operace použít přímo pro pancyklicitu a silnější vlastnosti, jak bylo ukázáno např. Brandtem a kol. [12](2000). Pro hamiltonovskou souvislost, P-souvislost, délku nejdelší kružnice, hamiltonovskost a existenci hamiltonovské cesty jsou známé uzávěrové operace, při kterých jsou tyto vlastnosti stabilní.

Nechť $\mathcal{G}$ je třída grafů. *Uzávěrem na $\mathcal{G}$* budeme rozumět binární relaci R na $\mathcal{G}$ takovou, že levý obor R je celá třída $\mathcal{G}$ a pro každou dvojici $[G_1, G_2]$ relace R platí $V(G_1) = V(G_2)$ a $E(G_1) \subset E(G_2)$.

4.2 Uzávěr se stabilní hamiltonovskostí

Patrně nejpoužívanější uzávěr v oblasti grafů bez $K_{1,3}$ je Ryjáčkův uzávěr [52](1997).

Uzel x v grafu G nazveme *k-uzavíratelný*, jestliže jeho okolí v G indukuje k-souvislý neúplný graf. Pro 1-uzavíratelné uzly budeme používat pouze označení uzavíratelný uzel. *Lokálním zúplněním uzlu x* nazveme graf G_x^* vytvořený z grafu G doplněním hran $\langle N_G(x)\rangle$ na kliku. *Ryjáčkovým uzávěrem* grafu G, značeno $\mathrm{cl}_R(G)$, rozumíme graf, který vznikne z grafu G opakovaným lokálním zúplňováním uzavíratelných uzlů, dokud je toto možné. U grafu $\mathrm{cl}_R(G)$ tedy každé souvislé okolí $\langle N_G(x)\rangle$ je klika.

Označme $c(G)$ délku nejdelší kružnice v grafu G. Označme $p(G)$ délku nejdelší cesty v grafu G. Body $(i)-(iii)$ následující věty dokázal Ryjáček [52], bod (iv) je dokázán Brandtem a kol. [12].

Věta 4.5 [52], [12]. *Buď G graf bez $K_{1,3}$. Pak*

(i) $\mathrm{cl}_R(G)$ *je jednoznačně definován,*

(ii) *existuje graf H bez trojúhelníků tak, že* $\mathrm{cl}_R(G) = L(H)$,

(iii) $c(G) = c(\mathrm{cl}_R(G))$,

(iv) $p(G) = p(\mathrm{cl}_R(G))$,

Uzávěr se také velmi dobře chová vůči kružnicím s $(*)$-vlastností.

Dále uvedenou větu a její důsledek využili Čada a kol. [19]* ke zkoumání hypotéz souvisejících s Tutteovými cykly na grafech bez $K_{1,3}$. Kružnici C v grafu G nazveme *Tutteovou maximální kružnicí* grafu G, jestliže C je Tutteova kružnice a maximální kružnice grafu G.

Věta 4.6 [19]*. *Buď G graf bez $K_{1,3}$ a buď v uzavíratelný uzel v G. Jestliže C' je kružnice s $(*)$-vlastností v grafu G_v^*, pak G má kružnici C s $(*)$-vlastností takovou, že $V(C) = V(C')$.*

Důsledek 4.7 [19]*. *Buď G graf bez $K_{1,3}$ Jestliže $\mathrm{cl}_R(G)$ má Tutteovu maximální kružnici s $(*)$-vlastností, pak G má Tutteovu maximální kružnici s $(*)$-vlastností.*

4.3 Uzávěr se stabilní hamiltonovskou souvislostí

Hlavní nástroj popsaný v této kapitole bude multigrafový uzávěr (zkráceně M-uzávěr) grafů bez $K_{1,3}$ zavedený v [54]*.

Označíme *k-uzávěr* grafu G, značený $\mathrm{cl}_k(G)$, graf vzniklý z G rekurzivním zúplňováním k-uzavíratelných uzlů, dokud je toto možné. Graf je *k-uzavřený*, jestliže $G = \mathrm{cl}_k(G)$. Pro nás jsou důležité následující vlastnosti.

Věta 4.8. *Pro každý graf G bez $K_{1,3}$,*

(i) *[9] $\mathrm{cl}_k(G)$ je definován jednoznačně pro každé $k \geq 1$,*

(ii) *[9] $\mathrm{cl}_2(G)$ je P-souvislý právě tehdy, když G je P-souvislý,*

(iii) *[53]* $\mathrm{cl}_2(G)$ je hamiltonovsky souvislý právě tehdy, když G je hamiltonovsky souvislý.*

Použitím charakterizace hranových grafů multigrafů od Bermonda a Meyera [6] je snadné ukázat, že $\mathrm{cl}_2(G)$ není hranový graf multigrafu, protože grafy G_2, G_4 charakterizace na obrázku 3.1 jsou 2-uzavřené. Ale 2-uzavřený graf neobsahuje žádný z 5 zbývajících zakázaných podgrafů charakterizace [54]*.

Buď $J = u_0u_1...u_{k+1}$ sled v G. Řekneme, že J je *dobrý v G*, jestliže $k \geq 4$, $J^2 \subset G$ a pro každé $i, 0 \leq i \leq k-4$ podgraf indukovaný $u_i, ..., u_{i+5}$ je izomorfní s G_2 nebo G_4 na obrázku 3.1.

Lemma 4.9 [54]*. *Buď G souvislý 2-uzavřený graf bez $K_{1,3}$, který není mocnina kružnice, a buď $J = u_0...u_{k+1}$ dobrý sled v G, $k \geq 5$. Pak*

(i) $d_G(u_i) = 4, i = 3, ..., k-2$,

(ii) $u_1...u_k$ *je cesta.*

Dobrý sled J je *maximální*, jestliže pro každý dobrý sled J' v G, J je část J', implikuje $J = J'$. Lze ukázat, že jestliže je G souvislý, 2-uzavřený a není druhá mocnina kružnice, pak každý dobrý sled je v nějakém maximálním dobrém sledu a dva různé maximální dobré sledy jsou uzlově disjunktní (viz [54]*).

Teď můžeme definovat M-uzávěr následujícím způsobem.

Buď G souvislý graf bez $K_{1,3}$, který není druhá mocnina kružnice.

1. *Polož* $G_1 = \mathrm{cl}_2(G)$, $i := 1$.

2. *Jestliže* G_i *obsahuje dobrý sled, pak*

   (a) *vyber maximální dobrý sled* $J = u_0 u_1 \dots u_{k+1}$,

   (b) *polož* $G_{i+1} = \mathrm{cl}_2(((G_i)^*_{u_1})^*_{u_k})$,

   (c) $i := i + 1$ *a jdi na* (2).

3. *Polož* $\mathrm{cl}^M(G) = G_i$.

Pokud G je druhá mocnina kružnice, definujeme $\mathrm{cl}^M(G)$ jako úplný graf.

Věta 4.10 [54]*. *Buď G souvislý graf bez $K_{1,3}$ a buď $\mathrm{cl}^M(G)$ M-uzávěr grafu G. Pak*

(i) $\mathrm{cl}^M(G)$ *je jednoznačně definován,*

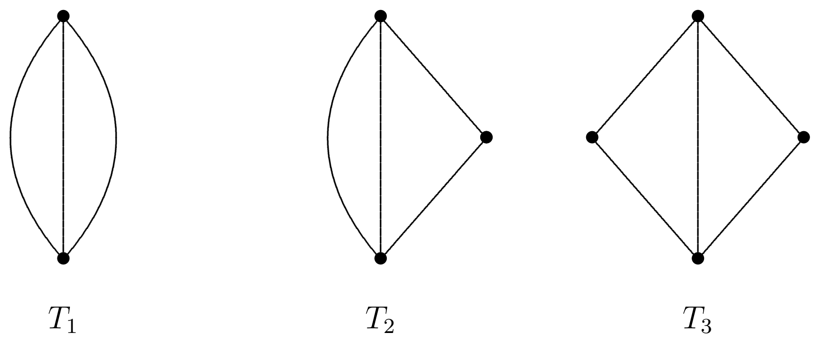
(ii) *existuje multigraf H tak, že* $\mathrm{cl}^M(G) = L(H)$,

(iii) $\mathrm{cl}^M(G)$ *je hamiltonovsky souvislý právě tehdy, když G je hamiltonovsky souvislý,*

(iv) $\mathrm{cl}^M(G)$ *je P-souvislý právě tehdy, když G je P-souvislý.*

Graf G bez $K_{1,3}$ nazveme *M-uzavřený*, pokud $G = \mathrm{cl}^M(G)$. Pokud vezmeme jednoznačně určený vzor H hranového grafu multigrafu G podle věty 3.5* můžeme M-uzavřené grafy snadno charakterizovat. Připomínáme zavedené značení takového vzoru $H = L_M^{-1}(G)$, hranu $\overline{a} \in E(H)$ odpovídající uzlu $a \in V(G)$ označíme $\overline{a} = L_{M(G)}^{-1}(a)$.

Věta 4.11 [54]*. *Buď G graf bez $K_{1,3}$ a T_1, T_2, T_3 grafy na obrázku 4.1. Pak G je M-uzavřený právě tehdy, když G je hranový graf multigrafu a $L_M^{-1}(G)$ neobsahuje podgraf S (ne nutně indukovaný) izomorfní s některým z grafů T_1, T_2, nebo T_3.*

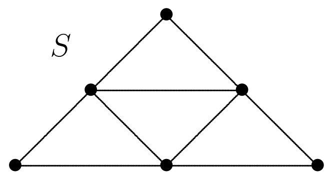


Obrázek 4.1

Multigrafový uzávěr lze dále zesílit a zachovat při tom stabilitu hamiltonovské souvislosti a P-souvislosti. Publikován byl uzávěr pro hamiltonovskou souvislost, ale obdobným způsobem lze zesílit i uzávěr pro P-souvislost. Uvedeme ho první kvůli větší jednoduchosti jako ukázku metody. Základním nástrojem pro výzkum v této oblasti je následující věta.

Věta 4.12 [12]. *Buď x uzavíratelný uzel grafu G bez $K_{1,3}$ a buďte a, b dva různé uzly grafu G. Pak pro každou nejdelší cestu $P'(a, b)$ s koncovými uzly a, b v grafu G^*_x existuje v grafu G cesta P taková, že $V(P') = V(P)$ a P má alespoň jeden koncový uzel z množiny $\{a,b\}$. Navíc v grafu G je cesta $P(a, b)$ s koncovými uzly a, b taková, že $V(P) = V(P')$ kromě možných následujících dvou situací (až na symetrii mezi a, b):*

- (i) *existuje indukovaný podgraf $F \subset G$ izomorfní s grafem S na obrázku 4.2 takový, že uzly a, x mají stupeň 4 v grafu F. V tom případě v grafu G je cesta P_b s koncovým uzlem b, pro kterou $V(P_b) = V(P')$. Jestliže navíc $b \in V(F)$, pak v grafu G je také cesta P_a s koncovým uzlem a taková, že $V(P_a) = V(P')$.*
- (ii) *$x = a, ab \in E(G)$. V tom případě graf G obsahuje cestu P_a s koncovým uzlem a, pro kterou $V(P_a) = V(P')$ i cestu P_b s koncovým uzlem b, pro kterou $V(P_b) = V(P')$.*



Obrázek 4.2

Nyní můžeme definovat uzávěr.

Pro daný graf G bez $K_{1,3}$, zkonstruujeme graf G^P následujícím způsobem.

(i) Jestliže G je P-souvislý, položíme $G^P = \mathrm{cl}_R(G)$.

(ii) Jestliže G není P-souvislý, rekurzivně provádíme lokální zúplnění v takovém uzavíratelném uzlu, pro který výsledný graf stále není P-souvislý, tak dlouho, dokud je to možné. Získáme posloupnost grafů $G_1, \ldots, G_k$ takovou, že

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ pro nějaký uzavíratelný uzel $x_i, i = 1, \ldots, k-1$,
- G_k není P-souvislý,
- pro každý uzavíratelný uzel $x \in V(G_k)^*_x$,$(G_k)^*_x$ je P-souvislý,

a položíme $G^P = G_k$.

Graf G^P získaný předchozí konstrukcí nazveme *P-uzávěr* grafu G a graf G rovný svému P-uzávěru nazveme P-uzavřený.

Věta 4.13*. *Buď G graf bez $K_{1,3}$ a buď G^P jeho P-uzávěr. Pak G^P má následující vlastnosti:*

(i) $V(G) = V(G^P)$ a $E(G) \subset E(G^P)$,

(ii) *G^P je získán z grafu G posloupností lokálních zúplnění v uzavíratelných uzlech,*

(iii) *G je P-souvislý právě když G^P je P-souvislý,*

(iv) *$G^P = L(H)$, kde H je hranový graf s nejvýše jedním trojúhelníkem,*

(v) *jestliže P nemá hamiltonovskou cestu s koncovým uzlem a pro nějaký $a \in V(G)$ a X je trojúhelník v H, pak $L^{-1}_{M(G^P)}(a) \in E(X)$.*

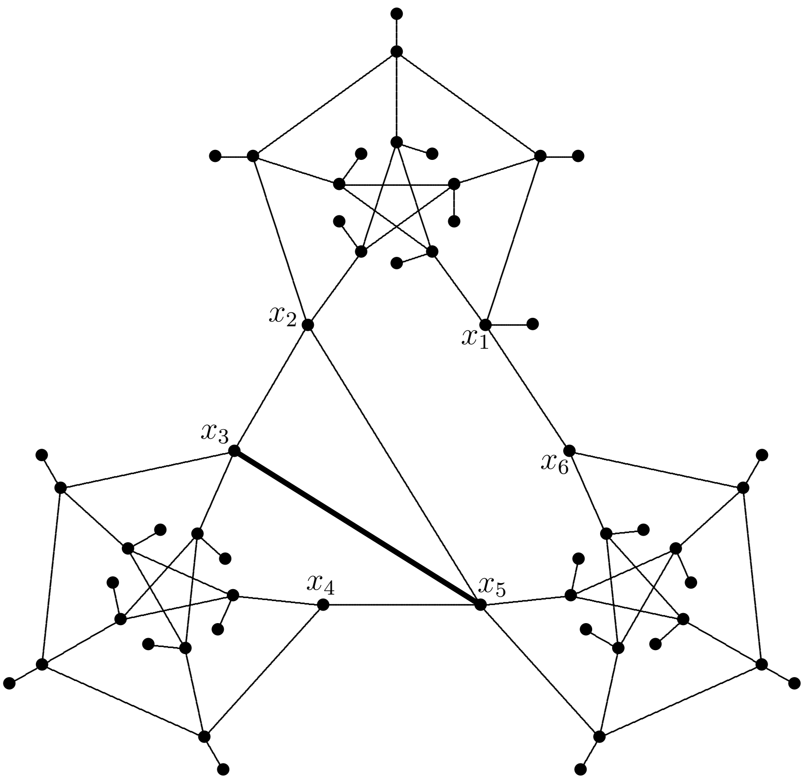
Důkaz. Body $(i)-(iii)$ plynou přímo z definice uzávěru. V důkazu nebudeme používat vzor hranových grafů odvozený z věty 3.9. Podle věty 4.10 je G^P M-uzavřený a tedy hranový graf multigrafu. Pokud $H = L_M^{-1}(G^P)$ obsahuje multihranu e_1, e_2, nejsou hrany e_1, e_2 v grafu H v trojúhelníku podle věty 4.11. Z toho plyne, že $L(e_1)$ je uzavíratelný uzel v grafu G a neexistuje podgraf S takový, že $d_S(L(e_1)) = 4$. Podle věty 4.12 můžeme $L(e_1)$ uzavřít při zachování P-souvislosti a dostáváme spor s P-uzavřeností.

Předpokládejme, že graf H obsahuje trojúhelník C, pro který neplatí tvrzení v bodu (v) věty. Pak obraz libovolné hrany C v grafu G^P je uzavíratelný uzel a podle věty 4.12 ho lze uzavřít při zachování P-souvislosti. To je spor s definicí G^P. Jestliže graf H obsahuje dva různé trojúhelníky C_1, C_2, podle věty 4.11 jsou hranově disjunktní a jeden z nich proto nesplňuje tvrzení v bodu (v) věty. Podle předchozího dostáváme spor. ∎

Teď ukážeme, že na třídě grafů bez $K_{1,3}$ neexistuje uzávěr se stabilní P-souvislostí na třídu hranových grafů grafů bez trojúhelníků.

Věta 4.14*. *Na třídě 3-souvislých hranových grafů neexistuje uzávěr* cl *takový, že každý graf* $\mathrm{cl}(G)$ *je hranový graf bez trojúhelníků a P-souvislost je stabilní.*

Důkaz. Předpokládejme, že věta neplatí a takový uzávěr existuje. Vezměme graf H na obrázku 4.3 a podle obrázku označme i uzly $x_1, \cdots, x_6$. Nejprve ukážeme, že v grafu H neexistuje HIDT s první hranou x_3x_5 (na obrázku vyznačena silně pro lepší orientaci) a tedy podle věty 4.3 graf $G = L(H)$ není P-souvislý. Nechť tedy naopak X je HIDT v grafu H s první hranou x_3x_5. V grafu H jsou tři podgrafy izomorfní s Petersenovým grafem bez hrany. Označme je P_1, P_2 (ty na obrázku dole) a P_3 (ten na obrázku nahoře). Protože Petersenův graf nemá hamiltonovskou kružnici, lze snadno odvodit, že požadovaný X končí v P_3. Pro P_1 (resp. P_2) pak existuje jediná možnost průchodu - kružnice s uzlem x_3 (resp. x_5) procházející všemi uzly P_1 (resp. P_2) kromě uzlu x_4 (resp. x_6). Tyto podmínky nelze splnit současně. Graf G tedy není P-souvislý.



Obrázek 4.3

Pokud se pokusíme přejít přidáním hran do grafu G na hranový graf grafu bez trojúhelníků, dostaneme podle věty 3.11, že všechny tři kliky hranového grafu G odpovídající uzlům trojúhelníku ve vzoru H je třeba uzavřít do společné kliky. Takový graf je ale P-souvislý (potřebné cesty lze snadno najít, v krajním případě pomocí počítače a jen ověřit správnost), což dává spor.

Nekonečnou třídu 3-souvislých hranových grafů, které nelze převést přidáním hran na hranové grafy grafů bez trojúhelníků při zachování P-souvislosti, dostaneme navýšením počtu simpliciálních uzlů v klikách se simpliciálními uzly grafu $G = L(H)$. ■

Teď uvedeme zesílení M-uzávěru pro hamiltonovskou souvislost dokázané Kuželem a kol. [41]*.

Pro daný graf G bez $K_{1,3}$ zkonstruujeme graf G^M následující konstrukcí.

(i) Jestliže G je hamiltonovsky souvislý, položíme $G^M = \mathrm{cl}_R(G)$.

(ii) Jestliže G není hamiltonovsky souvislý, rekurzivně opakujeme lokální zúplnění v takových lokálně souvislých uzlech, pro které výsledný graf stále není hamiltonovsky souvislý, dokud je to možné. Získáme posloupnost grafů $G_1, \ldots, G_k$ takovou, že

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ pro nějaký uzavíratelný uzel $x_i, i = 1, \ldots, k-1$
- G_k není hamiltonovsky souvislý
- pro každý uzavíratelný uzel $x \in V(G_k)^*_x$,$(G_k)^*_x$ je hamiltonovsky souvislý,

a položíme $G^M = G_k$.

Graf G^M získaný předchozí konstrukcí nazveme *silný M-uzávěr* (nebo zkráceně *SM-uzávěr*) grafu G, a graf G rovný svému SM-uzávěru nazveme *SM-uzavřený*. Následující věta dává přehled vlastností.

Věta 4.15 [41]*. *Buď G graf bez $K_{1,3}$. Pak existuje graf G^M takový, že*

(i) *existuje posloupnost uzlů $x_1, ..., x_{k-1} \in V(G)$ a grafy $G_1, ...G_k$ takové, že $G_1 = G, G_{i+1} = G^*_{x_i}, i = 1, ..., k-1$, a $G_k = G^M$,*

(ii) *G je hamiltonovsky souvislý právě tehdy, když G^M je hamiltonovsky souvislý,*

(iii) *existuje multigraf H takový, že*

(α) *$G^M = L(H)$,*

(β) *H neobsahuje 2 trojúhelníky se společnou hranou a žádnou multihranu s multiplicitou větší než 2,*

(γ) *H je buď bez multihran a obsahuje nejvýše 2 trojúhelníky, nebo neobsahuje žádný trojúhelník a nejvýše jednu multihranu,*

(δ) *jestliže H obsahuje trojúhelník T, pak H má IDT s koncovými hranami e, f pro každé $e, f \in E(H)$ s $e, f \cap T = \emptyset$,*

(ε) *jestliže H obsahuje multihranu ef, pak (e, f) je jediná dvojice, pro kterou v H není IDT s koncovými hranami e, f.*

Neznáme polynomiální algoritmus pro rozhodnutí zda je graf SM-uzavřený. Nevíme, zda nějaký SM-uzavřený graf obsahuje alespoň jeden trojúhelník. Ryjáček a Vrána ukázali následující větu.

Věta 4.16 [54]*. *Na třídě 3-souvislých hranových multigrafů neexistuje uzávěr* cl *takový, že každý graf* $\mathrm{cl}(G)$ *je hranový graf a hamiltonovská souvislost je stabilní.*

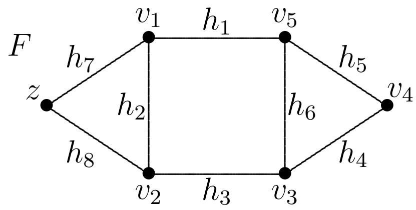
Poznamenejme, že definice uzávěru v článku [54]* je sice jiná, ale argument důkazu (přidání hrany do zakázaného podgrafu pro hranové grafy vede k hamiltonovské souvislosti grafu) projde i pro naši definici. Dále se podařilo dokázat několik následujících lemmat popisujících strukturu SM-uzavřených grafů.

Lemma 4.17 [54]*. *Buď G SM-uzavřený graf a buď $H = L_M^{-1}(G)$. Pak H neobsahuje trojúhelník s uzlem stupně 2 v H.*

Lemma 4.18 [54]*. *Buď G SM-uzavřený graf, buď $H = L_M^{-1}(G)$. Pak H neobsahuje podgraf $\overline{H}$ izomorfní s kružnicí C_5 s uzlem stupně 2 v H a s chordou.*

Lemma 4.19 [54]*. *Buď G SM-uzavřený graf a buď $H = L_M^{-1}(G)$. Pak H neobsahuje kružnici C délky 5 takovou, že některé dva uzly kružnice C mají stupeň 2 v H a některá hrana C je v multihraně s multiplicitou 2 nebo v trojúhelníku v H.*

Lemma 4.20 [54]*. *Buď G SM-uzavřený graf, buď $H = L_M^{-1}(G)$ a buď F graf s množinou uzlů $V(F) = \{v_1, v_2, v_3, v_4, v_5, z\}$ a množinou hran $E(F) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_3v_5, zv_1, zv_2\}$ (viz obrázek 4.4). Pak H neobsahuje podgraf $\overline{H}$ izomorfní s grafem F takový, že $N_H(\{v_1, v_2, v_3, v_5\}) \subset V(\overline{H})$.*



Obrázek 4.4

4.4 Uzávěr se stabilní 1-hamiltonovskou souvislostí

Z uvedných posledních dvou uzávěrů by se mohlo zdát, že klíčová pro celý postup je snaha uzavírat "co nejvíce". Poslední uzávěr dokázaný Ryjáčkem a Vránou [55]* ukazuje, že tomu tak zcela není.

Buď G graf bez $K_{1,3}$ a buď $x \in V(G)$ takové, že $G - x$ není hamiltonovsky souvislý. Buď $\widetilde{G}_x$ graf získaný následující konstrukcí.

(1) Polož $G_0 := G$, $i := 0$.

(2) Jestliže existuje $u_i \in V(G_i)$ takové, že u_i je uzavíratelný v grafu $G_i - x$ a současně $(G_i)^*_{u_i} - x$ není hamiltonovsky souvislý, pak polož $G_{i+1} = (G_i)^*_{u_i}$ a jdi na (3),

jinak polož $\widetilde{G}_x := G_i$ a zastav.

(3) Nastav $i := i + 1$ a jdi na (2).

Pak řekneme, že $\widetilde{G}_x$ je *částečný x-uzávěr* grafu G.

Podle následující věty 4.21* je graf $\widetilde{G}_x - x$ hranovým grafem multigrafu, a tedy v něm existuje jednoznačně určené Krauszovo pokrytí $\mathcal{K} = \{K_1, \ldots, K_m\}$ takové, že průnikový multigraf systému množin $\{V(K_1), \ldots, V(K_m)\}$ je H, $H = L_M^{-1}(G)$. Kdykoliv nadále v souvislosti s $\widetilde{G}_x - x$ hovoříme o Krauszových klikách, máme vždy na mysli prvky tohoto pokrytí $\mathcal{K}$.

Věta 4.21 shrnuje hlavní vlastnosti částečného x-uzávěru grafu bez $K_{1,3}$ a je nezbytně nutná pro oprávněnost konstrukce následujícího uzávěru.

Věta 4.21 [55]*. *Buď G graf bez $K_{1,3}$, buď $x \in V(G)$ takový, že $G-x$ není hamiltonovsky souvislý a buď $\widetilde{G}_x$ částečný x-uzávěr grafu G. Pak $\widetilde{G}_x - x$ je SM-uzavřený hranový graf a graf $\widetilde{G}_x$ splňuje jeden z následujících bodů:*

(*i*) *$\widetilde{G}_x$ je hranový graf multigrafu;*

(*ii*) *x je střed indukovaného W_5 a existují uzly $u_1, u_2 \in N_{\widetilde{G}_x}(x)$ takové, že*

- (α) *$\{u_1, u_2\}$ je řez multigrafu $\widetilde{G}_x - x$,*
- (β) *jedna z bikomponent multigrafu $\widetilde{G}_x - x$ určená $\{u_1, u_2\}$ je izomorfní s grafem $K_3 - e$,*
- (γ) *multigraf $(\widetilde{G}_x + \{u_1, u_2\}) - x$ neobsahuje indukované W_5 se středem x,*
- (δ) *multigraf $(\widetilde{G}_x + \{u_1, u_2\}) - x$ je SM-uzavřený;*

(*i*) *v grafu $\widetilde{G}_x - x$ jsou Krauszovy kliky K_1, K_2 takové, že*

- (α) *$N_{\widetilde{G}_x}(x) \subset K_1 \cup K_2$,*
- (β) *multigraf $(V(\widetilde{G}_x), E(\widetilde{G}_x) \cup \{xv | v \in K_1 \cup K_2\})$ je hranový graf multigrafu.*

Pokračujeme definicí hlavního uzávěru.

Buď G graf bez $K_{1,3}$ a buď $\overline{G}$ graf získaný následující konstrukcí:

(1) Jestliže G je 1-hamiltonovsky souvislý, polož $\overline{G} = \mathrm{cl}_R(G)$.

(2) Jestliže G není 1-hamiltonovsky souvislý, vyber uzel $x \in V(G)$, pro který $G - x$ není hamiltonovsky souvislý, a vyber částečný x-uzávěr $\widetilde{G}_x$ grafu G.

(3) Jestliže $\widetilde{G}_x$ splňuje (*ii*) věty 4.21 (tzn. x je střed indukovaného W_5 v grafu $\widetilde{G}_x$, vyber řez $\{u_1, u_2\}$ grafu $\widetilde{G}_x - x$, přidej hranu u_1u_2 do $\widetilde{G}_x$ (tzn. polož $\widetilde{G}_x := \widetilde{G}_x + u_1u_2$) a pokračuj na (4).

(4) Jestliže $\widetilde{G}_x$ je hranový graf multigrafu, polož $\overline{G} = \widetilde{G}_x$. Jinak $\widetilde{G}_x$ splňuje (*iii*) věty 4.21, tzn. některé dvě Krauszovy kliky K_1, K_2 v grafu $\widetilde{G}_x - x$ pokrývají všechny uzly okolí $N_{\overline{G}}(x)$, a pak polož $\overline{G} = (V(\widetilde{G}_x), E(\widetilde{G}_x) \cup \{xv | v \in K_1 \cup K_2\})$

Pak řekneme, že výsledný graf $\overline{G}$ je *1HC-uzávěr* grafu G.

Následující věta ukazuje základní vlastnosti 1HC-uzávěru grafu G.

Věta 4.22 [55]*. *Buď G graf bez $K_{1,3}$ a buď $\overline{G}$ jeho 1HC-uzávěr. Pak*

(*i*) *$\overline{G}$ je hranový graf multigrafu,*

(*ii*) *pro některé $x \in V(\overline{G})$ je graf $\overline{G} - x$ SM-uzavřený,*

(*iii*) *$\overline{G}$ je 1-hamiltonovsky souvislý právě tehdy, když G je 1-hamiltonovsky souvislý.*

Důležitá z hlediska podobnosti s cl_R uzávěrem a tedy i podobných důkazů je následující věta.

Věta 4.23 [55]*. *Buď G graf bez $K_{1,3}$. Pak existuje posloupnost grafů $G_0, \ldots, G_k$ taková, že*

(*i*) *$G_0 = G$,*

(*ii*) *$V(G_i) = V(G_{i+1})$ a současně $E(G_i) \subset E(G_{i+1}) \subset E((G_i)^*_{x_i})$ pro nějaký $x_i \in V(G_i)$ uzavíratelný v grafu G_i,*

(*iii*) *G_k je 1HC-uzávěr grafu G.*

Kapitola 5

Thomassenova hypotéza

5.1 Ekvivalentní verze Thomassenovy hypotézy

Už v roce 1981 byla zmíněna na straně 12 [7] následující hypotéza, která se v roce 1985 objevila v [57], Thomassen ji vznesl v roce 1986 [58] a jejíž platnost je stále otevřenou otázkou.

Hypotéza 5.1 [57]. **(Thomassen (1986))** *Každý* 4*-souvislý hranový graf je hamiltonovský.*

Hypotéza byla soustavně studována a časem se ukázala být ekvivalentní s následujícími hypotézami.

Hypotéza 5.2 [47]. **(Matthews, Sumner (1984))** *Každý* 4*-souvislý graf bez* $K_{1,3}$ *je hamiltonovský.*

Hypotéza 5.3 [1]. **(Ash, Jackson (1984))** *Každý cyklicky hranově* 4*-souvislý kubický graf má dominantní kružnici.*

Hypotéza 5.4 [23]. **(Fleischner (1984))** *Každý cyklicky hranově* 4*-souvislý kubický graf má hranové* 3*-obarvení nebo dominantní kružnici.*

Ekvivalenci hypotézy 5.1 a hypotézy 5.3 dokázali Fleischner a Jackson [25](1989). Naznačíme postup důkazu. Buď H multigraf, buď $v \in V(H)$ takový, že $d(v) \geq 4$. *Kubickou inflací* multigrafu H nazveme graf, který vznikne z H smazáním v, přidáním kružnice k s $d(v)$ uzly a spojením nových uzlů na původní sousedy v tak, že všechny nové uzly kružnice k mají stupeň 3 v novém grafu a ostatní uzly mají stejný stupeň jako v H. Ekvivalence plyne z věty 4.1 a následující věty.

Lemma 5.5 [25]. *Buď H esenciálně hranově* 4*-souvislý graf s minimálním stupněm $\delta(H) \geq 3$. Pak některá kubická inflace H je esenciálně hranově* 4*-souvislá.*

Poznamenejme, že kubická inflace se používá i v dalších důkazech, např. [41]*, [42]*.

První krok k ukázání ekvivalencí hypotéz 5.2 a 5.1 udělal Plummer (1993) [51], když dokázal ekvivalenci hypotézy 5.3 s následujícími dvěma hypotézami.

Hypotéza 5.6 [51]. *Každý* 4*-souvislý* 4*-regulární graf bez $K_{1,3}$ je hamiltonovský.*

Hypotéza 5.7 [51]. *Každý* 4*-souvislý* 4*-regulární graf bez $K_{1,3}$, ve kterém každý uzel leží právě ve dvou trojúhelnících, je hamiltonovský.*

Ryjáček ukázal [52](1997) ekvivalenci hypotéz 5.2, 5.1. Ekvivalenci hypotéz 5.4, 5.1 ukázal Kochol [35](2000) a později dokázal [37](2002), že hypotézy jsou ekvivalentní se zdánlivě slabšími verzemi se sublineárním defektem. Například hypotéza 5.1 je ekvivalentní s hypotézou: "každý 4-souvislý hranový graf G s počtem uzlů n lze uzlově pokrýt n_1 cestami s výjimkou n_2 uzlů tak, že n_1, n_2 rostou méně než lineárně v závislosti na n."V současnosti jsou již známé ekvivalentní verze hypotézy, které jsou zdánlivě silnější i zdánlivě slabší. V roce 2004 se podařilo ukázat ekvivalenci s následující hypotézou Kuželovi a Xiongovi [43].

Hypotéza 5.8 [43]. *Každý 4-souvislý hranový graf multigrafu je hamiltonovsky souvislý.*

Později Ryjáček a Vrána [54]* rozšířili ekvivalenci až na grafy bez $K_{1,3}$.

Hypotéza 5.9 [54]*. *Každý 4-souvislý graf bez $K_{1,3}$ je hamiltonovsky souvislý.*

V současnosti jednu ze zdánlivě nejsilnějších ekvivalentních verzí ukázali Kužel, Ryjáček a Vrána [42]*.

Hypotéza 5.10 [42]*. *Každý 4-souvislý graf multigrafu je 2-hranově hamiltonovsky souvislý.*

Pomocí uzávěru 4.22* se podařilo Ryjáčkovi a Vránovi rozšířit ekvivalentní verze z předchozí.

Hypotéza 5.11 [55]*. *Každý 4-souvislý graf bez $K_{1,3}$ je 1-hamiltonovsky souvislý.*

Tyto hypotézy naznačují možnou neplatnost Thomassenovy hypotézy (a všech ekvivalentních verzí), protože v případě platnosti je hranový graf s alespoň pěti uzly 2-hranově hamiltonovsky souvislý právě tehdy, když je 4-souvislý. Následující rozhodovací problémy jsou pak polynomiální.

1-HCL
Instance: *Hranový graf G*
Otázka: *Je G 1-hamiltonovsky souvislý?*

2-E-HCL
Instance: *Hranový graf G*
Otázka: *Je G 2-hranově hamiltonovsky souvislý?*

Je známo, že rozhodnutí, zda je hranový graf hamiltonovský, je NP-úplné [8]. Stejně tak je NP-úplné rozhodnutí, zda graf je 1-hamiltonovsky souvislý [42]* (respektive 2-hranově hamiltonovsky souvislý [50]*).

Pokud by taková situace byla i na hranových grafech, Thomassenova hypotéza neplatí, ledaže P=NP. Na druhou stranu rozhodnutí, zda je rovinný graf hamiltonovský, je NP-úplné (viz např. [8]) a rovinný graf s alespoň pěti uzly je 2-hranově hamiltonovsky souvislý právě tehdy, když je 4-souvislý [50]*.

V roce 1956 Tutte dokázal [59], že každý 4-souvislý rovinný graf má hamiltonovskou kružnici. V důkazu poprvé použil Tutteovy cykly (poznamenejme, že mají na rovinných grafech mírně jinou definici). V současné době je metoda Tutteových struktur hlavní důkazovou metodou hamiltonovských vlastností nejen pro rovinné grafy, ale i pro grafy na dalších plochách (např. torus, projektivní rovina). Jackson [31] (viz také [22], Hypotéza 2a.5) formuloval v roce 1992 následující hypotézu s cílem pokusit se dokázat Thomassenovu hypotézu obdobným způsobem, jaký se používá pro grafy na plochách.

Hypotéza 5.12 [31]. *Každý 2-souvislý graf bez $K_{1,3}$ má Tutteovu kružnici.*

Přesněji navrhoval pokusit se o důkaz zdánlivě slabší verze hypotézy pro vzory hranových grafů.

Hypotéza 5.13 [31]. *Každý hranově 2-souvislý graf G má eulerovský podgraf H s alespoň třemi hranami, pro který je každá komponenta grafu $G - H$ připojena nejvýše třemi hranami k H.*

Ekvivalenci předchozích dvou hypotéz s hypotézou 5.1 ukázali Čada a kol. [19]*. Zdá se tedy, že pokus navrhovaný Jacksonem je oprávněný přinejmenším ve smyslu nalezení protipříkladu.

Zdánlivě nejslabší je následující hypotéza. Ekvivalenci ukázali Broersma a kol. v [16]*. *Snark* je cyklicky hranově 4-souvislý kubický graf, který nemá hranové 3-obarvení, s délkou nejkratší kružnice alespoň 5.

Hypotéza 5.14 [16]*. *Každý snark má dominantní kružnici.*

Na kubických grafech ukázali ekvivalenci se zdánlivým zesílením Fouquet a Thuillier [27].

Hypotéza 5.15 [27]. *Libovolná dvojice disjunktních hran cyklicky hranově 4-souvislého kubického grafu leží na dominantní kružnici.*

Později ekvivalenci rozšířili Fleischner a Kochol [26].

Hypotéza 5.16 [26]. *Libovolná dvojice hran cyklicky hranově 4-souvislého kubického grafu leží na dominantní kružnici.*

Označme $V_i(H) = \{x \in V(H) | d_H(x) = i\}$ a buď H graf s $\delta(H) = 2$ a $|V_2(H)| = 4$. Řekneme, že H je $V_2(H)$-*dominovaný*, jestliže pro každé dvě hrany $e_1 = u_1v_1, e_2 = u_2v_2$ takové, že $V_2(H) = \{u_1, v_1, u_2, v_2\}$ graf $H + \{e_1, e_2\}$ má dominantní uzavřený tah obsahující e_1, e_2. Graf H nazveme *silně* $V_2(H)$-*dominovaný*, jestliže H je $V_2(H)$-dominovaný a pro každou hranu $e = uv$ takovou, že $u, v \in V_2(H)$ graf $H + \{e\}$ má dominantní uzavřený tah obsahující e.

Buď F graf a buď $A \subset V(F)$. Buď $\mathcal{A}$ rozdělení neprázdné sudé $X \subset A$ na dvouprvkové podmnožiny. Označíme $E(\mathcal{A})$ množinu všech hran a_1a_2 takových, že a_1, a_2 jsou ve stejné dvojici v $\mathcal{A}$ a F^A označíme multigraf s množinou uzlů $V(F^{\mathcal{A}}) = V(F)$ a množinou hran $E(F^{\mathcal{A}}) = E(F) \cup E(\mathcal{A})$. Graf F nazveme *slabě* A-*kontrahovatelný*, jestliže pro každou neprázdnou sudou $X \subset A$ a pro každé rozdělení $\mathcal{A}$ multigraf F^A má uzavřený dominantní tah obsahující všechny hrany $E(\mathcal{A})$.

Dalších několik ekvivalentních verzí je známo pro podgrafy kubických grafů. Odkazy jsou na články, ve kterých je dokázaná ekvivalence.

Hypotéza 5.17 [40]. *Libovolný podgraf H esenciálně hranově 4-souvislého kubického grafu s $\delta(H) = 2$ a $|V_2(H)| = 4$ je $V_2(H)$-dominovaný.*

Hypotéza 5.18 [42]*. *Libovolný podgraf H esenciálně hranově 4-souvislého kubického grafu s $\delta(H) = 2$ a $|V_2(H)| = 4$ je silně $V_2(H)$-dominovaný.*

Hypotéza 5.19 [16]*. *Každý cyklicky hranově 4-souvislý kubický graf obsahuje slabě A-kontrahovatelný podgraf F s $\delta(F) = 2$.*

5.2 Pozitivní výsledky dávající částečné řešení Thomassenovy hypotézy

Pokud omezíme třídu vzorů, je známých několik výsledků. Nejprve probereme konstrukci, která umožňuje převést hamiltonovské výsledky z grafů na plochách na jejich hranové grafy. Konstrukci objevil Lai (1994) [44] a v témže roce se objevila ještě v dalším článku [17].

Věta 5.20 [44]. *Každý 4-souvislý hranový graf rovinného grafu je hamiltonovský.*

Věta 5.21 [17]. *Každý 4-souvislý hranový graf grafu s nakreslením v projektivní rovině je hamiltonovský.*

Lze jí převést úplně stejným způsobem i další výsledky z grafů na plochách na jejich hranové grafy. Ukážeme jako příklad použití pro 2-hranovou hamiltonovskou souvislost na rovinných grafech. (Pouze vysvětlujeme známý postup bez nároku na autorství.) Zřejmě nebude snadné použít nějaké rozšíření této metody pro důkaz celé Thomassenovy hypotézy, protože například existují grafy s nakreslením v projektivní rovině, které jsou 4-souvislé a nejsou 2-hranově hamiltonovsky souvislé (viz [50]*). Další problém je, že pro kubické grafy konstrukce vytvoří pouze jejich hranové grafy.

Věta 5.22. *Každý 4-souvislý hranový graf rovinného grafu je 2-hranově hamiltonovsky souvislý.*

Důkaz. Nechť G je 4-souvislý hranový graf rovinného grafu H. Poznamenejme, že graf H je esenciálně hranově 4-souvislý. Pro rovinné nakreslení H sestrojíme nový graf G_P následující konstrukci.

(i) Odstraníme z grafu H všechny uzly stupně 1, podrozdělíme všechny hrany (tzn. každou hranu nahradíme cestou s jedním vnitřním uzlem) a výsledný graf označíme G_0.

(*ii*) Sestrojíme graf G_1 tak, že pro každý uzel x takový, že $d_{G_0}(x) \geq 3$ doplníme do grafu G_0 kružnici s uzly $N_H(x)$ tak, že v nakreslení G_1 odpovídajícímu rovinnému nakreslení H se žádné hrany neprotínají (podrobný popis s obrázky viz [44], [17]). Dále pro všechny uzly y grafu G_0 odpovídající uzlům stupně 2 grafu H spojíme uzly $N_{G_0}(y)$ hranou a odstraníme uzel y. Nakonec odstraníme všechny uzly z, pro které $d_{G_0}(z) = 3$.

(*iii*) V grafu G_1 pro všechny uzly w stupně alespoň 8 provedeme následující konstrukci.

(α) Podrozdělíme všechny hrany s uzlem w v grafu G_1, doplníme kružnici k se všemi novými uzly tak, že v nakreslení odpovídajícímu rovinnému nakreslením G_1 se žádné hrany neprotínají a odstraníme uzel w.

(β) Vezmeme maximální párování P v kružnici k takové, že pro každý blok B grafu G_0 existují alespoň dva páry, které mají sousední uzly jen na k nebo v uzlech odpovídajících uzlům bloku B. Pro každý pár p_ip_j v párování P odstraníme hranu p_ip_j a sloučíme uzly p_i, p_j do jednoho. Pro všechny uzly o na k, které nejsou v P vybereme jednoho souseda p_s uzlu o, odstraníme hranu op_s a sloučíme uzly o, p_s.

Z konstrukce je zřejmé, že graf G_P je rovinný a lze ho doplnit přidáním hran na hranový graf G_L, který se od grafu G liší v počtu simpliciálních uzlů v některých klikách, ale každá klika s neprázdnou množinou simpliciálních uzlů v grafu G má neprázdnou množinu simpliciálních uzlů i v grafu G_L (podrobnosti viz [44], [17]).

Ukážeme že G_P je 4-souvislý. Všechny uzly G_P mají podle konstrukce stupeň alespoň 4. Hranový graf G_L je 4-souvislý, protože se liší od G pouze počtem simpliciálních uzlů v klikách a podle konstrukce, každá klika se simpliciálním uzlem má alespoň 4 uzly, které nejsou simpliciální. Všechny podgrafy odpovídající klikám v G_L jsou 2-souvislé, takže řez R grafu G_P velikosti nejvýše 3 obsahuje uzly, jejichž odpovídající uzly v G_L jsou v jedné klice K. Pokud 2 různé komponenty K_1, K_2 grafu $G_P - R$ obsahují uzly, jimž odpovídající v G_L neleží v K, existují mezi K_1, K_2 v grafu G_P podle konstrukce 4 uzlově disjunktní cesty. (Klika K má alespoň 8 uzlů, které nejsou simpliciální.) Spor. Tedy jedna komponenta grafu $G_P - R$ obsahuje pouze uzly, jimž odpovídající uzly v G_L leží v klice K. Podle konstrukce snadno nalezneme, jak podgrafy odpovídající klice K vypadají, a ověříme spor.

Podle hlavního výsledku v [50] je G_P 2-hranově hamiltonovsky souvislý a přidání hran na tom nic nemění. ■

Obdobný výsledek dokázali Lai, Shao a Zhan pro *grafy kvazi bez* $K_{1,3}$ (z anglického quasi claw-free) tj. grafy, v nichž každý pár uzlů ve vzdálenosti 2 má společného souseda w sousedícího pouze s uzly v množině $N[u] \cup N[v]$.

Věta 5.23 [45]. *Každý 4-souvislý hranový graf grafu kvazi bez $K_{1,3}$ je hamiltonovsky souvislý.*

Pro hranové grafy a grafy bez $K_{1,3}$ s vyšší souvislostí se podařilo dokázat postupně následující věty. Existenci hranice pro k-souvislost hranových grafů, která zajišťuje hamiltonovskou souvislost stanovil Zhan [62](1991).

Věta 5.24 [62]. *Každý 7-souvislý hranový graf multigrafu je hamiltonovsky souvislý.*

Ryjáčkův uzávěr ukázal existenci hranice pro k-souvislost grafů bez $K_{1,3}$, která zajišťuje hamiltonovskost.

Věta 5.25 [52]. *Každý 7-souvislý graf bez $K_{1,3}$ je hamiltonovský.*

Předchozí výsledky zesílili Hu, Tian a Wei [30](2005).

Věta 5.26 [30]. *Buď G 6-souvislý hranový graf multigrafu s nejvýše 29 uzly stupně 6. Pak G je hamiltonovsky souvislý.*

Věta 5.27 [30]. *Buď G 6-souvislý graf bez $K_{1,3}$ s nejvýše 29 uzly stupně 6. Pak G je hamiltonovský.*

První hranice k-souvislosti zaručující hamiltonovskou souvislost pro grafy bez $K_{1,3}$ byla stanovena Brandtem [11], který dokázal, že každý 9-souvislý graf bez $K_{1,3}$ je hamiltonovsky souvislý.

Výsledek byl později zlepšen Hu, Tian and Wei [30].

Věta 5.28 [30]. *Každý 8-souvislý graf bez $K_{1,3}$ je hamiltonovsky souvislý.*

Důsledkem věty 5.26 a M-uzávěru je následující věta.

Věta 5.29 [54]*. *Buď G 6-souvislý graf bez $K_{1,3}$ s nejvýše 29 uzly stupně 6. Pak G je hamiltonovsky souvislý.*

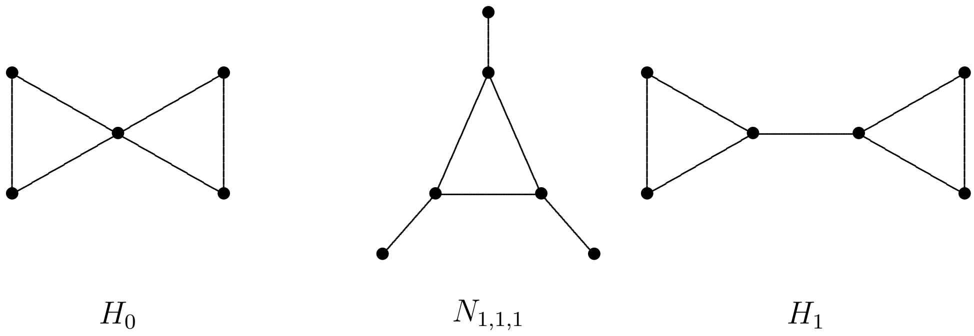
Další zlepšení pro 6-souvislé grafy hranové grafy s dodatečnou podmínkou (s možností rozšíření M-uzávěrem) dokázal Zhan [61]. Větší posun přinesla až práce Kaisera a Vrány [33]*.

Věta 5.30 [33]*. *Buď G 5-souvislý graf bez $K_{1,3}$ s minimálním stupněm alespoň 6. Pak G je hamiltonovsky souvislý.*

Mírně okrajovou oblastí výzkumu jsou zakázané dvojice pro hamiltonovské vlastnosti vyžadující 4-souvislost. Zmíníme je zde především, protože dávají určitý vhled pro platnost hypotézy na malých grafech stejně jako poslední směr výzkumů v této kapitole, generování úplných databází malých snarků pomocí počítače. Přímý výsledek dokázali Ryjáček a Vrána [55] pomocí 1HC-uzávěru. Jednoznačně určený graf s uzly stupně 4,2,2,2,2 (anglicky často nazývaný hourglass) označíme H_0.

Věta 5.31 [55]*. *Každý 4-souvislý graf bez $K_{1,3}$ a bez H_0 je 1-hamiltonovsky souvislý.*

Známé jsou ještě výsledky odvozené od zakázaných dvojic pro hamiltonovskou souvislost u 3-souvislých grafů. Odebrání uzlu ze 4-souvislého grafu sníží stupeň souvislosti nejvýše o jedna a můžeme proto tyto výsledky přímo použít i pro 1-hamiltonovskou souvislost a 4-souvislé grafy. Označme P_k cestu s počtem uzlů k. Označme $N_{i,j,k}$ graf složený trojúhelníku a tří cest délek i, j, k po dvou s prázdným průnikem takových, že každá cesta má s trojúhelníkem společný právě jeden uzel a to koncový. Označme H_i graf složený ze dvou trojúhelníků spojených právě jednou cestou délky i. Příklady popsaných grafů jsou na obrázku 5.1.



Obrázek 5.1

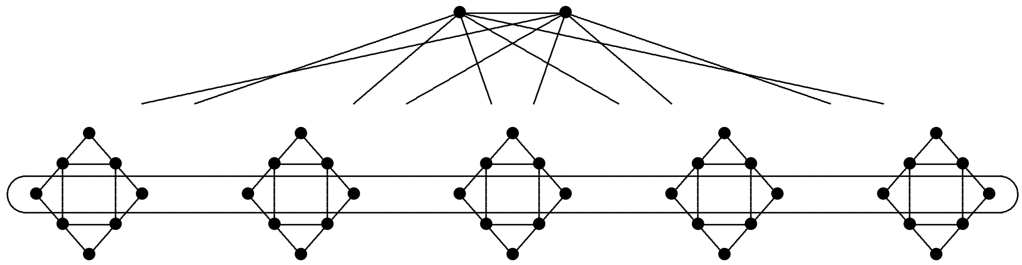
Věta 5.32 [21]*. *Jestliže G je 3-souvislý bez X a Y pro $X = K_{1,3}$ a $Y = P_8, N_{1,1,3}$ nebo $N_{1,2,2}$, pak G je hamiltonovsky souvislý.*

Věta 5.33 [15]. *Jestliže G je 3-souvislý bez X a Y pro $X = K_{1,3}$ a $Y = H_1$, pak G je hamiltonovsky souvislý.*

Velmi zajímavý pokus pro testování hypotéz jak pozitivně tak negativně je generování databází všech malých snarků. V současnosti se Brinkmannovi a kol. [13] podařilo vygenerovat všechny snarky s nejvýše 36 uzly. *Slabý snark* je cyklicky hranově 4-souvislý kubický graf, který nemá hranové 3-obarvení.

Pozorování 5.34 [13]. *Neexistují protipříklady na hypotézu 5.16 mezi snarky s nejvýše 36 uzly, slabými snarky s nejvýše 34 uzly a obecnými kubickými cyklicky hranově 4-souvislými grafy s nejvýše 26 uzly.*

5.3 Vyvrácené hypotézy implikující Thomassenovu hypotézu



Obrázek 5.2

S Thomassenovou hypotézou částečně souvisí následující hypotéza, kterou vyslovil Chvátal v [18]. Počet komponent grafu G označíme $\omega(G)$. Graf G je t-tuhý ($t \in \mathbb{R}, t \geq 0$), jestliže $|S| \geq t * \omega(G-S)$ pro každou $S \subset V(G)$ takovou, že $\omega(G-S) > 1$. Tuhost $\tau(G)$ grafu G je největší číslo t, pro které je G t-tuhý.

Hypotéza 5.35 [18]. **(Chvátal (1973))** *Existuje t takové, že každý t-tuhý graf je hamiltonovský.*

V současné době je známo, že hypotéza neplatí pro $t = 2$. Protipříklad nalezli Bauer a kol. v [2]. Konstrukce je naznačená na obrázku 5.2. Horní dva uzly jsou spojené hranami se všemi zbylými uzly grafu. Oblast nakreslená kolem uzlů představuje kliku na všech uzlech uvnitř.

Ačkoliv obecně zůstává hypotéza otevřená, pro některé speciální třídy grafů se ji podařilo dokázat. Souvislost $\kappa(G)$ grafu G je největší číslo k, pro které je G k-souvislý. Z našeho hlediska je důležitý následující výsledek Matthewse a Sumnera v [47], který umožňuje převést v grafech bez $K_{1,3}$ tuhost na souvislost.

Věta 5.36 [47]. *Pro každý graf G bez $K_{1,3}$ platí: $\tau(G) = \frac{1}{2}\kappa(G)$.*

Připomeňme, že v současnosti je podle věty 5.30 a předchozího věty každý 3-tuhý graf bez $K_{1,3}$ hamiltonovsky souvislý. Další obdobné výsledky mimo naši oblast zájmu lze nalézt například v [14].

V roce 2002 vyslovil Fleischner v [24] hypotézu BMC (bipartizing matching conjecture), která se pokouší spojit Thomassenovu hypotézu s dalšími slavnými hypotézami. Mějme kubický graf G s dominantní kružnicí D. Budeme značit (G, D), abychom vyjádřili, kterou dominantní kružnici v G máme na mysli. Označíme $\{q_1, \ldots, q_k\} := V(G) - V(D)$, a pro uzel $v \in V(G)$ označíme E_v množinu hran obsahujících uzel v. Potlačením uzlu $u \in V(G)$ stupně 2 budeme rozumět odstranění uzlu u z grafu G a spojení sousedních uzlů u v G hranou. Buď $M \subset E(G) - E(D)$ párování v grafu (G, D). Definujeme graf G_M tak, že z grafu G odstraníme M a potlačíme vzniklé uzly u stupně 2. Pokud G je kubický a $V(M) = V(G)$, definujeme $G_M = \emptyset$.

Bipartizující párování pro (G, D) je párování $M \subset E(G) - E(D)$ takové, že graf G_M je bipartitní a $E_{q_i} \cap M \neq \emptyset$ pro $i = 1, \ldots, k$. Definujeme G_M jako bipartizující jestliže $V(G_M) = \emptyset$.

Hypotéza 5.37 [24]. *Každý cyklicky 4-souvislý kubický graf G, který není hranově 3-obarvitelný, má pro každou dominantní kružnici D dvě disjunktní bipartizující párování.*

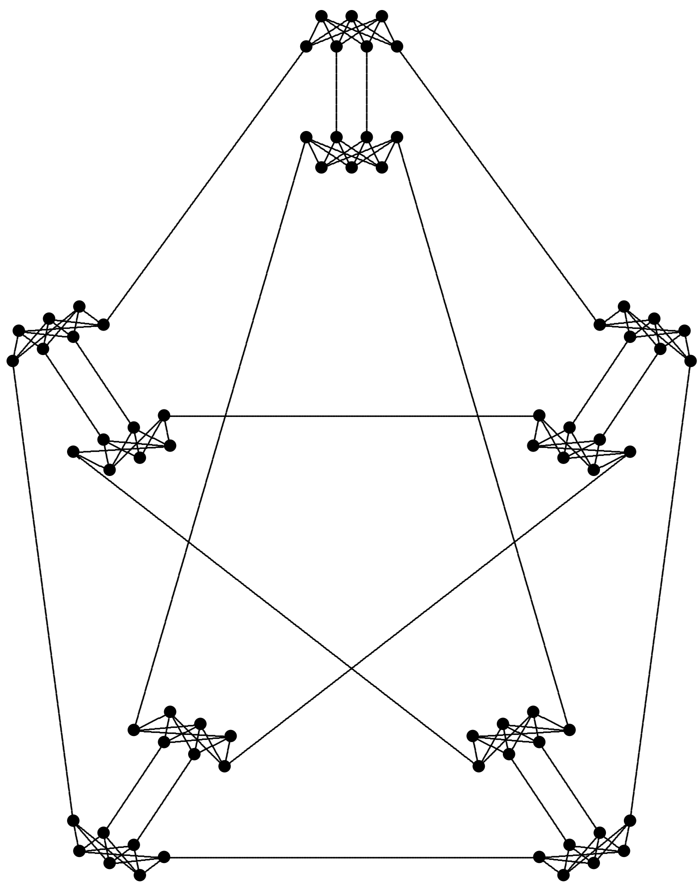
V článku [29] sestrojil Hoffmann-Ostenhof protipříklad na 5.37 a upravil hypotézu do následující formy.

Hypotéza 5.38 [29]. *Každý cyklicky 4-souvislý kubický graf G má alespoň jednu dominantní kružnici D takovou, že (G, D) má dvě disjunktní bipartizující párování.*

Buď A podmnožina přirozených čísel. Graf G nazveme A-pokrytelný, jestliže G má podgraf se všemi uzly sudého stupně, který obsahuje alespoň jednu hranu každého hranového řezu grafu G, pro který platí $|T| \in A$. Množinu A nazveme pokrytelnou, jestliže každý graf je A-pokrytelný. V článku [32] Kaiser a Škrekovski vyslovili hypotézu, že $\mathbb{N} + 3 = \{4, 5, 6, \ldots\}$ je pokrytelná.

Čada a kol. v [20]* sestrojili nekonečnou třídu protipříkladů pro $A = \{4, 5\}$ a zmíněnou hypotézu tím vyvrátili. Pro úplnost uvádíme, že Thomassenova hypotéza lze vyjádřit tímto jazykem následujícím způsobem (zmíněno v [20]*). Každý cyklicky 4-souvislý kubický graf G je $\mathbb{N} + 3$-pokrytelný.

V roce 1967 Kotzig (viz [10]) položil otázku, zda každý 4-regulární graf má dekompozici na dva hamiltonovské cykly. Nezávisle v roce 1969 Nash-Williams v [49] položil stejnou otázku (jinak formulovanou), zda je každý 4-souvislý 4-regulární graf hamiltonovský (po odebrání hamiltonovské kružnice by pak ze 4-souvislosti zbyla ve 4-regulárním grafu druhá). Protipříklad našel Meredith [48] již v roce 1973 a je po něm pojmenován - Meredithův graf. Je na obrázku 5.3.



Obrázek 5.3

Kapitola 6

Závěr

Tato práce není přehledem výsledků kolem Thomassenovy hypotézy, ale pokusem o její objasnění. Nejedná se o pokus neúspěšný, pouze neukončený. Součásti postupu nedílně tvoří dokazování slabších hypotéz i vyvracení silnějších hypotéz, stejně jako důkazy ekvivalencí hypotéz zdánlivě slabších či silnějších. Důležitou součástí práce je i přehled všech dosažených výsledků.

Ukázali jsme jak zdánlivě nejslabší verzi hypotéz, která říká, že každý snark má dominantní kružnici, tak verze zdánlivě velmi silné:

- (i) Každý 4-souvislý graf bez $K_{1,3}$ je 1-hamiltonovsky souvislý.
- (ii) Každý 4-souvislý hranový graf multigrafu je 2-hranově hamiltonovsky souvislý.
- (iii) Každý 2-souvislý graf bez $K_{1,3}$ má Tutteovu kružnici.
- (iv) Libovolný podgraf H esenciálně hranově 4-souvislého kubického grafu. s $\delta(H) = 2$ a $|V_2(H)| = 4$ je silně $V_2(H)$-dominantní.

Dále jsme ukázali částečná řešení:

- (i) Každý 5-souvislý graf bez $K_{1,3}$ s minimálním stupněm alespoň 6 je hamiltonovsky souvislý.
- (ii) Každý 4-souvislý graf bez $K_{1,3}$ a bez H_0 je 1-hamiltonovsky souvislý.
- (iii) Každý 4-souvislý hranový graf roviného grafu je 2-hranově hamiltonovsky souvislý.

Snaha o nalezení protipříkladu vedla k vyvrácení hypotézy, že každý graf je $\mathbb{N}+3$-pokrytelný.

K dosažení výsledků byly využity nové důkazové techniky:

(i) uzávěr grafů bez $K_{1,3}$ na hranové grafy multigrafů zachovávající 1-hamiltonovskou souvislost,

(ii) uzávěr grafů bez $K_{1,3}$ na hranové grafy multigrafů zachovávající hamiltonovskou souvislost,

(iii) uzávěr grafů bez $K_{1,3}$ na hranové grafy zachovávající P-souvislost,

(iv) jednoznačně definovaný vzor hranových grafů multigrafů.

V současné době neznáme techniku, která by přímo mohla vést k důkazu Thomassenovy hypotézy nebo k jejímu vyvrácení. Hlavní motivací současného úsilí o pokrok v této oblasti je zlepšování postupů a důkazových technik, nikoliv výsledků samotných. Hlavním cílem této práce je ukázat na příkladech takové zlepšování (nikoliv samotný důkaz hypotézy, i když se i o něj pokoušíme). Začíná se rozvíjet zkoumání souvislostí s grafy na plochách a technika "zobecněných koster". Problém je v současnosti intenzivně studován a je možné, že se v blízké budoucnosti dočkáme výrazného posunutí hranic známého.

Kapitola 7

Shrnutí

Thomassenova hypotéza se časem ukázala jako zásadní problém v teorii grafů. V současné době je publikováno přes dvacet ekvivalentních hypotéz se širokým záběrem od hamiltonovských vlastností přes dominantní tahy až po Tutteovy struktury.

V této práci jsme ukázali ekvivalence jak se zdánlivě slabšími hypotézami, tak hypotézami zdánlivě velmi silnými. Zdánlivé oslabování ekvivalentních hypotéz směřuje hlavně na vlastnosti podtříd kubických grafů. Zdánlivé zesilování vede k rozšiřování ekvivalentních hypotéz na grafy bez $K_{1,3}$, zesilování na silnější vlastnosti nebo k oslabování podmínky souvislosti grafů.

Dále jsme ukázali částečná řešení spočívající ve vyslovení dodatečných podmínek na vzor hranových grafů, zesílení podmínky na souvislost, nebo výsledky z oblasti zakázaných dvojic indukovaných podgrafů. Snaha o nalezení protipříkladu vedla k vyvrácení hypotézy, že každý graf je $\mathbf{N} + 3$-pokrytelný.

K dosažení výsledků byly využity nové důkazové techniky v oblasti uzávěrů grafů bez $K_{1,3}$ a v oblasti jednoznačné korespondence vzorů hranových grafů multigrafů. Důkazy využívají nové charakterizace podtříd grafů bez $K_{1,3}$ zejména grafů bez $K_{1,3}$, u kterých každé 2-souvislé okolí uzlu indukuje kliku.

Kapitola 8

Summary

Thomassen's conjecture turned out to be a fundamental problem in graph theory. Currently, more than twenty equivalent conjectures have been published with a wide range from hamiltonian properties through dominating trails to Tutte structures.

In the present thesis we show both equivalences with seemingly weaker conjectures and equivalences with seemingly very strong conjectures. Apparent weakening of the equivalent conjectures is oriented mainly towards the properties of subclasses of cubic graphs, while apparent strengthening leads to extending the equivalent conjectures to claw-free graphs, to strengthening the conjectures to stronger properties or to weakening of the connectivity assumption on the graph.

Furthermore, we have shown partial solutions consisting in imposing additional conditions on the root graph or in strengthening the connectivity condition, and we also present results on pairs of forbidden subgraphs. Our effort to find a counterexample led to disproving a conjecture, that every graph is $\mathbf{N}+3$-coverable.

The results were obtained using new proof techniques for closures of claw-free graphs and for uniqueness of the root graph of a line graphs of a multigraph. The proofs use new characterizations of subclasses of claw-free graphs, especially claw-free graphs in which every 2-connected neighbourhood induces a clique.

Kapitola 9

Zusammenfassung

Thomassen-Vermutung hat sich im Laufe der Zeit als ein grundlegendes Problem der Graphentheorie erwiesen. Es sind derzeit über 20 äquivalente Vermutungen publiziert, die durch ein breites Spektrum von hamiltonschen Eigenschaften, dominanten Wegen und Tutte-Strukturen charakterisiert sind.

In dieser Arbeit zeigen wir Äquivalenz mit sowohl scheinbar schwächeren als auch scheinbar sehr starken Vermutungen. Eine scheinbare Schwächung der Vermutung ist durch Eigenschaften von Unterklassen der kubischen Graphen charakterisiert. Eine scheinbare Verschärfung der Vermutung entschpricht einer Verallgemeinerung auf klauenfreie Graphen und eine Schwächung der Bedingung für Zusammenhang des Graphen.

Darüber hinaus haben wir partielle Lösungen gezeigt, die zusätzliche Bedingungen beinhalten wie stärkeren Bedingungen für Zusammenhang des Graphen. oder Paare von verbotenen Untergraphen. Ein Versuch, ein Gegenbeispiel zu finden, hat zum Widerlegen der Vermutung über $\mathbf{N}+3$-Abdeckbarkeit geführt.

Um diese Ergebnisse zu erreichen, haben wir neue Beweistechniken im Bereich der Hüllen für klauenfreie Graphen eingephiert Wir haben auch eine eindeutige Korrespondenz von Kantengraphen ausgenutzt. In Beweisen haben wir dazu noch eine neue Charakterisierung von klauenfreien Graphen, in jeden jede zweifach zusammenhängende Nachbarschaft eine Clique induziert, benutzt.

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Kapitola 10

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Přílohy

1. Hajo Broersma, Gašper Fijavž, Tomáš Kaiser, Roman Kužel, Zdeněk Ryjáček a Petr Vrána: Contractible Subgraphs, Thomassen's Conjecture and the Dominating Cycle Conjecture for Snarks, Discrete Mathematics 308 (2008), 6064-6077. (IF 0.519)

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Contractible subgraphs, Thomassen's conjecture and the dominating cycle conjecture for snarks

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Abstract

We show that the conjectures by Matthews and Sumner (every 4-connected claw-free graph is Hamiltonian), by Thomassen (every 4-connected line graph is Hamiltonian) and by Fleischner (every cyclically 4-edge-connected cubic graph has either a 3-edge-coloring or a dominating cycle), which are known to be equivalent, are equivalent to the statement that every snark (i.e. a cyclically 4-edge-connected cubic graph of girth at least five that is not 3-edge-colorable) has a dominating cycle.

We use a refinement of the contractibility technique which was introduced by Ryjáček and Schelp in 2003 as a common generalization and strengthening of the reduction techniques by Catlin and Veldman and of the closure concept introduced by Ryjáček in 1997.

Keywords: Dominating cycle; Contractible graph; Cubic graph; Snark; Line graph; Hamiltonian graph

1. Introduction

In this paper we consider finite undirected graphs. All the graphs we consider are loopless (with one exception in Section 3); however, we allow the graphs to have multiple edges. We follow the most common graph-theoretic terminology and notation, and for concepts and notation not defined here we refer the reader to [2]. If F, G are graphs then $G - F$ denotes the graph $G - V(F)$ and by an a, b-*path* we mean a path with end vertices a, b. A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to the claw $K_{1,3}$.

In 1984, Matthews and Sumner [8] posed the following conjecture.

Conjecture A (*[8]*). *Every 4-connected claw-free graph is Hamiltonian.*

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Since every line graph is claw-free (see [1]), the following conjecture by Thomassen is a special case of Conjecture A.

Conjecture B ([12]). *Every 4-connected line graph is Hamiltonian.*

A closed trail T in a graph G is said to be *dominating*, if every edge of G has at least one vertex on T, i.e., the graph $G - T$ is edgeless (a closed trail is defined as usual, except that we allow a single vertex to be such a trail). The following result by Harary and Nash-Williams [6] shows the relation between the existence of a dominating closed trail (abbreviated DCT) in a graph G and Hamiltonicity of its line graph $L(G)$.

Theorem 1 ([6]). *Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G contains a DCT.*

Let k be an integer and let G be a graph with $|E(G)| > k$. The graph G is said to be *essentially k-edge-connected* if G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ are nontrivial (i.e. containing at least one edge). If G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ contain a cycle, G is said to be *cyclically k-edge-connected*.

It is well-known that G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected. Thus, the following statement is an equivalent formulation of Conjecture B.

Conjecture C. *Every essentially 4-edge-connected graph contains a DCT.*

By a *cubic* graph we will always mean a regular graph of degree 3 without multiple edges. It is easy to observe that if G is cubic, then a DCT in G becomes a dominating cycle (abbreviated DC), and that every essentially 4-edge-connected cubic graph must be triangle-free, with a single exception of the graph K_4. To avoid this exceptional case, we will always consider only essentially 4-edge-connected cubic graphs on at least five vertices.

Since a cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected (see [5], Corollary 1), the following statement, known as the Dominating Cycle Conjecture, is a special case of Conjecture C.

Conjecture D. *Every cyclically 4-edge-connected cubic graph has a DC.*

Restricting to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Fleischner [4].

Conjecture E ([4]). *Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a DC.*

In [10], a closure technique was used to prove that Conjectures A and B are equivalent. Fleischner and Jackson [5] showed that Conjectures B–D are equivalent. Finally, Kochol [7] established the equivalence of these conjectures with Conjecture E. Thus, we have the following result.

Theorem 2 ([5,7,10]). *Conjectures A–E are equivalent.*

A cyclically 4-edge-connected cubic graph G of girth $g(G) \geq 5$ that is not 3-edge-colorable is called a *snark*. Snarks have turned out to be an important class of graphs, for example in the context of nowhere zero flows. For more information about snarks see the paper [9]. Restricting our considerations to snarks, we obtain the following special case of Conjecture E.

Conjecture F. *Every snark has a DC.*

The following theorem, which is the main result of this paper, shows that Conjecture F is equivalent to the previous ones.

Theorem 3. *Conjecture F is equivalent to Conjectures A–E.*

The proof of Theorem 3 is postponed to Section 4.

As already noted, every cyclically 4-edge-connected cubic graph other than K_4 must be triangle-free. Thus, the difference between Conjectures E and F consists in restricting to graphs which do not contain a 4-cycle. For the proof of the equivalence of these conjectures in Section 4 we first develop in Section 2 a refinement of the technique of contractible subgraphs that was developed in [11] as a common generalization of the closure concept [10] and Catlin's collapsibility technique [3], and in Section 3 a technique that allows us to handle the (non)existence of a DC while replacing a subgraph of a graph by another one.

2. Weakly contractible graphs

In this section we introduce a refinement of the contractibility technique from [11] under a special assumption which is automatically satisfied in cubic graphs. We basically follow the terminology and notation of [11].

For a graph H and a subgraph $F \subset H$, $H|_F$ denotes the graph obtained from H by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges (i.e. edges with one vertex of degree 1). Note that $H|_F$ may contain multiple edges and $|E(H|_F)| = |E(H)|$. For a subset $X \subset V(H)$ and a partition $\mathcal{A}$ of X into subsets, $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1 and a_2 are in the same element of $\mathcal{A}$, and $H^{\mathcal{A}}$ denotes the graph with vertex set $V(H^{\mathcal{A}}) = V(H)$ and edge set $E(H^{\mathcal{A}}) = E(H) \cup E(\mathcal{A})$ (here the sets $E(H)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e. if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $H^{\mathcal{A}}$).

Let F be a graph and $A \subset V(F)$. Then F is said to be *A-contractible*, if for every even subset $X \subset A$ (i.e. with $|X|$ even) and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$. In particular, the case $X = \emptyset$ implies that an A-contractible graph has a DCT containing all vertices of A.

If H is a graph and $F \subset H$, then a vertex $x \in V(F)$ is said to be a *vertex of attachment of F in H* if x has a neighbor in $V(H) \setminus V(F)$. The set of all vertices of attachment of F in H is denoted by $A_H(F)$. Finally, $\mathrm{dom}_{tr}(H)$ denotes the maximum number of edges of a graph H that are dominated by (i.e. have at least one vertex on) a closed trail in H. Specifically, H has a DCT if and only if $\mathrm{dom}_{tr}(H) = |E(H)|$.

The following theorem shows that a contraction of an $A_H(F)$-contractible subgraph of a graph H does not affect the value of $\mathrm{dom}_{tr}(H)$.

Theorem 4 (*[11]*). *Let F be a connected graph and let $A \subset V(F)$. Then F is A-contractible if and only if*

$$\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$$

for every graph H such that $F \subset H$ and $A_H(F) = A$.

Specifically, F is A-contractible if and only if, for any H such that $F \subset H$ and $A_H(F) = A$, H has a DCT if and only if $H|_F$ has a DCT (the "only if" part follows by Theorem 4; the "if" part can be easily seen by the definition of A-contractibility).

Let F be a graph and let $A \subset V(F)$. The graph F is said to be *weakly A-contractible*, if for every *nonempty* even subset $X \subset A$ and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$.

Thus, in comparison with the contractibility concept as introduced in [11], we do not include the case $X = \emptyset$. This means that we do not require that a weakly A-contractible graph has a DCT containing all vertices of A.

Clearly, every A-contractible graph is also weakly A-contractible. It is easy to see that if F is weakly A-contractible and $|A| \geq 3$, then $d_F(x) \geq 2$ for every $x \in A$.

Examples. 1. The graphs in Fig. 1 are examples of graphs that are weakly A-contractible but not A-contractible (vertices of the set A are double-circled).

2. The triangle C_3 is A-contractible for any subset A of its vertex set.
3. Let C be a cycle of length $\ell \geq 4$, let $x, y \in V(C)$ be nonadjacent and set $A = V(C)$, $X = \{x, y\}$ and $\mathcal{A} = \{\{x, y\}\}$. Then there is no DCT in C containing the edge $xy \in C^{\mathcal{A}}$ and all vertices of A. Hence no cycle C of length at least 4 is weakly $V(C)$-contractible.

If H is a graph and $F \subset H$, then H_{-F} denotes the graph with vertex set $V(H_{-F}) = V(H) \setminus (V(F) \setminus A_H(F))$ and with edge set $E(H_{-F}) = E(H) \setminus E(F)$ (equivalently, H_{-F} is the graph determined by the edge set $E(H) \setminus E(F)$).

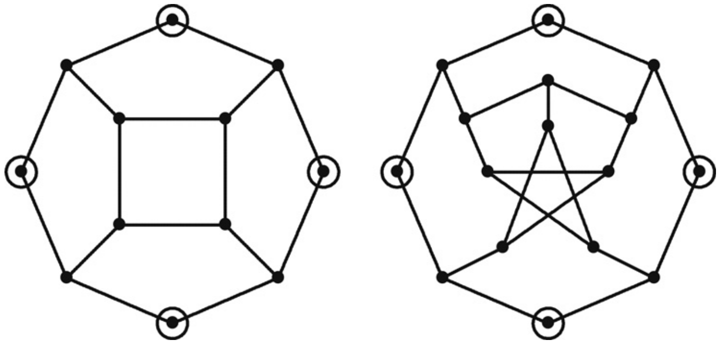

Fig. 1.

Our next theorem shows that, in a special situation, weak contractibility is sufficient to obtain the equivalence of Theorem 4.

Theorem 5. *Let F be a graph and let $A \subset V(F)$, $|A| \geq 2$. Then F is weakly A-contractible if and only if*

$$\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$$

for every graph H such that $F \subset H$, $A_H(F) = A$, $d_{H_{-F}}(a) = 1$ for every $a \in A$, and $|V(K) \cap A| \geq 2$ for at least one component K of H_{-F}.

Proof. The proof of Theorem 5 basically follows the proof of Theorem 2.1 of [11].

Let F be a graph and let H be a graph satisfying the assumptions of the theorem. Then every closed trail T in H corresponds to a closed trail in $H|_F$, dominating at least as many edges as T. Hence immediately $\mathrm{dom}_{tr}(H) \leq \mathrm{dom}_{tr}(H|_F)$.

Suppose that F is weakly A-contractible and let T' be a closed trail in $H|_F$ such that T' dominates $\mathrm{dom}_{tr}(H|_F)$ edges and, subject to this condition, T' has maximum length. If $v_F \notin V(T')$, then T' is also a closed trail in H, implying $\mathrm{dom}_{tr}(H|_F) \leq \mathrm{dom}_{tr}(H)$, as requested. Hence we can suppose $v_F \in V(T')$.

If T' is nontrivial, i.e. contains an edge, then the edges of T' determine in H a system of trails $\mathcal{P} = \{P_1, \ldots, P_k\}$, $k \geq 1$, such that every $P_i \in \mathcal{P}$ has end vertices in A (note that all trails in $\mathcal{P}$ are open since $d_{H_{-F}}(a) = 1$ for all $a \in A$). Since $d_{H_{-F}}(a) = 1$ for all $a \in A$, every $x \in A$ is an end vertex of at most one trail from $\mathcal{P}$, and we set $X = \{x \in A_H(F) | x \text{ is an end vertex of some } P_i \in \mathcal{P}\}$ and $\mathcal{A} = \{A_1, \ldots, A_k\}$, where A_i is the (two-element) set of end vertices of P_i, $i = 1, \ldots, k$.

If T' is trivial (i.e., a one-vertex trail), then we consider a component K of H_{-F} for which $|V(K) \cap A_H(F)| \geq 2$. Let $x_1, x_2 \in V(K) \cap A_H(F)$. If $V(K) \setminus \{x_1, x_2\} \neq \emptyset$ then, since K is connected, K contains a path of length at least 2 with end vertices x_1, x_2, but then we have a contradiction with the maximality of T'. Hence $V(K) = \{x_1, x_2\}$ and $E(K) = \{x_1x_2\}$, and we set $P_1 = x_1x_2$, $\mathcal{P} = \{P_1\}$, $X = \{x_1, x_2\}$ and $\mathcal{A} = \{\{x_1, x_2\}\}$. Note that in both cases the set X is nonempty.

By the weak A-contractibility of F, $F^{\mathcal{A}}$ has a DCT Q, containing all vertices of A and all edges of $E(\mathcal{A})$. The trail Q determines in F a system of trails $Q_1, \ldots, Q_k$ such that every Q_i has its two end vertices in two different elements of $\mathcal{A}$. Now, the trails Q_i together with the system $\mathcal{P}$ form a closed trail in H, dominating at least as many edges as T'. Hence $\mathrm{dom}_{tr}(H|_F) \leq \mathrm{dom}_{tr}(H)$, implying $\mathrm{dom}_{tr}(H|_F) = \mathrm{dom}_{tr}(H)$.

Next suppose that F is not weakly A-contractible (possibly even disconnected). Then, for some nonempty $X \subset A$ and a partition $\mathcal{A}$ of X into two-element sets, $F^{\mathcal{A}}$ has no DCT containing all vertices of A and all edges of $E(\mathcal{A})$. Let $\mathcal{A} = \{\{x'_1, x''_1\}, \ldots, \{x'_k, x''_k\}\}$, and construct a graph H with $F \subset H$ by replacing the edges of $E(\mathcal{A})$ by k vertex disjoint x'_i, x''_i-paths P_i of length at least 3, $i = 1, \ldots, k$, and by attaching a pendant edge to every vertex in $A \setminus X$. Since $X \neq \emptyset$, at least one component K of H_{-F} is a path with end vertices in A, implying $|V(K) \cap A| \geq 2$. Since $F^{\mathcal{A}}$ has no DCT containing all vertices of A and all edges of $E(\mathcal{A})$, H has no DCT. However, clearly $H|_F$ has a DCT and we have $\mathrm{dom}_{tr}(H) < \mathrm{dom}_{tr}(H|_F)$. ■

In the special case of cubic graphs, we have the following corollary.

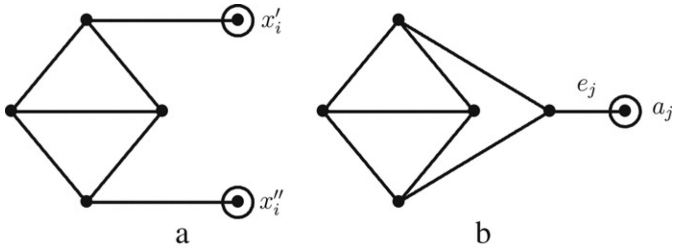

Fig. 2.

Corollary 6. *Let F be a graph with $\delta(F) = 2$, $\Delta(F) \leq 3$ and $|A| \geq 2$, where $A = \{x \in V(F) \mid d_F(x) = 2\}$. Then F is weakly A-contractible if and only if*

$$\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$$

for every cubic graph H such that $F \subset H$, $A_H(F) = A$, and $|V(K) \cap A| \geq 2$ for at least one component K of H_{-F}.

Proof. Clearly $d_{H_{-F}} = 1$ for every $a \in A$, since H is cubic. If F is weakly A-contractible, then $\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$ immediately by Theorem 5. For the rest of the proof, it is sufficient to modify the last part of the proof of Theorem 5 such that the constructed graph H is cubic. To achieve this, it is sufficient to use a copy of the graph in Fig. 2(a) instead of each of the paths P_i, and a copy of the graph in Fig. 2(b) instead of each of the pendant edges attached to the vertices $a_j \in A \setminus X$. Then there is a component K of H_{-F} with $|V(K) \cap A| \geq 2$ since X is nonempty. The graph $H|_F$ has a closed trail dominating all edges except for the edges different from e_j in the copies attached to the vertices in $A \setminus X$, while in H there is no such closed trail. ■

We say that a subgraph $F \subset H$ is a *weakly contractible subgraph of* H if F is weakly $A_H(F)$-contractible. We then have the following corollary.

Corollary 7. *Let H be a cubic graph and let F be a weakly contractible subgraph of H with $\delta(F) = 2$. Then H has a DC if and only if $H|_F$ has a DCT.*

Proof. First note that in a cubic graph every closed trail is a cycle and that a cubic graph with a DC must be essentially 2-edge-connected. Since H is cubic and $\delta(F) = 2$, $A_H(F) = \{x \in V(F) \mid d_F(x) = 2\}$ and the weak contractibility assumption implies F is connected. If every component of H_{-F} contains one vertex from $A_H(F)$, then clearly neither H nor $H|_F$ is essentially 2-edge-connected (since H is cubic) and hence neither H nor $H|_F$ has a DCT. The rest of the proof follows from Corollary 6. ■

Example. Let H be the graph obtained from three vertex-disjoint copies F_1, F_2, F_3 of the graph F_i from Fig. 2(a) by adding edges $x_1'x_2'$, $x_1'x_3'$, $x_2'x_3'$, $x_1''x_2''$, $x_1''x_3''$, $x_2''x_3''$. Then H is cubic, $F_1 \subset H$ is weakly contractible, $H|_{F_1}$ has a DCT, but H has no DC. This example shows that the assumption $\delta(F) = 2$ in Corollaries 6 and 7 cannot be omitted.

3. Replacement of a subgraph

In this section we develop a technique to replace certain subgraphs by others without affecting the (non)existence of a DCT.

Let G be a graph and let $F \subset G$ be a subgraph of G. Let F' be a graph such that $V(F') \cap V(G) = \emptyset$, let $A' \subset V(F')$ be such that $|A'| = |A_G(F)|$ and let $\varphi : A_G(F) \to A'$ be a bijection. Let H be the graph obtained from G_{-F} and F' by identifying each $x \in A_G(F)$ with its image $\varphi(x) \in A'$. We say that the graph H is obtained by *replacement* (*in* G) *of* F *by* F' *modulo* φ and denote $H = G[F \overset{\varphi}{\to} F']$.

Note that if $H = G[F \overset{\varphi}{\to} F']$ then also clearly $G = H[F' \xrightarrow{\varphi^{-1}} F]$.

Let F be a graph and let $A = \{a_1, \ldots, a_k\} \subset V(F)$. Let $\overline{A}$ be a set with $\overline{A} \cap V(F) = \emptyset$, $|\overline{A}| = |A|$, and set $\overline{A} = \{\overline{a}_1, \ldots, \overline{a}_k\}$. Then $\overline{F}^A$ denotes the graph with vertex set $V(\overline{F}^A) = V(F) \cup \overline{A}$ and with edge set $E(\overline{F}^A) = E(F) \cup \{a_i\overline{a}_i | i = 1, \ldots, k\}$ (i.e., $\overline{F}^A$ is obtained from F by attaching a pendant edge to every vertex of A).

The following observation shows that, under certain conditions, the replacement in a graph G of a weakly contractible subgraph by another one affects neither the existence nor the nonexistence of a DCT in G.

Proposition 8. *Let G be a graph with $\delta(G) \geq 1$ and let $F \subset G$ be a weakly contractible subgraph of G such that $|E(F)| \geq 1$, $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Let F', $|E(F')| \geq 1$, be a weakly A'-contractible graph for an $A' \subset V(F')$, and let $\varphi : A_G(F) \to A'$ be a bijection. Then G has a DCT if and only if $G[F \stackrel{\varphi}{\to} F']$ has a DCT.*

Proof. Set $H = G[F \stackrel{\varphi}{\to} F']$. For $|A_G(F)| = 0$ the assumptions $G \not\simeq \overline{F}^{A_G(F)}$ and $\delta(G) \geq 1$ imply that G is disconnected and neither G nor H has a DCT. If $|A_G(F)| = 1$ or if $|A_G(F)| \geq 2$ and $|V(K) \cap A_G(F)| = 1$ for every component K of G_{-F}, then neither G nor H can have a DCT since $|E(F)| \geq 1$, $|E(F')| \geq 1$, $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Thus, we can assume that $|A_G(F)| \geq 2$ and there is a component K of G_{-F} such that $|V(K) \cap A_G(F)| \geq 2$. Then, by Theorem 5, G has a DCT if and only if $G|_F$ has a DCT. Similarly, H has a DCT if and only if $H|_{F'}$ has a DCT, but the graphs $G|_F$ and $H|_{F'}$ are, up to the number of pendant edges at v_F ($v_{F'}$), isomorphic. ■

In the special case of cubic graphs, we obtain the following consequence.

Corollary 9. *Let G be a cubic graph and let $F \subset G$ be a weakly contractible subgraph of G with $\delta(F) = 2$. Let F' be a graph with $\delta(F') = 2$ and $\Delta(F') \leq 3$, let $A' = \{x \in V(F') | d_{F'}(x) = 2\}$ and suppose that F' is weakly A'-contractible. Let $\varphi : A_G(F) \to A'$ be a bijection. Then the graph $H = G[F \stackrel{\varphi}{\to} F']$ is cubic and G has a DC if and only if H has a DC.*

Proof. Clearly $A_G(F) = \{x \in V(F) | d_F(x) = 2\}$ and since G is cubic, we have $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Since φ is a bijection, H is cubic. By Proposition 8, G has a DCT if and only if H has a DCT, but in cubic graphs every DCT is a DC. ■

Now we consider a similar question if F and/or F' are not contractible. We restrict our observations to cubic graphs.

A connected graph F without multiple edges with $\Delta(F) \leq 3$ will be called a *cubic fragment*. For any cubic fragment F and $i = 1, 2$ we set $A_i(F) = \{x \in V(F) | d_F(x) = i\}$ and $A(F) = A_1(F) \cup A_2(F)$ (note that if $F \subset H$, F is connected and H is cubic, then F is a cubic fragment and $A_H(F) = A(F)$). A cubic fragment F is said to be *essential* if $|V(F) \setminus A_1(F)| \geq 2$. It is easy to observe that if F is an essential cubic fragment, the set $V(F) \setminus A_1(F)$ induces (in F) a connected subgraph with at least one edge.

For a cubic fragment F we now introduce the concept of an F-linkage. An F-linkage will be allowed to contain loops. A loop on a vertex v is considered as an edge joining v to itself, and is denoted by an element vv of the edge set. Edges of an F-linkage that are not loops will be referred to as *open edges*.

Let F be a cubic fragment and let B be a graph with $V(B) \subset A(F)$, $E(B) \cap E(F) = \emptyset$, and with components $B_1, \ldots, B_k$. We say that B is an *F-linkage*, if $E(B)$ contains at least one open edge and, for any $i = 1, \ldots, k$,

(i) every B_i is a path (of length at least one) or a loop,
(ii) if B_i is a path of length at least two, then all interior vertices of B_i are in $A_1(F)$,
(iii) if B_i is a loop at a vertex x, then $x \in A_2(F)$.

Let F be a cubic fragment and let B be an F-linkage. Then F^B denotes the graph with vertex set $V(F^B) = V(F)$ and edge set $E(F^B) = E(F) \cup E(B)$. Note that $E(B)$ and $E(F)$ are assumed to be disjoint, i.e. if $h_1 = x_1x_2 \in E(F)$ and $h_2 = x_1x_2 \in E(B)$, then h_1, h_2 are parallel edges of the graph F^B.

Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. For any F_1-linkage B, $\varphi(B)$ denotes the graph with vertex set $V(\varphi(B)) = \{\varphi(x) | x \in V(B)\}$ and edge set $E(\varphi(B)) = \{\varphi(x)\varphi(y) | xy \in E(B)\}$ (note that the sets $E(F_2)$ and $E(\varphi(B))$ are again considered to be disjoint, and we admit $x = y$ in which case $\varphi(x)\varphi(x)$ is a loop at $\varphi(x)$). Note that $\varphi(B)$ is an F_2-linkage.

Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. We say that φ is a *compatible mapping* if

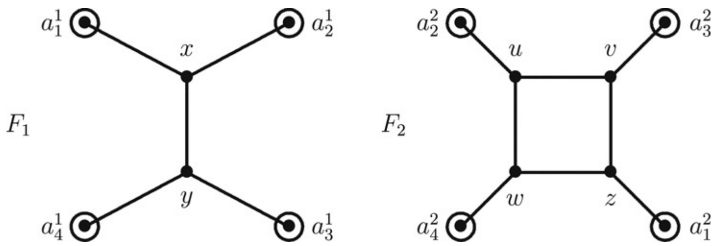

Fig. 3.

(i) $\varphi(A_i(F_1)) = A_i(F_2)$, $i = 1, 2$,

(ii) if B is an F_1-linkage such that F_1^B has a DC containing all open edges of B, then $F_2^{\varphi(B)}$ has a DC containing all open edges of $\varphi(B)$.

For a compatible mapping $\varphi : A(F_1) \to A(F_2)$ we will simply write $\varphi : F_1 \to F_2$.

Let F_1, F_2 be cubic fragments and let $\varphi : A(F_1) \to A(F_2)$ be a bijection such that $\varphi(A_i(F_1)) = A_i(F_2)$, $i = 1, 2$. It is easy to observe that if F_2 is weakly $A(F_2)$-contractible then φ is compatible, and if moreover F_1 is weakly $A(F_1)$-contractible then both φ and φ^{-1} are compatible (note that B cannot contain a path of length at least 2 in this case — this is clear for $|A(F_i)| \leq 2$, and for $|A(F_i)| \geq 3$ this follows from the fact that weak $A(F_i)$-contractibility of F_i then implies $A(F_i) = A_2(F_i)$).

The following example shows that the compatibility of a mapping φ does not imply φ^{-1} is compatible if the F_i's are not weakly contractible.

Example. Let F_1, F_2 be the graphs in Fig. 3 and let $\varphi : A(F_1) \to A(F_2)$ be the mapping that maps a_j^1 on a_j^2, $j = 1, 2, 3, 4$. By a straightforward check of all possible F_1-linkages B and the corresponding DC's in F_1^B and in $F_2^{\varphi(B)}$, we easily see that there are, up to symmetry, the following possibilities.

$E(B)$	DC in F_1^B	DC in $F_2^{\varphi(B)}$
$a_1^1a_4^1$	$a_1^1a_4^1yxa_1^1$	$a_1^2a_4^2wuvza_1^2$
$a_1^1a_2^1$	not existing	not existing
$a_1^1a_2^1, a_2^1a_4^1$	$a_1^1a_2^1a_4^1yxa_1^1$	$a_1^2a_2^2a_4^2wuvza_1^2$
$a_1^1a_3^1, a_3^1a_2^1$	not existing	$a_1^2a_3^2a_2^2uwza_1^2$
$a_1^1a_2^1, a_2^1a_3^1, a_3^1a_4^1$	$a_1^1a_2^1a_3^1a_4^1yxa_1^1$	$a_1^2a_2^2a_3^2a_4^2wuvza_1^2$
$a_1^1a_4^1, a_4^1a_3^1, a_3^1a_2^1$	$a_1^1a_4^1a_3^1a_2^1xa_1^1$	$a_1^2a_4^2a_3^2a_2^2uwza_1^2$
$a_1^1a_4^1, a_2^1a_3^1$	$a_1^1a_4^1ya_3^1a_2^1xa_1^1$	$a_1^2a_4^2wua_2^2a_3^2vza_1^2$
$a_1^1a_2^1, a_3^1a_4^1$	not existing	$a_1^2a_2^2uva_3^2a_4^2wza_1^2$

We conclude that $\varphi : A(F_1) \to A(F_2)$ is a compatible mapping, but there is no compatible mapping of $A(F_2)$ onto $A(F_1)$. Note that this mapping φ will play an important role in the proof of our main result in Section 4.

The following result shows that the replacement of a subgraph of a cubic graph modulo a compatible mapping does not affect the existence of a DC.

Theorem 10. *Let G be a cubic graph and let C be a DC in G. Let $F \subset G$ be an essential cubic fragment such that $G - F$ is not edgeless, and let F' be a cubic fragment such that $V(F') \cap V(G) = \emptyset$ and there is a compatible mapping $\varphi : F \to F'$. Then the graph $G' = G[F \overset{\varphi}{\to} F']$ is a cubic graph having a DC C' such that $E(C) \setminus E(F) = E(C') \setminus E(F')$.*

(Note that if both φ and φ^{-1} are compatible and both F and F' are essential, then G has a DC if and only if $G' = G[F \overset{\varphi}{\to} F']$ has a DC.)

Proof. By the compatibility of φ, $A_1(F') = \varphi(A_1(F))$ and $A_2(F') = \varphi(A_2(F))$, hence G' is cubic. Let C be a DC in G. We show that G' has a DC C' with $E(C) \setminus E(F) = E(C') \setminus E(F')$.

We first observe that $E(C) \cap E(F) \neq \emptyset$. Since F is essential, there is an edge $xy \in E(F)$ with $d_F(x) \geq 2$ and $d_F(y) \geq 2$. Then one of x, y (say, x) is on C. Since $d_F(x) \geq 2$, x has a neighbor x_1 in F, $x_1 \neq y$. Then, since $d_G(x) = 3$, the edge xy or xx_1 is in $E(C) \cap E(F)$.

Let C_F and C_{-F} denote the subgraph of C induced by the edge set $E(C) \cap E(F)$ and $E(C) \cap E(G_{-F})$, respectively. Since $E(C) \cap E(F) \neq \emptyset$ and $G - F$ is not edgeless, C_{-F} is a nonempty system of paths. Let $P_1, \ldots, P_k$ be the components of C_{-F}. Then:

- the end vertices of every P_i are in $A(F)$,
- the interior vertices of every P_i are in $A_1(F)$ or in $V(G) \setminus V(F)$,

where $i = 1, \ldots, k$.

We define an F-linkage B as follows:

(i) for every P_i, let P_i^B be the path obtained from P_i by replacing every maximal subpath of P_i with all interior vertices in $V(G) \setminus V(F)$ by a single edge (with both vertices in $A(F)$),

(ii) for every vertex $x \in A(F) \setminus V(C_{-F})$ which is on C_F (note that such a vertex x must be in $A_2(F)$), let e_x be a loop at x,

(iii) B is the graph with components $\{P_i^B | i = 1, \ldots, k\} \cup \{e_x | x \in A_2(F) \setminus V(C_{-F}) \cap V(C)\}$.

It is immediate to observe that the graph F^B has a DC C^B containing all open edges of B. By the compatibility of φ, the graph $(F')^{\varphi(B)}$ has a DC C'^B containing all open edges of the graph $\varphi(B)$.

Let $C'_{F'}$ denote the subgraph of C'^B induced by the edge set $E(C'^B) \cap E(F')$. Then $C'_{F'}$ is a system of paths, and the edges in $E(C'_{F'}) \cup E(C_{-F})$ determine a cycle C' in $G' = G[F \xrightarrow{\varphi} F']$ with $E(C) \setminus E(F) = E(C') \setminus E(F')$. Note that, by the construction, $V(C) \cap A(F) \subset V(C') \cap A(F')$ (this is clear for vertices x with $d_{C_{-F}}(x) \geq 1$, and for vertices x with $d_{C_{-F}}(x) = 0$ this follows from the fact that both C^B and C'^B dominate all loops in B and in $\varphi(B)$, respectively).

It remains to show that C' is a DC in G'. Thus, let $xy \in E(G')$.

If $x, y \in V(G') \setminus V(F') = V(G) \setminus V(F)$, then x or y is on C_{-F}, implying x or y is on C' since $C_{-F} \subset C'$. If $x, y \in V(F') \setminus A(F')$, then x or y is on $C'_{F'}$, implying x or y is on C' since $C'_{F'} \subset C'$.

Up to symmetry, it remains to consider the case $x \in A(F') = \varphi(A(F))$. If $x \in V(C)$, then also $x \in V(C')$ since $V(C) \cap A(F) \subset V(C') \cap A(F')$, as observed above. Hence we can suppose that $x \notin V(C)$, implying $y \in V(C)$. If $y \in A(F')$, then similarly $y \in V(C')$ and we are done; hence $y \notin A(F')$. Then either $y \in V(F') \setminus A(F')$, or $y \in V(G') \setminus V(F')$. But then, in the first case y is on $C'_{F'}$ since C' is dominating in $(F')^{\varphi(B)}$, and in the second case y is on C_{-F} since C is dominating in G. In either case this implies $y \in V(C')$. ■

The following result shows that the existence of a compatible mapping is not affected by a replacement of a subgraph by another one modulo a compatible mapping.

Proposition 11. *Let X, F be essential cubic fragments such that there is a compatible mapping $\psi : X \to F$. Let $F_1 \subset F$ be an essential cubic fragment, and let F_2 be a cubic fragment such that $V(F) \cap V(F_2) = \emptyset$ and there is a compatible mapping $\varphi : F_1 \to F_2$. Let $F' = F[F_1 \xrightarrow{\varphi} F_2]$. Then there is a compatible mapping $\psi' : X \to F'$.*

Proof. For any $x \in A(X)$ set

$$\psi'(x) = \begin{cases} \psi(x) & \text{if } x \in \psi^{-1}(A(F) \setminus A(F_1)), \\ \varphi(\psi(x)) & \text{if } x \in \psi^{-1}(A(F) \cap A(F_1)). \end{cases}$$

Then $\psi' : A(X) \to A(F')$ is a bijection, and $\psi' : A_i(X) \to A_i(F')$, $i = 1, 2$, by the compatibility of ψ and φ. Let B be an X-linkage such that X^B has a DC containing all open edges of B. By the compatibility of ψ, the graph $F^{\psi(B)}$ has a DC C containing all open edges of $\psi(B)$. We need to show that $(F')^{\psi'(B)}$ has a DC containing all open edges of $\psi'(B)$. We will construct a cubic graph H such that $F \subset H$, H has a DC that coincides with C on F, and the structure of $H - F$ implies that an application of Theorem 10 to H yields the required DC in $(F')^{\psi'(B)}$.

Let $B_1, \ldots, B_k$ be the components of $\psi(B)$, and choose the notation such that

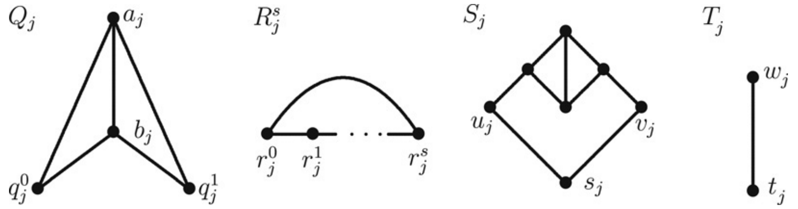

Fig. 4.

- $B_1, \dots, B_p$ ($p \geq 1$) are paths, $V(B_j) = \{x_j^0, \dots, x_j^{\ell_j}\}$ (i.e. B_j is of length ℓ_j), $j = 1, \dots, p$;
- if none of $B_1, \dots, B_k$ is a loop, then $\ell = 0$, otherwise $B_{p+1}, \dots, B_{p+\ell}$ are loops, $V(B_{p+j}) = \{x_{p+j}\}$, $j = 1, \dots, \ell$;
- if $A(F) \setminus V(\psi(B)) = \emptyset$, then $f = 0$, otherwise $A(F) \setminus V(\psi(B)) = \{x_{p+\ell+1}, \dots, x_{p+\ell+f}\}$.

Thus, we have $k = p + \ell$ and $V(\psi(B)) = \cup_{j=1}^{p+\ell}(V(B_j))$.

Let Q_j, R_j^s ($s \geq 2$), S_j and T_j be the graphs shown in Fig. 4. We construct a cubic graph H containing F by the following construction:

- take the graph F with the labeling of vertices of $A(F)$ defined above;
- for each B_j with $1 \leq j \leq p$, $\ell_j = 1$, take one copy of Q_j and for $i = 0, 1$ identify $x_j^i = q_j^i$ if $x_j^i \in A_1(F)$ or add the edge $x_j^i q_j^i$ if $x_j^i \in A_2(F)$, respectively,
- for each B_j with $1 \leq j \leq p$, $\ell_j > 1$, take one copy of R_j^s for $s = \ell_j$ and
  - for $i = 0$ and $i = \ell_j$ identify $x_j^i = r_j^i$ if $x_j^i \in A_1(F)$ or add the edge $x_j^i r_j^i$ if $x_j^i \in A_2(F)$, respectively,
  - for $1 \leq i \leq \ell_j - 1$ identify $x_j^i = r_j^i$;
- for each B_j with $p + 1 \leq j \leq p + \ell$ (if $\ell > 0$) take one copy of S_j, add the edge $x_j s_j$, and if $\ell \geq 2$, then for $j \geq p + 2$ add the edge $v_{j-1}u_j$;
- for each x_j with $p + \ell + 1 \leq j \leq p + \ell + f$ (if $f > 0$) do the following:
  - if $x_j \in A_1(F)$, take one copy of S_j, identify $x_j = s_j$ and if $f \geq 2$, then for $j \geq p + \ell + 2$ add the edge $v_{j-1}u_j$ (if $x_{j-1} \in A_1(F)$), or the edge $w_{j-1}u_j$ (if $x_{j-1} \in A_2(F)$), respectively;
  - if $x_j \in A_2(F)$, take one copy of T_j, identify $x_j = t_j$ and if $f \geq 2$, then for $j \geq p + \ell + 2$ add the edge $v_{j-1}w_j$ (if $x_{j-1} \in A_1(F)$), or the edge $w_{j-1}w_j$ (if $x_{j-1} \in A_2(F)$), respectively;
  - if $x_{p+\ell+1} \in A_2(F)$, then relabel $w_{p+\ell+1}$ as $u_{p+\ell+1}$ and if $x_{p+\ell+f} \in A_2(F)$, then relabel $w_{p+\ell+f}$ as $v_{p+\ell+f}$;
- if $\ell \neq 0$, then
  - for $\ell_1 = 1$ remove the edge $q_1^0 a_1$ and add the edges $q_1^0 u_{p+1}$ and $a_1 v_{p+\ell}$,
  - for $\ell_1 > 1$ remove the edge $r_1^0 r_1^1$ and add the edges $r_1^0 u_{p+1}$ and $r_1^1 v_{p+\ell}$;
- if $f \neq 0$, then
  - for $\ell_1 = 1$ remove the edge $b_1 q_1^1$ and add the edges $b_1 u_{p+\ell+1}$ and $q_1^1 v_{p+\ell+f}$,
  - for $\ell_1 > 1$ remove the edge $r_1^{\ell_1 - 1} r_1^{\ell_1}$ and add the edges $r_1^{\ell_1 - 1} u_{p+\ell+1}$ and $r_1^{\ell_1} v_{p+\ell+f}$.

Then H is a cubic graph, $F \subset H$, $A_H(F) = A(F)$, and it is straightforward to check that H has a DC C^H such that $E(C^H) \cap E(F) = E(C) \cap E(F)$.

Let C_{-F}^H denote the subgraph of C^H induced by the edge set $E(C^H) \cap E(H_{-F})$. Then the structure of the graphs Q_j, R_j^s, S_j and T_j implies the following properties of C_{-F}^H:

- if $1 \leq j \leq p$ and $i = 0$ or $i = \ell_j$, then $d_{C_{-F}^H}(x_j^i) = 1$,
- if $1 \leq j \leq p$ and $1 \leq i \leq \ell_j - 1$, then $d_{C_{-F}^H}(x_j^i) = 2$,
- if $\ell > 0$ and $p + 1 \leq j \leq p + \ell$, then $d_{C_{-F}^H}(x_j) = 0$ and x_j has no neighbor on C_{-F}^H,
- if $f > 0$ and $p + \ell + 1 \leq j \leq p + \ell + f$, then $d_{C_{-F}^H}(x_j) = 0$ and all neighbors of x_j in H_{-F} are on C_{-F}^H.

Set $H' = H[F_1 \xrightarrow{\varphi} F_2]$. By the compatibility of φ and by Theorem 10, H' has a DC $C^{H'}$ such that $E(C^{H'}) \setminus E(F_2) = E(C^H) \setminus E(F_1)$. Specifically, $F' \subset H'$ and $E(C^{H'}) \setminus E(F') = E(C^H) \setminus E(F)$. Let $C^{H'}_{F'}$ and $C^{H'}_{-F'}$ denote the subgraph of $C^{H'}$ induced by $E(C^{H'}) \cap E(F')$ and $E(C^{H'}) \cap E(H'_{-F'})$, respectively. Then $C^{H'}_{-F'} = C^H_{-F}$, and from the above properties of C^H_{-F} we obtain the following properties of $C^{H'}_{F'}$:

- if $1 \le j \le p$ and $i = 0$ or $i = \ell_j$, then $d_{C^{H'}_{F'}}(x^i_j) = 1$,
- if $1 \le j \le p$ and $1 \le i \le \ell_j - 1$, then $d_{C^{H'}_{F'}}(x^i_j) = 0$ and all edges of F' with at least one vertex in $N_{F'}(x^i_j)$ have at least one vertex on $C^{H'}$,
- if $\ell > 0$ and $p + 1 \le j \le p + \ell$, then $d_{C^{H'}_{F'}}(x_j) = 2$,
- if $f > 0$ and $p + \ell + 1 \le j \le p + \ell + f$, then either $d_{C^{H'}_{F'}}(x_j) = 2$, or $d_{C^{H'}_{F'}}(x_j) = 0$ and all neighbors of x_j in F' are on $C^{H'}_{F'}$.

This implies that $C^{H'}_{F'}$ together with the open edges of $\psi'(B)$ determines the required DC in $(F')^{\psi'(B)}$ containing all open edges of $\psi'(B)$. ■

For a cubic fragment F with $A(F) = A_2(F)$ we will simply write $\overline{F}^{A(F)} = \overline{F}$. If F_1, F_2 are cubic fragments with $A(F_i) = A_2(F_i)$, $i = 1, 2$ and $\varphi : A(F_1) \to A(F_2)$ is a bijection, then $\overline{\varphi}$ denotes the bijection $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ defined by $\overline{\varphi}(\overline{a}) = \overline{\varphi(a)}$, $a \in A(F_1)$.

In the proof of Proposition 14 we will also need the following statement showing that the existence (or nonexistence) of a compatible mapping is not affected by adding pendant edges to vertices of attachment.

Proposition 12. *Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and $A(F_i) = A_2(F_i)$, $i = 1, 2$, and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. Then φ is compatible if and only if $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ is compatible.*

Proof. Set $A(F_1) = \{a_1, \ldots, a_k\}$. Suppose first that φ is compatible and let $\overline{B}$ be an $\overline{F_1}$-linkage such that there is a DC $\overline{C}$ in $(\overline{F_1})^{\overline{B}}$ containing all open edges of $\overline{B}$. Since $A(\overline{F_1}) = A_1(\overline{F_1})$, all components of $\overline{B}$ are paths. We define an F_1-linkage B as follows:

(i) $a_i a_j \in E(B)$, $i \ne j$, if and only if $\overline{B}$ has a component which is an $\overline{a_i}, \overline{a_j}$-path,
(ii) $a_i a_i \in E(B)$ if and only if $\overline{a_i} \in A(\overline{F_1}) \setminus V(\overline{B})$.

(This means that vertices in $A(F)$ corresponding to internal vertices of paths in $\overline{B}$ will not be in $V(B)$, and vertices corresponding to vertices not in $V(\overline{B})$ will have loops in B.)

Since $\overline{C}$ dominates all edges of $\overline{F_1}$ (including the edges $a_i\overline{a_i}$ with $\overline{a_i} \notin V(\overline{B})$), it is straightforward to see that removing from $\overline{C}$ the edges of $\overline{B}$ and the pendant edges of $\{a_i\overline{a_i}, i = 1, \ldots, k\} \cap E(\overline{C})$, and adding the open edges of B results in a DC C in F_1^B, containing all open edges of B. Using the compatibility of φ we obtain a DC in $F_2^{\varphi(B)}$ containing all open edges of $\varphi(B)$, and adding the pendant edges and all edges of $\overline{\varphi}(\overline{B})$ yields a required DC in $(\overline{F_2})^{\overline{\varphi}(\overline{B})}$.

Conversely, let $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ be compatible and let B be an F_1-linkage. Since $A(F_1) = A_2(F_1)$, B contains no paths of length more than one. Suppose the notation is chosen such that $E(B) = \{a_1a_2, \ldots, a_{2p-1}a_{2p}, a_{2p+1}a_{2p+1}, \ldots, a_{2p+\ell}a_{2p+\ell}\}$, where $2p+\ell \le k$. Then we define $\overline{B}$ as the graph which has as components the path $a_1 a_{2p+\ell+1} \ldots a_k a_2$ and (if $p > 1$) the edges $a_{2i-1}a_{2i}$, $i = 2, \ldots, p$. The rest of the proof is similar to that above. ■

4. Equivalence of Conjectures A–F

Before proving our main result, Theorem 3, we first prove several auxiliary statements that describe the structure of potential counterexamples to Conjecture D.

Proposition 13. *If Conjecture D is not true, then there is an essential cubic fragment F such that*

(i) $|A_2(F)| = |A(F)| = 4$,
(ii) *there is a cyclically 4-edge-connected cubic graph G such that $F \subset G$,*

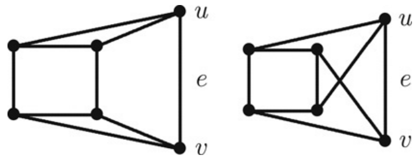

Fig. 5.

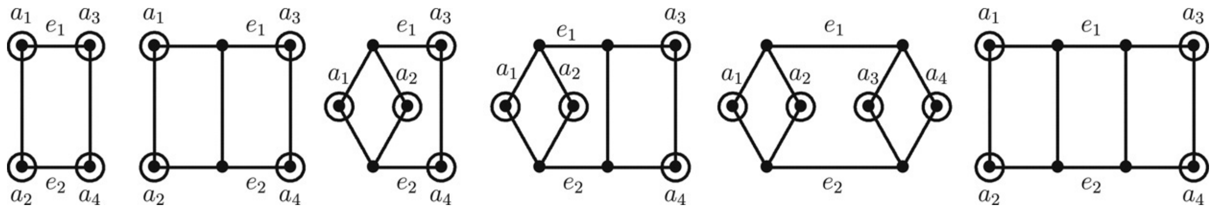

Fig. 6.

(iii) *there is no compatible mapping* $\varphi : C_4 \to F$.

Proof. Let G be a counterexample to Conjecture D, i.e. a cyclically 4-edge-connected cubic graph having no DC, let $e = uv \in E(G)$ and set $F = G - \{u, v\}$. Then F is an essential cubic fragment with $|A_2(F)| = |A(F)| = 4$. Let, to the contrary, $\varphi : C_4 \to F$ be a compatible mapping and set $G' = G[F \xrightarrow{\varphi^{-1}} C_4]$. Then G' is isomorphic to one of the graphs in Fig. 5, and hence G' has a DC. But then, by Theorem 10, the graph $G = G'[C_4 \overset{\varphi}{\to} F]$ has a DC, a contradiction. ■

Proposition 14. *Let F be an essential cubic fragment such that*

(i) $|A_2(F)| = |A(F)| = 4$,
(ii) *there is a cyclically 4-edge-connected cubic graph G such that* $F \subset G$,
(iii) *there is no compatible mapping* $\varphi : C_4 \to F$,
(iv) *subject to* (i), (ii) *and* (iii), $|V(F)|$ *is minimal.*

Then F is essentially 3*-edge-connected and contains no cycle of length* 4.

Proof. Recall that a cubic graph is cyclically 4-edge-connected if and only if it is essentially 4-edge-connected (see [5]).

We first show that F is essentially 3-edge-connected. Suppose the contrary. By definition, F is connected. Denote $A(F) = \{a_1, a_2, a_3, a_4\}$, and let f_i denote the edge in $E(G) \setminus E(F)$ incident with a_i, $i = 1, 2, 3, 4$. If F has a cut edge e, then some nontrivial (i.e. containing at least one edge) component of $F - e$ contains at most two vertices a_i, but then e together with the corresponding edges f_i is an essential edge cut in G of size at most 3, a contradiction. Hence F has no cut edge. (Note that F has also no cut vertex since G is cubic.)

Thus, let $R = \{e_1, e_2\} \subset E(F)$ be an essential edge cut of F, and let F_1, F_2 be nontrivial components of $F - R$. Denote $e_i = b_i^1 b_i^2$ with $b_i^j \in V(F_j)$, $i, j = 1, 2$. If $|V(F_1) \cap A(F)| = 1$, then we set $V(F_1) \cap A(F) = \{x\}$ and observe that the edges e_1, e_2 and the only edge of G_{-F} incident to x form an essential edge cut of G of size 3, a contradiction. We obtain a similar contradiction for $|V(F_1) \cap A(F)| = 0$; hence $|V(F_1) \cap A(F)| \geq 2$. Symmetrically, $|V(F_2) \cap A(F)| \geq 2$, implying $|V(F_1) \cap A(F)| = |V(F_2) \cap A(F)| = 2$. Thus, we can suppose that the notation is chosen such that $a_1, a_2 \in V(F_1)$ and $a_3, a_4 \in V(F_2)$.

If $|V(F_1)| > 4$, then there is a compatible mapping $\varphi : C_4 \to F_1$ by the minimality of F. Let $\widetilde{C}$ be a copy of C_4 and set $H = F[F_1 \xrightarrow{\varphi^{-1}} \widetilde{C}]$. Then $|V(H)| < |V(F)|$ and, by the minimality of F, there is a compatible mapping $\psi : C_4 \to H$. By Proposition 11 (with $X := C_4$, $F := H$, $F_1 := \widetilde{C}$ and $F_2 := F_1$), there is a compatible mapping $\psi' : C_4 \to H[\widetilde{C} \overset{\varphi}{\to} F_1] = F$, a contradiction. Hence $|V(F_1)| \leq 4$ and, symmetrically, $|V(F_2)| \leq 4$.

Now, since G is cyclically 4-edge-connected, either $\{a_1, a_2\} \cap \{b_1^1, b_2^1\} = \emptyset$, or (up to symmetry), $a_1 = b_1^1$ and $a_2 = b_2^1$. Hence F_1 is a single edge or a cycle of length 4. Similarly, F_2 is a single edge or a cycle of length 4. Thus,

F is isomorphic to one of the graphs shown in Fig. 6. However, it is straightforward to check that for each of these graphs there is a compatible mapping $\varphi : C_4 \to F$, a contradiction. Thus, F is essentially 3-edge-connected.

Next we show that

(∗) F contains no subgraph $\widetilde{F}$, $\widetilde{F} \neq F$, with $|V(\widetilde{F})| > 4$ and $|A_2(\widetilde{F})| = |A(\widetilde{F})| = 4$.

Thus, let $\widetilde{F}$ be such a subgraph. By the minimality of F, there is a compatible mapping $\varphi : C_4 \to \widetilde{F}$. Let $\widetilde{C}$ be a copy of C_4 and set $H = F[\widetilde{F} \xrightarrow{\varphi^{-1}} \widetilde{C}]$. By the minimality of F, there is a compatible mapping $\psi : C_4 \to H$. By Proposition 11 (with $X := C_4$, $F := H$, $F_1 := \widetilde{C}$ and $F_2 := \widetilde{F}$), there is a compatible mapping $\psi' : C_4 \to H[\widetilde{C} \xrightarrow{\varphi} \widetilde{F}] = F$, a contradiction. Hence there is no such $\widetilde{F}$.

Finally, we show that F contains no cycle of length 4. Let, to the contrary, $Y \subset F$ be a copy of C_4 (note that possibly $V(Y) \cap A(F) \neq \emptyset$). Let $\overline{F}$ be the graph obtained from F by attaching a pendant edge to each vertex in $A(F)$, and let F_1 and F_2 be the graphs shown in Fig. 3 (recall that we already know there is a compatible mapping $\varphi : F_1 \to F_2$). Let $\overline{Y}$ be the (only) subgraph of $\overline{F}$ such that $Y \subset \overline{Y}$ and $\overline{Y}$ is isomorphic to F_2, let T be a copy of F_1 and let $\varphi : T \to \overline{Y}$ be a compatible mapping. Set $\overline{F}' = \overline{F}[\overline{Y} \xrightarrow{\varphi^{-1}} T]$ (i.e., $\overline{F} = \overline{F}'[T \xrightarrow{\varphi} \overline{Y}]$), and let F' be the graph obtained from $\overline{F}'$ by removing the four pendant edges. Then F' is a cubic fragment with $|A(F')| = |A_2(F')| = 4$.

We show that there is no compatible mapping $\psi : C_4 \to F'$. Let, to the contrary, $\psi : C_4 \to F'$ be compatible. By adding pendant edges to $A(C_4)$ and $A(F')$ and by Proposition 12, there is a compatible mapping $\overline{\psi} : \overline{C_4} \to \overline{F}'$. Thus, we have $\overline{\psi} : \overline{C_4} \to \overline{F}'$, $T \subset \overline{F}'$ and $\varphi : T \to \overline{Y}$. By Proposition 11, there is a compatible mapping $\overline{\psi}' : \overline{C_4} \to \overline{F}$. By removing the pendant edges and by Proposition 12 we obtain a compatible mapping $\psi' : C_4 \to F$, a contradiction. Thus, there is no compatible mapping $\psi : C_4 \to F'$.

By the minimality of F, the graph F' (and hence also $\overline{F}'$) cannot be a subgraph of a cyclically 4-edge-connected cubic graph. Thus, there is an edge cut R' of $\overline{F}'$ such that $|R'| \leq 3$ and at least one component X' of $\overline{F}' - R'$ contains a cycle and has minimum degree 2 (if such an R' does not exist then, identifying the vertices of degree 1 of $\overline{F}'$ with vertices of a C_4, we get a cyclically 4-edge-connected cubic graph containing $\overline{F}'$, a contradiction). However, there is no such edge cut in $\overline{F}$. Since $\overline{F}' = \overline{F}[\overline{Y} \xrightarrow{\varphi^{-1}} T]$, R' contains the edge $e = xy \in E(T)$ with $d_T(x) = d_T(y) = 3$ and some two edges $f_1, f_2 \in E(\overline{F}') \setminus E(T)$. Suppose the vertices of T are labeled such that $A_1(T) = \{a_1, a_2, a_3, a_4\}$, $E(T) = \{a_1x, a_2x, a_3y, a_4y, xy\}$ and $a_1, a_2, x \in V(X')$. Then $R'' = \{f_1, f_2, a_3y, a_4y\}$ is an edge cut in $\overline{F}'$ such that $|R''| = 4$ and $X' + e$ is a component of $\overline{F}' - R''$. Let e_1 (e_2, e_3, e_4) denote the pendant edge of $\overline{Y}$ which corresponds to the edge a_1x (a_2x, a_3y, a_4y) $\in E(T)$, respectively, in the mapping φ. Then $R = \{f_1, f_2, e_3, e_4\}$ is an edge cut of $\overline{F}$ such that the component X of $\overline{F} - R$ containing X' and Y has $|V(X)| > 4$ and $|A_2(X)| = |A(X)| = 4$.

By (∗) (and since $F \not\simeq C_4$, implying $e_1, e_2 \in E(F)$), F contains no such graph as a proper subgraph; hence $X = F$. But then $\{e_1, e_2\}$ is an edge cut of F, contradicting the fact that F is essentially 3-edge-connected. Hence F contains no cycle of length 4. ■

Proposition 15. *If Conjecture D is not true, then there is an essential cubic fragment F such that*

(i) *F contains no cycle of length 4,*
(ii) *there is a cyclically 4-edge-connected cubic graph G such that $F \subset G$,*
(iii) *$|A_2(F)| = |A(F)| = 4$ and $A(F)$ is independent,*
(iv) *there is a compatible mapping $\varphi : F \to C_4$.*

Proof. By Propositions 13 and 14, there is an essential cubic fragment H such that H contains no cycle of length 4, $|A_2(H)| = |A(H)| = 4$, there is a cyclically 4-edge-connected cubic graph G such that $H \subset G$, and there is no compatible mapping $\psi : C_4 \to H$. Let H be minimal with these properties. Since $A(H) = A_2(H)$, by the nonexistence of a compatible mapping $\psi : C_4 \to H$, H is not weakly $A(H)$-contractible. Hence there is a nonempty even set $X \subset A(H)$ and a partition $\mathcal{A}$ of X into two-element subsets such that $H^{\mathcal{A}}$ has no DCT containing all vertices of $A(H)$ and all edges of $E(\mathcal{A})$. Set $A(H) = \{a_1, a_2, a_3, a_4\}$ and suppose the notation is chosen such that $\mathcal{A} = \{\{a_1, a_2\}\}$ if $|X| = 2$ or $\mathcal{A} = \{\{a_1, a_2\}, \{a_3, a_4\}\}$ if $|X| = 4$. Then the graph H^B has no DC containing all open edges of B for either $E(B) = \{a_1a_2, a_3a_3, a_4a_4\}$ or $E(B) = \{a_1a_2, a_3a_4\}$.

Let H, H' be two copies of H (with a corresponding labeling $A(H') = \{a_1', a_2', a_3', a_4'\}$), and let F be the cubic fragment obtained from H and H' by adding the edges a_1a_1' and a_2a_2'. Recall that H contains no cycle of length 4.

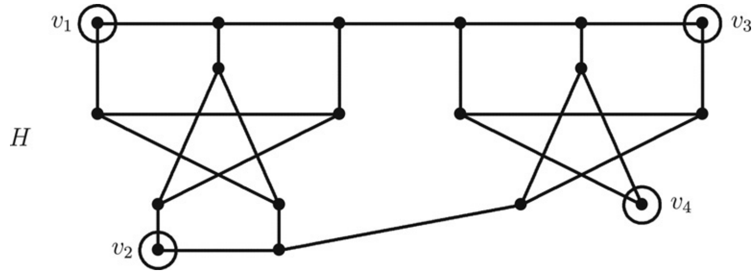

Fig. 7.

Since H is essentially 3-edge-connected by Proposition 14, the set $\{a_1, a_2, a_3, a_4\}$ (and hence also $\{a'_1, a'_2, a'_3, a'_4\}$) is independent. Hence F also contains no cycle of length 4, and the set $A(F) = \{a_3, a_4, a'_3, a'_4\}$ is independent. It remains to prove that there is a compatible mapping $\varphi : F \to C_4$.

First we show that the graph F^B has no DC containing all open edges of B for $E(B) = \{a_3a_3, a_4a_4, a'_3a'_4\}$. To the contrary, let C be such a DC. Then $(E(C) \cap E(H)) \cup \{a_1a_2\}$ is a DC in H^B containing all open edges of B for $E(B) = \{a_1a_2, a_3a_3, a_4a_4\}$, and $(E(C) \cap E(H')) \cup \{a'_1a'_2, a'_3a'_4\}$ is a DC in $H'^{B'}$ containing all open edges of B' for $E(B') = \{a'_1a'_2, a'_3a'_4\}$, which is not possible. Thus, there is no such DC in F^B. Symmetrically, $F^{B'}$ has no DC containing all open edges of B' for $E(B') = \{a'_3a'_3, a'_4a'_4, a_3a_4\}$. Let Y be a copy of C_4 with vertices labeled b_3, b_4, b'_3, b'_4 such that $b_3b_4 \notin E(Y)$ and $b'_3b'_4 \notin E(Y)$. Then it is straightforward to check that $Y^{B''}$ has a DC containing all open edges of B'' for all Y-linkages B'' except for the cases $E(B'') = \{b_3b_3, b_4b_4, b'_3b'_4\}$ and $E(B'') = \{b'_3b'_3, b'_4b'_4, b_3b_4\}$. Hence the mapping $\varphi : A(F) \to A(Y)$ that maps a_i on b_i and a'_i on b'_i, $i = 3, 4$, is a compatible mapping. ■

Note that we do not know any example of a cubic fragment with the properties given in Proposition 15. Moreover, we believe that such a graph in fact does not exist.

Now we are ready to prove the main result of this paper, Theorem 3.

Proof of Theorem 3. Clearly, Conjecture E implies Conjecture F. By Theorem 2, it is sufficient to show that Conjecture F implies Conjecture D. Thus, suppose Conjecture D is not true, and let F be an essential cubic fragment as given by Proposition 15. Let G be a counterexample to Conjecture D, i.e. a cyclically 4-edge-connected cubic graph without a DC. For any cycle C of length 4 in G, choose a compatible mapping of F on C, and let G' be the graph obtained by recursively replacing every cycle of length 4 by a copy of F. Then G' is a cubic graph of girth $g(G') \geq 5$ and, by Theorem 10, G' has no DC. Moreover, G' is cyclically 4-edge-connected since any cycle-separating edge cut in G' of size at most 3 would imply the existence of such an edge cut in G. If G' is not 3-edge-colorable, G' is a snark and we are done. Otherwise, we use the following fact and construction by Kochol [7].

Claim ([7]). *If a cubic graph G contains the graph H of Fig. 7 as an induced subgraph, then G is not 3-edge-colorable.*

We use the claim as follows. Let $xy \in E(G')$, let x', x'' (y', y'') be the neighbors of x (of y) different from y (x), respectively, and let G'_i, $i = 1, 2, 3$, be three copies of the graph $G' - x - y$ (where x'_i, x''_i, y'_i, y''_i are the copies of x', x'', y', y'' in G'_i), $i = 1, 2, 3$. Then the graph $\bar{G}$ obtained from G'_1, G'_2, G'_3 and H by adding the edges x'_1v_3, x''_1v_4, $y'_1x'_2$, $y''_1x''_2$, $y'_2x'_3$, $y''_2x''_3$, y'_3v_1 and y''_3v_2 is a cyclically 4-edge-connected graph of girth $g(\bar{G}) \geq 5$. By the claim, $\bar{G}$ is not 3-edge-colorable. It remains to show that $\bar{G}$ has no DC.

Let, to the contrary, C be a DC in $\bar{G}$. Then it is easy to check that for some $i \in \{1, 2, 3\}$, the intersection of C with G'_i is either a path with one end in $\{x'_i, x''_i\}$ and the second in $\{y'_i, y''_i\}$, or two such paths. But, in both cases, the path(s) can be easily extended to a DC in G', a contradiction. ■

5. Concluding remarks

1. Note that our proof of the equivalence of Conjecture F with Conjectures A–E is based on properties (compatible mappings) that are specific for the C_4. This means that our proof cannot be directly extended to obtain higher girth restrictions.

2. We pose the following conjecture and show it is equivalent to Conjectures A–F.

Conjecture G. *Every cyclically 4-edge-connected cubic graph contains a weakly contractible subgraph F with $\delta(F) = 2$.*

Theorem 16. *Conjecture G is equivalent to Conjectures A–F.*

Proof. We first show that Conjecture G implies Conjecture D. Suppose Conjecture G is true and let G be a minimum counterexample to Conjecture D. Hence G has no DC. Let $F \subset G$ be a weakly contractible subgraph of G with $\delta(F) = 2$ and set $A = A_G(F)$. Note that $A \neq \emptyset$ since $\delta(F) = 2$. By Corollary 7, the graph $G|_F$ has no DCT. If $|A| \leq 3$, then every edge in G_{-F} has at least one vertex in A since G is essentially 4-edge-connected. But then $G|_F$ has a (trivial) DCT, a contradiction. Hence $|A| \geq 4$.

We use the following operation (see [5]). Let H be a graph, let $v \in V(H)$ be of degree $d = d_H(v) \geq 4$, and let $x_1, \ldots, x_d$ be an ordering of the neighbors of v (allowing repetition in case of multiple edges). Let H' be the graph obtained by adding edges $x_i y_i$, $i = 1, \ldots, d$, to the disjoint union of the graph $H - v$ and the cycle $y_1 y_2 \ldots y_d y_1$. Then H' is said to be an *inflation of* H *at* v. The following fact was proved in [5].

Claim (*[5]*). *Let H be an essentially 4-edge-connected graph of minimum degree $\delta(G) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Then some inflation of H at v is essentially 4-edge-connected.*

Now let G' be an essentially 4-edge-connected inflation at v_F of the graph obtained from $G|_F$ by deleting its pendant edges. Then G' is a cubic graph having no DC (since otherwise $G|_F$ would have a DCT). Since no cycle of length $\ell \geq 4$ is weakly contractible, F is not a cycle, and since $\delta(F) = 2$, we have $|A_G(F)| < |E(F)|$. But then $|E(G')| < |E(G)|$, contradicting the minimality of G.

For the rest of the proof, it is sufficient to show that Conjecture D implies Conjecture G. Indeed, if C is a dominating cycle in G, $e = uv \in E(C)$ and $A = \{u, v\}$, then the graph F with $V(F) = V(G)$ and $E(F) = E(G) \setminus \{e\}$ is a weakly A-contractible subgraph of G. ■

It should be noted here that the last part of the proof of Theorem 16 is based on a construction with $|A| = 2$, which forces $G - F$ be empty (G_{-F} is a one edge graph) since G is cubic and cyclically 4-edge-connected. It is straightforward to observe that the following stronger statement implies Conjectures A–G. However, we do not know whether these statements are equivalent.

Conjecture H. *Every cyclically 4-edge-connected cubic graph G contains a weakly contractible subgraph F with $|A_G(F)| \geq 4$.*

Acknowledgements

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Line Graphs of Multigraphs and Hamilton-Connectedness of Claw-Free Graphs

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Abstract: We introduce a closure concept that turns a claw-free graph into the line graph of a multigraph while preserving its (non-)Hamilton-connectedness. As an application, we show that every 7-connected claw-free graph is Hamilton-connected, and we show that the well-known conjecture by Matthews and Sumner (every 4-connected claw-free graph is hamiltonian) is equivalent with the statement that every 4-connected claw-free graph is Hamilton-connected. Finally, we show a natural way to avoid the non-uniqueness of a preimage of a line graph of a multigraph, and we prove that the closure operation is, in a sense, best possible.
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1. NOTATION AND TERMINOLOGY

In this article, by a *graph* we mean a finite simple undirected graph $G=(V(G),E(G))$; whenever we allow multiple edges we say that G is a *multigraph*.

For a vertex $x\in V(G)$, $d_G(x)$ denotes the *degree of x in G*, $N_G(x)$ denotes the *neighborhood of x* in G (i.e. $N_G(x)=\{y\in V(G)|\ xy\in E(G)\}$) and $N_G[x]$ denotes the *closed neighborhood of x in G* (i.e. $N_G[x]=N_G(x)\cup\{x\}$). For $x,y\in V(G)$, $\mathrm{dist}_G(x,y)$ denotes the *distance of x,y* in G. A *universal vertex* of G is a vertex that is adjacent to all other vertices of G. By a *clique* we mean a (not necessarily maximal) complete subgraph of G; $\alpha(G)$ denotes the *independence number* of G and $\kappa(G)$ denotes the (vertex) *connectivity* of G. By the *square* of a graph G we mean the graph G^2 with $V(G^2)=V(G)$ and $E(G^2)=\{xy\in V(G)|\mathrm{dist}_G(x,y)\le 2\}$.

If G, H are (multi-)graphs, then $H\subset G$ or $H\overset{\mathrm{IND}}{\subset}G$ means that H is a *subgraph* or an *induced subgraph* of G, respectively, and $H\simeq G$ stands for the *isomorphism* of H and G. The *induced subgraph* of G on a set $M\subset V(G)$ is denoted $\langle M\rangle_G$.

A path with endvertices a,b will be referred to as an *(a,b)-path*. If P is a path and $u\in V(P)$, then u^- and u^+ denote the *predecessor* and *successor* of u on P. A path on k vertices is denoted P_k.

For a graph G and $a,b\in V(G)$, $p(G)$ denotes the length of a longest path in G, $p_a(G)$ the length of a longest path in G with one endvertex at $a\in V(G)$, and $p_{ab}(G)$ the length of a longest (a,b)-path in G. A graph G is *homogeneously traceable* if, for any $a\in V(G)$, G has a hamiltonian path with one endvertex at a (i.e. for any $a\in V(G)$, $p_a(G)=|V(G)|$), and G is *Hamilton-connected* if, for any $a,b\in V(G)$, G has a hamiltonian (a,b)-path (i.e. for any $a,b\in V(G)$, $p_{ab}(G)=|V(G)|$).

A *walk* (in G) is a sequence of vertices $u_1u_2\dots u_k$ such that $u_iu_{i+1}\in E(G)$, $i=1,\dots,k-1$. For a walk $J=u_1u_2\dots u_k$ we denote $V(J)=\{u_1,u_2,\dots,u_k\}$ the corresponding set of vertices, and $|V(J)|=|\{u_1,u_2,\dots,u_k\}|$ (thus, $|V(J)|=k$ if and only if J is a path). Finally, G is *claw-free* if G does not contain an induced subgraph that is isomorphic to the *claw* $K_{1,3}$.

For further concepts and notations not defined here we refer the reader to [4].

2. INTRODUCTION

A vertex $x\in V(G)$ is *eligible* if $N_G(x)$ induces a connected non-complete graph, and x is *simplicial* if the subgraph induced by $N_G(x)$ is complete. The *local completion* of G at a vertex x is the graph G_x^* obtained from G by adding all edges with both vertices in $N_G(x)$ (note that the local completion at x turns x into a simplicial vertex, and preserves the claw-free property of G).

The *closure* $\mathrm{cl}(G)$ of a claw-free graph G is the graph obtained from G by recursively performing the local completion operation at eligible vertices as long as this is possible. We say that G is *closed* if $G=\mathrm{cl}(G)$.

The following was proved in [14].

Theorem A (Ryjáček [14]). *For every claw-free graph G:*

(i) $\mathrm{cl}(G)$ *is uniquely determined*,

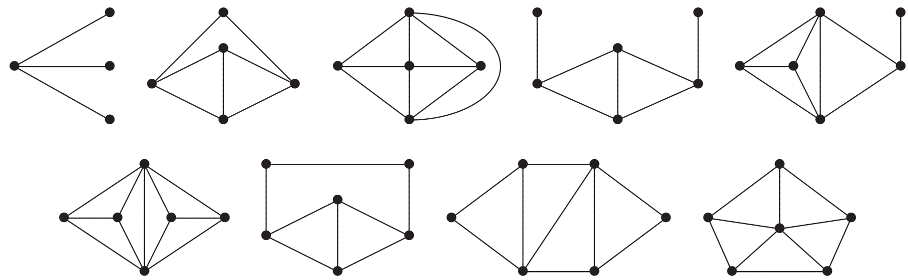

FIGURE 1. Forbidden subgraphs for line graphs.

(ii) cl(G) *is the line graph of a triangle-free graph*,
(iii) cl(G) *is hamiltonian if and only if G is hamiltonian.*

Note that the fact that cl(G) is a line graph can be seen e.g. also from the well-known Beineke's characterization of line graphs in terms of forbidden induced subgraphs.

Theorem B (Beineke [1]). *A graph G is a line graph* (*of some graph*) *if and only if G does not contain a copy of any of the graphs in Figure* 1 *as an induced subgraph.*

A class $\mathcal{C}$ is *stable* if $G \in \mathcal{C}$ implies cl(G)$\in \mathcal{C}$. A graph property π is *stable* in a stable class $\mathcal{C}$ if, for any $G \in \mathcal{C}$, G has π if and only if cl(G) has π.

Thus, Theorem A says that hamiltonicity is a stable property in the class of claw-free graphs.

Zhan [17] proved the following.

Theorem C (Zhan [17]). *Every* 7*-connected line graph of a multigraph is Hamilton-connected.*

Using the fact that hamiltonicity is a stable property, combining Theorems A and C the following was obtained.

Theorem D (Ryjáček [14]). *Every* 7*-connected claw-free graph is hamiltonian.*

The line graph of the multigraph H in Figure 2 shows that Hamilton-connectedness is not stable in 3-connected claw-free graphs (there is no hamiltonian (u_1, u_2)-path in $L(H)$, where u_1, u_2 are the vertices of $L(H)$ that correspond to the edges u_1, u_2 in H). Thus, the closure technique does not give a similar result for Hamilton-connectedness.

The existence of a connectivity bound for Hamilton-connectedness in claw-free graphs was established by Brandt [5] who proved that every 9-connected claw-free graph is Hamilton-connected. This result was later on improved by Hu et al. [8] as follows.

Theorem E (Hu et al. [8]). *Every* 8*-connected claw-free graph is Hamilton-connected.*

In the same article, Zhan's result (Theorem C) was improved as follows.

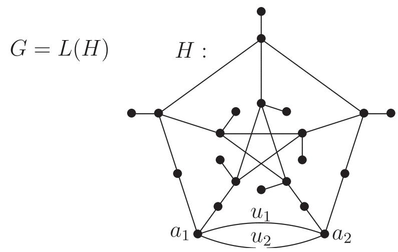

FIGURE 2. A graph with non-Hamilton-connected line graph.

Theorem F (Hu et al. [8]). *Let G be a* 6*-connected line graph of a multigraph with at most* 29 *vertices of degree* 6*. Then G is Hamilton-connected.*

On the other hand, the following conjectures by Matthews and Sumner (Conjecture G) and by Thomassen (Conjecture H) are still wide open.

Conjecture G (Matthews and Sumner [16]). *Every* 4*-connected claw-free graph is hamiltonian.*

Conjecture H (Thomassen [16]). *Every* 4*-connected line graph is hamiltonian.*

Note that Theorem A immediately implies that Conjectures G and H are equivalent. More equivalent versions of these conjectures (among others, on cycles in cubic graphs), can be found e.g. in [7].

Another equivalence was established by Kužel and Xiong [10] (see also [11]), who proved that Conjectures G and H are equivalent with the following statement.

Conjecture I (Kužel and Xiong [10]). *Every* 4*-connected line graph of a multigraph is Hamilton-connected.*

It is natural to pose the following question.

Conjecture J. *Every* 4*-connected claw-free graph is Hamilton-connected.*

For a similar reason as with the extension of Theorem D to Hamilton-connectedness, the closure technique as introduced in [14] does not establish the equivalence of Conjecture J with the previous ones.

In Section 4 we develop a closure concept for Hamilton-connectedness from which, as immediate applications, we obtain the following statements (see Theorems 15 and 17).

(i) *Every* 6*-connected claw-free graph with at most* 29 *vertices of degree* 6 *is Hamilton-connected.*
(ii) *Every* 7*-connected claw-free graph is Hamilton-connected.*
(iii) *Conjecture* J *is equivalent with Conjectures* G, H *and* I.

3. *k*-CLOSURE AND STRUCTURE OF 2-CLOSED GRAPHS

The closure concept was extended in [3] as follows.

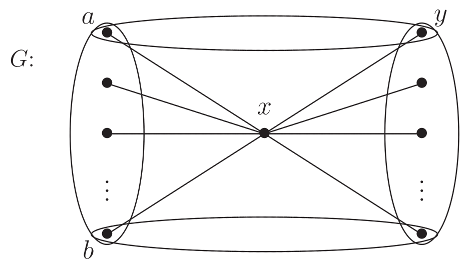

FIGURE 3. A graph with no Hamiltonian (a, b)-path.

A vertex $x \in V(G)$ is *k-eligible* if its neighborhood induces a k-connected non-complete graph, and the *k-closure* of G, denoted $\mathrm{cl}_k(G)$, is the graph obtained from G by recursively performing the local completion operation at k-eligible vertices as long as this is possible. A graph G is *k-closed* if $G = \mathrm{cl}_k(G)$.

A class $\mathcal{C}$ is *k-stable* if $G \in \mathcal{C}$ implies $\mathrm{cl}_k(G) \in \mathcal{C}$. A graph property π is *k-stable* in a k-stable class $\mathcal{C}$ if, for any $G \in \mathcal{C}$, G has π if and only if $\mathrm{cl}_k(G)$ has π.

Theorem K (Bollobás et al. [3]). *For every claw-free graph G,*

(i) $\mathrm{cl}_k(G)$ *is uniquely determined,*
(ii) $\mathrm{cl}_2(G)$ *is homogeneously traceable if and only if G is homogeneously traceable,*
(iii) $\mathrm{cl}_3(G)$ *is Hamilton-connected if and only if G is Hamilton-connected.*

Thus, homogeneous traceability is 2-stable and hamilton-connectedness is 3-stable in the class of claw-free graphs.

Let G be the graph in Figure 3 (where the ovals represent cliques on at least three vertices). Then G has no hamiltonian (a,b)-path, the vertex x is 2-eligible, and there is a hamiltonian (a,b)-path in the local completion G^*_x of G at x. This shows that the property "having a hamiltonian (a,b)-path for given $a,b \in V(G)$" is not 2-stable. However, neither G nor its 2-closure are Hamilton-connected. This motivated the following conjecture.

Conjecture L (Bollobás et al. [2]). *Hamilton-connectedness is* 2*-stable in the class of claw-free graphs.*

Note that in [9] the author claimed to give an infinite family of counterexamples to Conjecture L. However, this statement is not true, since it is not difficult to observe that the graphs constructed in [9] have similar behavior as the graphs in Figure 3 (i.e. they show that the property "having a hamiltonian (a,b)-path for given $a,b \in V(G)$" is not 2-stable, but do not disprove Conjecture L).

Affirmative answer to Conjecture L was given in [15].

Theorem M (Ryjáček and Vrána [15]). *Hamilton-connectedness is* 2*-stable in the class of claw-free graphs.*

A natural question is whether a 2-closure of a claw-free graph belongs to some "nice" class of graphs. It is easy to see that, in general, $\mathrm{cl}_2(G)$ is not a line graph, since e.g. the second or fourth graph in Figure 1 is an example of a 2-closed claw-free graph

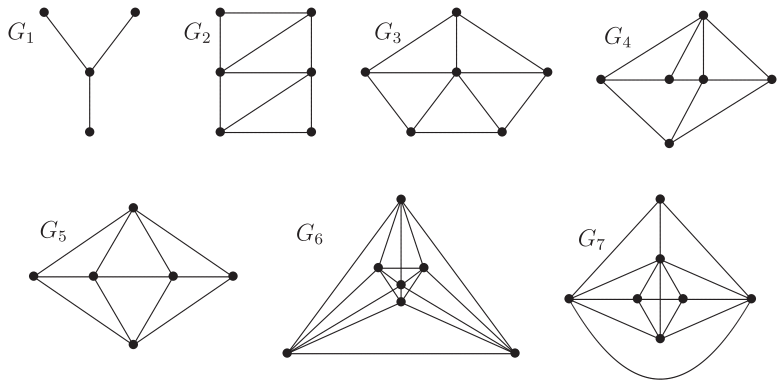

FIGURE 4. Forbidden subgraphs for line graphs of multigraphs.

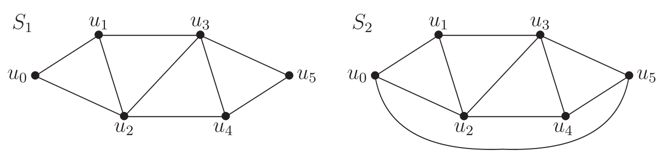

FIGURE 5. The graphs S_1 and S_2.

that is not a line graph. Thus, a next question is whether a 2-closure of a claw-free graph is a line graph of a multigraph.

Line graphs of multigraphs were characterized by Bermond and Meyer [2] (see also Zverovich [18]).

Theorem N (Bermond and Meyer [3]). *A graph G is a line graph of a multigraph if and only if G does not contain a copy of any of the graphs in Figure* 4 *as an induced subgraph.*

We see that, in general, $\mathrm{cl}_2(G)$ is not a line graph of a multigraph, since the graphs G_2 and G_4 of Figure 4 are 2-closed, i.e. they can be induced subgraphs in $\mathrm{cl}_2(G)$.

We now consider the structure of $\mathrm{cl}_2(G)$ in more detail. We include here only those results that are needed for introducing the closure concept in Section 4. Proofs and further necessary auxiliary results are postponed to Section 6.

Lemma 1. *Let G be a* 2*-closed claw-free graph, and let G_i, $i = 1, \ldots, 7$ be the graphs from Figure* 4. *Then G is $\{G_1, G_3, G_5, G_6, G_7\}$-free.*

Thus, a 2-closed claw-free graph can contain only induced G_2 and/or G_4. In the rest of the article we will keep the notation of these graphs as shown in Figure 5.

Let $J = u_0u_1 \ldots u_{k+1}$ be a walk in G. We say that J is *good in G*, if $k \geq 4$, $J^2 \subset G$ and for any i, $0 \leq i \leq k-4$, $\langle\{u_i, u_{i+1}, \ldots, u_{i+5}\}\rangle_G$ is isomorphic to S_1 or to S_2.

Similarly, a cycle $C \subset G$ is said to be *good in G*, if every set of six consecutive vertices of C induces in G the graph S_1 or S_2.

Lemma 2. *Let G be a 2-closed claw-free graph and $J = u_0 u_1 \ldots u_{k+1}$ a good walk in G, $k \geq 5$. Then $d_G(u_i) = 4$, $i = 3, \ldots, k-2$.*

Thus, for $i = 3, \ldots, k-2$, $\langle N_G(u_i)\rangle_G$ is a path of length 3 with vertices u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}.

Corollary 3. *Let G be a connected 2-closed claw-free graph and let $C \subset G$ be a good cycle in G. Then $G = C^2$.*

Corollary 3 specifically implies that a connected 2-closed claw-free graph either is isomorphic to the square of a cycle (and hence is trivially Hamilton-connected), or contains no good cycle. In the rest of the article we concentrate on the second (non-trivial) case.

Let J be a good walk in G. We say that J is *maximal* if, for every good walk J' in G, J being a subsequence of J' implies $J = J'$.

Lemma 4. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, and let $J = u_0 u_1 \ldots u_{k+1}$ be a maximal good walk in G. Then $\langle N_G[u_1] \setminus \{u_3\}\rangle_G = \langle N_G[u_2] \setminus \{u_3, u_4\}\rangle_G$ and this subgraph is a clique.*

Note that symmetrically also $\langle N_G[u_k] \setminus \{u_{k-2}\}\rangle_G = \langle N_G[u_{k-1}] \setminus \{u_{k-2}, u_{k-3}\}\rangle_G$ is a clique.

Lemma 5. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, and let $J = u_0 u_1 \ldots u_{k+1}$ be a good walk in G. Then $u_1 \ldots u_k$ is a path.*

Let J_i^k be the graphs in Figure 6. We set:

$$\mathcal{J}_1 = \{J_1^k \mid k \geq 4\},$$

$$\mathcal{J}_2 = \{J_2^k \mid k \geq 4\},$$

$$\mathcal{J}_3 = \{J_3^k \mid k \geq 6\},$$

$$\mathcal{J}_4 = \{J_4^k \mid k \geq 8\}.$$

Our next lemma describes the structure of subgraphs induced by good walks.

Lemma 6. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, let $J = u_0 u_1 \ldots u_{k+1}$ be a maximal good walk in G, and let J be chosen such that*

$$|V(J)| = \min\{|\{x, u_1, \ldots, u_k, y\}| \mid xu_1 \ldots u_k y \text{ is a maximal good walk in } G\}.$$

Then

$$\langle V(J)\rangle_G \in \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3 \cup \mathcal{J}_4.$$

The following lemma shows that the sets of interior vertices of maximal good walks in a 2-closed graph are vertex-disjoint.

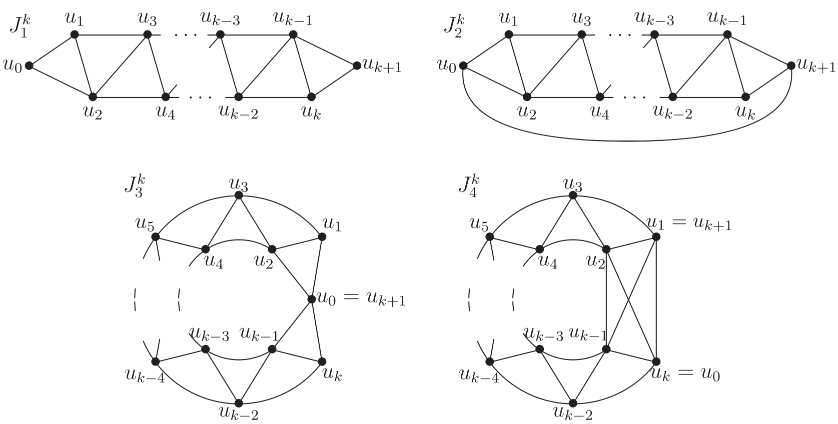

FIGURE 6. Subgraphs induced by maximal good walks.

Lemma 7. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle and let $J^1 = u_0^1 u_1^1 \ldots u_{k+1}^1$, $J^2 = u_0^2 u_1^2 \ldots u_{k'+1}^2$ be maximal good walks in G such that $u_s^1 = u_t^2$ for some s,t, $1 \le s \le k$, $1 \le t \le k'$. Then*

(i) $\{u_1^1, \ldots, u_k^1\} = \{u_1^2, \ldots, u_{k'}^2\}$,
(ii) *$k = k'$ and $u_i^1 = u_i^2$ or $u_i^1 = u_{k-i+1}^2$, $i = 1, \ldots, k$.*

4. CLOSURE CONCEPT AND HAMILTON-CONNECTEDNESS

Before introducing the main concept of this article, the closure operation, we first introduce some notations and recall some helpful definitions and facts from [9].

For any $X \subset V(G)$ let G_X^* denote the local completion of G at X, i.e. the graph with $V(G_X^*) = V(G)$ and $E(G_X^*) = E(G) \cup \{uv \mid u, v \in X\}$. Thus, the previous notation G_x^* means that, for a vertex $x \in V(G)$, we simply write G_x^* for $G_{N_G(x)}^*$. Similarly, for a sequence of vertices $x_1, \ldots, x_k$ we will simply write $G_{x_1 \ldots x_k}^*$ for $((G_{x_1}^*)_{x_2}^* \ldots)_{x_k}^*$.

Let $\mathcal{C}$ be a class of graphs and let $\mathcal{P}$ be a function on $\mathcal{C}$ such that, for any $G \in \mathcal{C}$, $\mathcal{P}(G) \subset 2^{V(G)}$ (i.e. $\mathcal{P}(G)$ is a set of subsets of $V(G)$). We say that a graph F is a *$\mathcal{P}$-extension* of G, denoted $G \preceq F$, if there is a sequence of graphs $G_0 = G, G_1, \ldots, G_k = F$ such that $G_i \in \mathcal{C}$, $i = 1, \ldots, k$, and $G_{i+1} = (G_i)_{X_i}^*$ for some $X_i \in \mathcal{P}(G_i)$, $i = 1, \ldots, k-1$. Clearly, for any graph G a $\preceq$-maximal $\mathcal{P}$-extension H exists, and in this case we say that H is a $\mathcal{P}$-closure of G. If a $\mathcal{P}$-closure is uniquely determined then it is denoted by $\mathrm{cl}_{\mathcal{P}}(G)$. Finally, a function $\mathcal{P}$ is *non-decreasing* (on a class $\mathcal{C}$), if, for any $H, H' \in \mathcal{C}$, $H \preceq H'$ implies that for any $X \in \mathcal{P}(H)$ there is an $X' \in \mathcal{P}(H')$ such that $X \subset X'$.

The following result was proved in [9]. For the sake of completeness, we include its (short) proof here.

Theorem O (Kelmans [9]). *If $\mathcal{P}$ is a non-decreasing function on a class $\mathcal{C}$, then, for any $G \in \mathcal{C}$, a $\mathcal{P}$-closure of G is uniquely determined.*

Proof. Let $H \neq H'$ be $\mathcal{P}$-closures of G, let $G=G_0, G_1, \ldots, G_k=H'$ be such that $G_{i+1}=(G_i)^*_{X_i}$ for some $X_i \in \mathcal{P}(G_i)$, and let s be a smallest integer such that $G_s \not\subset H$. Since $G_{s-1} \subset H$ and $\mathcal{P}$ is non-decreasing, there is $X \in \mathcal{P}(H)$ such that $X_{s-1} \subset X$. Since H is $\preceq$-maximal, we have $H^*_X = H$, a contradiction. ■

For a given graph G, let $\mathcal{C}_G$ denote the class of graphs with vertex set $V(G)$. The following two facts are easy to observe.

Lemma 8. *Let G be a graph.*

(i) *Let $\mathcal{P}$ be a non-decreasing function on $\mathcal{C}_G$, let $X \subset V(G)$, and for any $H \in \mathcal{C}_G$ set $\mathcal{P}^X(H) = \mathcal{P}(H) \cup \{N_H(x) |\ x \in X\}$. Then $\mathcal{P}^X$ is a non-decreasing function on $\mathcal{C}_G$.*
(ii) *For any integer $k \geq 1$, the function $\mathcal{P}_k(H) = \{N_H(x) |\ \langle N_H(x) \rangle_H$ is $k-$connected$\}$ is a non-decreasing function on $\mathcal{C}_G$.* ■

Consequently, for any graph G, integer $k \geq 1$ and a set $X \subset V(G)$, the function $\mathcal{P}^X_k$, defined (for any $H \in \mathcal{C}_G$) by $\mathcal{P}^X_k(H) = (\mathcal{P}_k)^X(H)$, is a non-decreasing function on $\mathcal{C}_G$.

Let now G be a connected claw-free graph that is not the square of a cycle and let $J_1, \ldots, J_t$ be all maximal good walks in $\mathrm{cl}_2(G)$. For any $J_i = u^i_0 u^i_1 \ldots u^i_{k+1}$ set

$$X_i = \{u^i_1, \ldots, u^i_{r-1}\} \cup \{u^i_{r+2} \ldots u^i_{2r}\} \quad \text{if } k=2r$$

or

$$X_i = \{u^i_1, \ldots, u^i_{r-1}\} \cup \{u^i_{r+3} \ldots u^i_{2r+1}\} \quad \text{if } k=2r+1,$$

respectively, and set $X = \bigcup_{i=1}^t X_i$ (note that the sets X_i are pairwise disjoint by Lemma 7). Then, by Lemma 8, the function $\mathcal{P}^M(H) = \mathcal{P}^X_2(H)$ is a non-decreasing function on $\mathcal{C}_G$. The corresponding $\mathcal{P}^M$-closure of G (which is unique by Lemma 8) will be called the *multigraph closure* (or simply *M-closure*) of G and denoted $\mathrm{cl}^M(G)$. If G is the square of a cycle, we define $\mathrm{cl}^M(G)$ as the complete graph on $V(G)$. If $G = \mathrm{cl}^M(G)$ then we say that G is *M-closed.*

Theorem 9. *Let G be a connected claw-free graph and let* $\mathrm{cl}^M(G)$ *be the M-closure of G. Then*

(i) $\mathrm{cl}^M(G)$ *is uniquely determined,*
(ii) *there is a multigraph H such that* $\mathrm{cl}^M(G) = L(H)$,
(iii) *for every $a \in V(G)$, $p_a(\mathrm{cl}^M(G)) = p_a(G)$,*
(iv) $\mathrm{cl}^M(G)$ *is Hamilton-connected if and only if G is Hamilton-connected.*

Proof. If $G = C^2$ for some cycle C then the statement is trivial, hence we suppose that G is not the square of a cycle. Part (i) then follows immediately from Lemma 8, and part (ii) follows immediately from Lemma 1, from the construction of $\mathrm{cl}^M(G)$, from Lemma 25 and from Theorem N. ■

Before proving parts (iii) and (iv) of Theorem 9, we first show that if G is not the square of a cycle, then $\mathrm{cl}^M(G)$ can be equivalently constructed by the following algorithm.

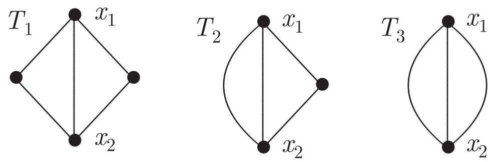

FIGURE 7. Forbidden subgraphs for preimages of M-closed graphs.

Algorithm 10. *Let G be a connected claw-free graph that is not the square of a cycle.*

1. *Set* $G_1 = \mathrm{cl}_2(G)$, $i := 1$.
2. *If G_i contains a good walk, then*
   (a) *choose a maximal good walk* $J = u_0u_1 \dots u_{k+1}$,
   (b) *set* $G_{i+1} = \mathrm{cl}_2((G_i)^*_{u_1u_k})$,
   (c) $i := i+1$ *and go to* (2).
3. *Set* $\overline{G} = G_i$.

Proposition 11. *Let G be a connected claw-free graph that is not the square of a cycle and let $\overline{G}$ be the graph constructed by Algorithm* 10. *Then* $\overline{G} = \mathrm{cl}^M(G)$.

Proof. By Lemma 28, Algorithm 10 closes all vertices with neighborhood in some $\mathcal{P}^M(G_i)$, hence $\mathrm{cl}^M(G) \subset \overline{G}$. By Lemma 25, every vertex with neighborhood in some $\mathcal{P}^M(G_i)$ is closed by Algorithm 10. Hence $\overline{G}$ is a special case of one possible construction of $\mathcal{P}^M(G)$ and, by Theorem 9(i), $\overline{G} = \mathrm{cl}^M(G)$. ■

Proof. Proof of parts (iii), (iv) of Theorem 9 now immediately follows from Proposition 27. ■

Let T_1, T_2, T_3 be the graphs in Figure 7. It is easy to observe that if $G = L(H)$ and $x \in V(G)$ is 2-eligible, then the edge $x_1x_2 \in E(H)$, corresponding to x, is contained in a copy of T_i for some i, $1 \le i \le 3$, such that $d_{T_i}(x_1) = d_{T_i}(x_2) = 3$. However, the converse is not true in general, unless x_1 and/or x_2 have an appropriate neighbor outside. More specifically, it is straightforward to verify the following observation.

Proposition 12. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in Figure 7. Then G is M-closed if and only if there is a multigraph H such that $G = L(H)$ and H does not contain a subgraph S (not necessarily induced) with any of the following properties*:

(i) $S \simeq T_1$,
(ii) $S \simeq T_2$ *and there is a* $u \in V(H) \setminus V(S)$ *such that* $|N_H(u) \cap \{x_1, x_2\}| = 1$,
(iii) $S \simeq T_3$ *and there are* $u_1, u_2 \in V(H) \setminus V(S)$ *such that* $u_1 \neq u_2$ *and* $u_ix_i \in E(H)$, $i = 1, 2$

(*where x_1, x_2 are the only vertices in S with* $d_S(x_i) = 3$).

A well-known drawback of line graphs of multigraphs is the fact that there can be multigraphs H_1, H_2 such that $H_1 \not\simeq H_2$ but $L(H_1) \simeq L(H_2)$ (i.e. the "preimage" is not uniquely determined). However, this problem can be avoided by a slight modification of an approach given in [18]. Namely, we show that the preimage $H = L_M^{-1}(G)$ of a line graph G of a multigraph is uniquely determined under an (very natural) additional assumption that simplicial vertices in G correspond to edges in H with one vertex of degree 1 (called *pendant edges*).

The *basic graph* of a multigraph H is the graph with the same vertex set, in which two vertices are adjacent if and only if they are adjacent in H. A *multitriangle* (*multistar*) is a multigraph such that its basic graph is a triangle (star). The *center* of a multistar S with m edges is the vertex $x \in V(S)$ with $d_S(x) = m$ (for $|V(S)| = 2$ we choose the center arbitrarily), and all other vertices of S are its *leaves*. An induced multistar S in H is *pendant* if none of its leaves has a neighbor in $V(G) \setminus V(S)$, and similarly a multitriangle T is pendant if exactly one of its vertices (called the *root*) has neighbors in $V(G) \setminus V(S)$. We will use the following operations introduced in [18].

Operation A. Choose a pendant multistar in H and identify all its leaves.
Operation B. Choose a pendant multitriangle H with vertices $\{v,x,y\}$ and root v, delete all edges joining v and x, and add the same number of edges between v and y.

Now, for a multigraph H, $AB(H)$ denotes the multigraph obtained by recursively repeating operations A and B. The following result was proved in [18].

Theorem P (Zverovich [18]). *Let H, H' be connected multigraphs such that $L(H) \simeq L(H')$. Then $AB(H) = AB(H')$ unless one of H, H' is a multitriangle and the other one is a non-isomorphic multitriangle or a multistar.*

We will need one more operation.

Operation C. Choose a pendant multistar in H and replace every leaf of degree $k \geq 2$ by k leaves of degree 1.

Similarly as before, let $BC(H)$ denote the multigraph obtained from a multigraph H by recursively repeating operations B and C. Theorem P then easily implies the following result.

Theorem 13. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L_M^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

Proof. Let $G = L(H)$. It is easy to see that every edge $e \in E(H)$ corresponding to a simplicial vertex $e \in V(G)$ is in a pendant multitriangle or in a pendant multistar. Thus, $BC(H)$ has the required properties. Uniqueness follows from Theorem P. ■

Note that if, specifically, G is a line graph of a graph, then the multigraph preimage $L_M^{-1}(G)$ of G, given by Theorem 13, and the obvious line graph preimage $L^{-1}(G)$ can be different. For example, for the graph T_1 of Figure 7, $L_M^{-1}(T_1)$ and $L^{-1}(T_1)$ are shown in Figure 8.

The following result shows that, with the use of the (uniquely determined) preimage $L_M^{-1}(G)$ of a line graph of a multigraph G, Proposition 12 can be simplified.

Proposition 14. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in Figure 7. Then G is M-closed if and only if G is a line graph of a multigraph and*

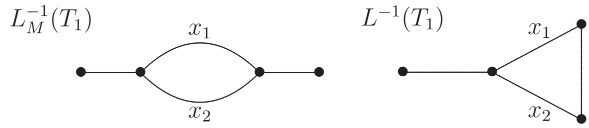

FIGURE 8. Preimages of the graph T_1.

$L_M^{-1}(G)$ *does not contain a subgraph* (*not necessarily induced*) *isomorphic to any of the graphs* T_1, T_2 *or* T_3.

Proof. If $L_M^{-1}(G)$ does not contain any of T_1, T_2, T_3, then clearly the conditions (i), (ii) and (iii) of Proposition 12 are satisfied and hence G is M-closed by Proposition 12.

Conversely, suppose that G is M-closed and let H be a multigraph given by Proposition 12. Then clearly T_1 is not a subgraph of H and any T_2 or T_3 in H not satisfying (ii) or (iii) is turned by Operations B and/or C into a star. Hence $BC(H)$ does not contain any of T_1, T_2, T_3. ■

5. APPLICATIONS AND SHARPNESS

Combining Theorems F and 9(iv), we immediately obtain the following result.

Theorem 15. *Every* 6-*connected claw-free graph with at most* 29 *vertices of degree* 6 *is Hamilton-connected.*

Proof. If G is a counterexample to Theorem 15, then $H = \mathrm{cl}^M(G)$ is a counterexample to Theorem F. ■

Corollary 16. *Every* 7-*connected claw-free graph is Hamilton-connected.*

Similarly, Theorem 9(iv) immediately implies the following result.

Theorem 17. *Conjecture J is equivalent with Conjectures G, H and I.*

Proof. Conjecture J implies Conjecture I since every line graph (of a multigraph) is claw-free. Conversely, if G is a counterexample to Conjecture J, then $H = \mathrm{cl}^M(G)$ is a counterexample to Conjecture I. ■

Note that Corollary 16 was conjectured in [12].

We conclude by showing that the closure operation $\mathrm{cl}^M(G)$ is, in a sense, best possible; more specifically, there is no closure operation that turns a 3-connected line graph of a multigraph into a line graph (of a graph) and preserves Hamilton-connectedness.

If $\mathcal{C}$ is a class of graphs, then by a *closure on* $\mathcal{C}$ we mean a mapping $\mathfrak{cl}: \mathcal{C} \to \mathcal{C}$ such that, for any $G \in \mathcal{C}$, $V(G) = V(\mathfrak{cl}(G))$ and $E(G) \subset E(\mathfrak{cl}(G))$. Let $\mathcal{L}_k$ denote the class of k-connected line graphs (of graphs) and let $\mathcal{L}_k^M$ denote the class of k-connected line graphs of multigraphs.

Theorem 18. *There is no closure* $\mathfrak{cl}$ *on* $\mathcal{L}_3^M$ *such that* $\mathfrak{cl}: \mathcal{L}_3^M \to \mathcal{L}_3$ *and Hamilton-connectedness is stable under* $\mathfrak{cl}$.

Proof. Let H be the multigraph shown in Figure 2 and let $G = L(H)$. Then G is not Hamilton-connected, and the vertices of G that correspond to edges of H adjacent to some of the vertices a_1, a_2 induce in G a subgraph F isomorphic to the sixth graph in Figure 1. Thus, for any closure $\mathfrak{cl}: \mathcal{L}_3^M \to \mathcal{L}_3$, $\mathfrak{cl}(G)$ contains at least one edge joining two non-adjacent vertices of F. However, adding any such edge turns G into a graph that is Hamilton-connected. ■

6. PROOFS AND LEMMAS

Lemma 19. *Let G be a claw-free graph, $x \in V(G)$, let $y \in V(G)$ be a cutvertex of $\langle N_G(x)\rangle_G$ and let K_1, K_2 be components of $\langle N_G(x)\rangle_G - y$. Then (up to a relabeling of K_1, K_2),*

(i) *$\langle V(K_1) \cup \{y\}\rangle_G$ is a clique and K_2 is a clique,*
(ii) *if $H \subset \langle N_G(x)\rangle_G$ is 2-connected non-complete, then $H \subset \langle V(K_2) \cup \{y\}\rangle_G$.*

Proof. If (i) fails, then $\alpha(\langle N_G(x)\rangle_G) \geq 3$ and x is a center of an induced claw, a contradiction. Part (ii) follows immediately from (i). ■

Corollary 20. *Let G be a claw-free graph, $x \in V(G)$, let $H \overset{\mathrm{IND}}{\subset} \langle N_G(x)\rangle_G$ be a 2-connected graph containing two distinct pairs of independent vertices. Then $\langle N_G(x)\rangle_G$ is 2-connected.*

Proof. Proof follows immediately from Lemma 19. ■

Corollary 21. *Let G be a 2-closed claw-free graph, $H \subset G$ (not necessarily induced), $H \simeq S_1$. If $\{u_\ell u_{\ell+3} | \ell = 0,1,2\} \cap E(G) = \emptyset$, then*

(i) *either $H \overset{\mathrm{IND}}{\subset} G$,*
(ii) *or $H + u_0u_5 \overset{\mathrm{IND}}{\subset} G$ (and $H + u_0u_5 \simeq S_2$).*

Proof. If $u_\ell u_{\ell+4} \in E(G)$ for some $\ell \in \{0,1\}$, then $u_{\ell+2}$ is 2-eligible by Corollary 20, a contradiction. ■

Lemma 22. *Let G be a 2-closed claw-free graph, $x \in V(G)$, $H \overset{\mathrm{IND}}{\subset} \langle N_G(x)\rangle_G$ 2-connected, $u, v \in V(H)$ independent. Then u or v is a cutvertex of $\langle N_G(x)\rangle_G$.*

Proof. Since G is 2-closed and u, v are independent, $\langle N_G(x)\rangle_G$ cannot be 2-connected. If $\langle N_G(x)\rangle_G$ is disconnected, then, for an arbitrary vertex w in the component of $\langle N_G(x)\rangle_G$ not containing H, $\langle\{x,u,v,w\}\rangle_G \simeq K_{1,3}$, a contradiction. Hence $\kappa(\langle N_G(x)\rangle_G) = 1$. Rest of the proof follows from Lemma 19. ■

Proof of Lemma 1. Each of the graphs G_i, $i \in \{1,3,5,6,7\}$, contains a vertex x_i satisfying the assumptions of Corollary 20, i.e. such that x_i is 2-eligible in any claw-free graph G such that $G_i \overset{\mathrm{IND}}{\subset} G$. Hence none of the G_i can be an induced subgraph of a 2-closed graph. ■

Lemma 23. *Let G be a 2-closed claw-free graph, $H \overset{\text{IND}}{\subset} G$, $H \simeq S_1$ or $H \simeq S_2$. Then there is no vertex $z \in V(G) \setminus V(H)$ such that $\{u_1, u_3\} \subset N_G(z)$ or $\{u_2, u_3\} \subset N_G(z)$ (and, symmetrically, neither $\{u_2, u_4\} \subset N_G(z)$).*

Proof.

1. We first show that there is no $z \in V(G) \setminus V(H)$ such that $\{u_1, u_2, u_3, u_4\} \subset N_G(z)$. Let, to the contrary, $z \in V(G) \setminus V(H)$ and $u_i \in N_G(z)$ for $i = 1, 2, 3, 4$. Then $\langle\{u_1, u_2, u_3, u_4\}\rangle_G$ is a 2-connected subgraph of $\langle N_G(z)\rangle_G$ and u_1, u_4 are independent. By Lemma 22, u_1 or u_4 is a cutvertex of $\langle N_G(z)\rangle_G$.

   Suppose u_4 is a cutvertex of $\langle N_G(z)\rangle_G$ (the other case is symmetric), and let $w \in N_G(z)$ be in the component of $\langle N_G(z)\rangle_G - u_4$ not containing u_1, u_2 and u_3. Since $\langle\{u_4, u_5, u_2, w\}\rangle_G \not\simeq K_{1,3}$, we have $u_5 w \in E(G)$. Then $\langle\{u_2, u_3, u_5, w, z\}\rangle_G$ is a 2-connected subgraph of $\langle N_G(u_4)\rangle_G$ containing two distinct pairs of independent vertices, hence u_4 is 2-eligible by Corollary 20, a contradiction.
2. We show that there is no $z \in V(G) \setminus V(H)$ such that $\{u_1, u_2, u_3\} \subset N_G(z)$ or $\{u_2, u_3, u_4\} \subset N_G(z)$. Let, to the contrary, $\{u_1, u_2, u_3\} \subset N_G(z)$ (the second case is symmetric). By part 1 of the proof, $zu_4 \notin E(G)$ and from $\langle\{u_2, u_0, z, u_4\}\rangle_G \not\simeq K_{1,3}$ we have $zu_0 \in E(G)$. Then $\langle\{u_0, u_2, u_3, z\}\rangle_G$ is 2-connected, u_0, u_3 are independent and, by Lemma 22, either u_0 or u_3 is a cutvertex of $\langle N_G(u_1)\rangle_G$. Choose a vertex w in the component of $\langle N_G(u_1)\rangle_G - u_0$ ($\langle N_G(u_1)\rangle_G - u_3$) not containing u_2 and z, respectively.
   (i) If u_0 is a cutvertex of $\langle N_G(u_1)\rangle_G$, then $\langle\{w, u_0, u_1, u_2, u_3, u_4\}\rangle_G$ is isomorphic to S_1 or S_2 and we have a contradiction with part 1 of the proof (for the vertex z).
   (ii) If u_3 is a cutvertex of $\langle N_G(u_1)\rangle_G$, then from $\langle\{u_3, w, u_2, u_5\}\rangle_G \not\simeq K_{1,3}$ we have $wu_5 \in E(G)$, but then $\langle N_G(u_3)\rangle_G$ contains a 2-connected induced subgraph with two distinct pairs of independent vertices. By Corollary 20, u_3 is 2-eligible, a contradiction.
3. (a) Let now $\{u_1, u_3\} \subset N_G(z)$ (but $u_2 z \notin E(G)$). From $\langle\{u_3, z, u_2, u_5\}\rangle_G \not\simeq K_{1,3}$ we have $zu_5 \in E(G)$, but then again u_3 is 2-eligible by Corollary 20, a contradiction.
   (b) The case $\{u_2, u_4\} \subset N_G(z)$ is symmetric.
   (c) Finally, if $\{u_2, u_3\} \subset N_G(z)$ (but $u_1 z \notin E(G)$), then from $\langle\{u_3, z, u_1, u_4\}\rangle_G \not\simeq K_{1,3}$ we have $zu_4 \in E(G)$, which is not possible by part 2 of the proof. ■

Corollary 24. *Let G be a 2-closed claw-free graph, $H \overset{\text{IND}}{\subset} G$, $H \simeq S_1$ or $H \simeq S_2$. Then*

(i) *both $\langle N_G[u_1] \setminus \{u_2, u_3\}\rangle_G$ and $\langle N_G[u_2] \setminus \{u_3, u_4\}\rangle_G$ are cliques,*
(ii) *$N_G[u_2] \setminus \{u_3, u_4\} \subset N_G[u_1] \setminus \{u_3\}$,*
(iii) *the only neighbor of u_4 in $N_G(u_2)$ is u_3.*

Note that also symmetrically $\langle N_G[u_4] \setminus \{u_3, u_2\}\rangle_G$ and $\langle N_G[u_3] \setminus \{u_2, u_1\}\rangle_G$ are cliques.

Proof.

(i) If $\langle N_G[u_1] \setminus \{u_2, u_3\}\rangle_G$ is not a clique, then there is a $z \in N_G(u_1)$ such that $zu_0 \notin E(G)$, but then by Lemma 23 $\langle\{u_1, z, u_0, u_3\}\rangle_G \simeq K_{1,3}$, a contradiction. The proof for $\langle N_G[u_2] \setminus \{u_3, u_4\}\rangle_G$ is symmetric.

(ii) By (i), every neighbor of u_2 is adjacent to u_1.
(iii) If $z \in N_G(u_2)$, $z \neq u_3$, is adjacent to u_4, then $z \notin V(H)$ since H is induced, but this contradicts Lemma 23. ■

Lemma 25. *Let G be a claw-free graph, $F \subset V(G)$, $F = \{u_0, u_1, u_2, u_3, u_4, u_5\}$. If F induces S_1 or S_2 in $\mathrm{cl}_2(G)$, then there are vertices $v_0, v_5 \in V(G)$ such that the set $\{v_0, u_1, u_2, u_3, u_4, v_5\}$ induces S_1 or S_2 in G.*

Proof. Let $\mathrm{cl}_2(G) = G^*_{x_1 \ldots x_k}$, set $G_i = G^*_{x_1 \ldots x_i}$, $i = 1, \ldots, k$ (i.e. $G_k = \mathrm{cl}_2(G)$), and let $F = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ be such that $\langle F \rangle_{G_k} \simeq S_1$ or $\langle F \rangle_{G_k} \simeq S_2$. The proof then follows by induction from the following fact.

If $v_0, v_5 \in V(G)$ are such that $\{v_0, u_1, u_2, u_3, u_4, v_5\}$ induces S_1 or S_2 in G_{i+1} for some i, $1 \leq i \leq k-1$, then there are $w_0, w_5 \in V(G)$ such that $\{w_0, u_1, u_2, u_3, u_4, w_5\}$ induces S_1 or S_2 in G_i.

Thus, suppose that $\{v_0, u_1, u_2, u_3, u_4, v_5\}$ induces S_1 or S_2 in $G_{i+1} = (G_i)^*_{x_i}$, and set $B = E(G_{i+1}) \setminus E(G_i)$.

Since x_i is adjacent to both vertices of all edges in B and F induces S_1 or S_2 in $G_k = \mathrm{cl}_2(G)$, by Lemma 23, $B \cap \{u_1u_3, u_2u_3, u_2u_4\} = \emptyset$. Since $\langle N_{G_k}(x_i) \rangle_{G_k}$ is a clique, and by symmetry, we can suppose that $B \subset \{v_0u_1, v_0u_2, u_1u_2\}$. If $u_1u_2 \in B$, then $\langle \{u_3, u_1, u_2, u_5\} \rangle_{G_i}$ is a claw; hence $u_1u_2 \in E(G_i)$ and $|B| \leq 2$. If x_i is adjacent in G_i to both u_1 and u_2, then $\{x_i, u_1, u_2, u_3, u_4, u_5\}$ induces S_1 or S_2 in G_i, we set $w_0 = x_i$, $w_5 = v_5$ and we are done. Hence it remains to consider the case when x_i is adjacent in G_i to at most one of u_1, u_2 and, consequently, $|B| = 1$. But then for $B = \{v_0u_1\}$ we have $\langle \{u_2, v_0, u_1, u_4\} \rangle_{G_i} \simeq K_{1,3}$ and for $B = \{v_0u_2\}$ we have $\langle \{u_2, x_i, u_1, u_4\} \rangle_{G_i} \simeq K_{1,3}$, a contradiction. ■

Proof of Lemma 2. Let $d_G(u_i) \geq 5$ for some i, $3 \leq i \leq k-2$, and let $w \in V(G)$ be a neighbor of u_i, $w \notin \{u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}\}$. By Lemma 23 and since J is good, we have $wu_{i-2} \notin E(G)$ and $wu_{i+2} \notin E(G)$. From $\langle \{u_i, w, u_{i-2}, u_{i+2}\} \rangle_G \not\simeq K_{1,3}$ we then have $u_{i-2}u_{i+2} \in E(G)$, contradicting the fact that J is good. ■

Proof of Corollary 3. If $|V(C)| \leq 6$, then C cannot be good, hence $|V(C)| \geq 7$. Then, by Lemma 2, all vertices of C are of degree 4 in G, implying $C^2 = G$.

Proof Lemma 4. By Corolary 24 (i), $\langle N_G[u_2] \setminus \{u_3, u_4\} \rangle_G$ is a clique and by Corollary 24 (ii), $N_G[u_2] \setminus \{u_3, u_4\} \subset N_G[u_1] \setminus \{u_3\}$. Thus, it remains to show that $N_G[u_1] \setminus \{u_3\} \subset N_G[u_2] \setminus \{u_3, u_4\}$. If this is not the case, then there is a vertex $x \in V(G)$ such that $xu_1 \in E(G)$ and $xu_2 \notin E(G)$. By Corollary 24 (i) then $xu_0 \in E(G)$ and, by Corollary 21, $J' = xu_0u_1 \ldots u_{k+1}$ is a good walk in G, contradicting the maximality of J. ■

Proof of Lemma 5. Suppose that $u_i = u_j$ for some i, j, $1 \leq i < j \leq k$, and choose i, j such that $j-i$ is minimum. Then $u_i \ldots u_{j-1}u_j$ is a cycle, and by the minimality of $j-i$, $u_{i+1} \neq u_{j-1}$.

1. Let first $3 \leq j \leq k-2$. Then, by Lemma 2, $\langle N_G(u_j) \rangle_G \simeq P_4$.

   If $2 \leq i \leq k-2$, then also the neighborhood of u_i in J^2 is a P_4, and these neighborhoods coincide. Since $u_{i+1} \neq u_{j-1}$, we have $u_{i+1} = u_{j+1}$, from which $u_{i+2} = u_{j+2}$, $u_{i-1} = u_{j-1}$ and $u_{i-2} = u_{j-2}$. Then $u_i \ldots u_{j-1}u_j$ is a good cycle, a contradiction by Corollary 3.

If $i=1$, then the equality $u_0=u_{j-1}$ follows from $u_2=u_{j+1}$ and from the equality of neighborhoods, and the cycle $u_i \dots u_{j-1}u_j$ is good by Corollary 21.

2. The case $3 \le i \le k-2$ is symmetric.
3. Thus, it remains to consider the possibility $i \in \{1,2\}$, $j \in \{k-1,k\}$. This specifically implies that for every good walk $J=u_0u_1 \dots u_{k+1}$ we have $k \le |V(G)|+2$, hence for every good walk J there is a maximal good walk J' such that J is a subsequence of J'. Hence we can without loss of generality suppose that J is maximal. We distinguish 4 cases.
   (a) $i=1$, $j=k-1$. Then, by Lemma 4 and by the fact that J is good, $\langle N_G(u_1)\rangle_G$ consists of a clique and one edge while $\langle N_G(u_{k-1})\rangle_G$ consists of a clique and a P_3, a contradiction.
   (b) $i=2$, $j=k$. This case is symmetric to the previous one.
   (c) $i=2$, $j=k-1$. Then the only possible vertices of degree 1 in $\langle N_G(u_2)\rangle_G$ are u_0 and u_4, and, in $\langle N_G(u_{k-1})\rangle_G$ only u_{k-3} and u_{k+1}. Since $u_{k-3} \neq u_4$ (by the choice of i and j), we have $u_{k+1}=u_4$, and hence $u_{k-3}=u_0$. Since clearly $k \ge 5$, we have $d_G(u_3)=4$ and u_3 is the only common neighbor of u_2, u_4, but then, since u_k is a common neighbor of $u_{k-1}=u_2$ and $u_{k+1}=u_4$, necessarily $u_k=u_3$ and we are in Case 2.
   (d) $i=1$, $j=k$. The only universal vertex in $\langle N_G(u_1)\rangle_G$ is u_2 and in $\langle N_G(u_k)\rangle_G$ is u_{k-1}. Hence $u_2=u_{k-1}$, contradicting the choice of i, j.

Lemma 26. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, $J=u_0u_1 \dots u_{k+1}$ a maximal good walk in G, $u \in V(G)$, $u \notin \{u_0,u_1,u_2,u_3,u_4\}$, such that $uu_1 \in E(G)$ or $uu_2 \in E(G)$. Then*:

(i) *both $uu_1 \in E(G)$ and $uu_2 \in E(G)$,*
(ii) *$uu_1 \dots u_{k+1}$ is a good walk in G,*
(iii) *if $u \in V(J)$, then $k \ge 6$ and $u \in \{u_{k-1},u_k,u_{k+1}\}$.*

Proof. (i) follows immediately from Lemma 4.
(ii), (iii) if $u \notin V(J)$, then Lemma 23 implies $uu_3 \notin E(G)$ and we are done by Corollary 21. Hence suppose $u \in V(J)$. Since $uu_2 \in E(G)$ and J is good, necessarily $u=u_j$ for some $j \ge 7$, implying $k \ge 6$. Since $d_G(u_3)=4$ (by Lemma 2), $uu_3 \notin E(G)$ and hence $uu_1 \dots u_{k+1}$ is good by Corollary 21. Since $d_G(u_j)=4$ for $3 \le j \le k-2$ (by Lemma 2), we have $u \in \{u_{k-1},u_k,u_{k+1}\}$.

Proof of Lemma 6. First observe that by Lemma 2 the only edges to be considered are those between u_0,u_1,u_2 and u_{k-1},u_k,u_{k+1}.

Case 1. J is not a path. Since $u_1,\dots,u_k$ is a path by Lemma 5, the only possibilities are $u_0 \in \{u_{k-1},u_k,u_{k+1}\}$, and, symmetrically, $u_{k+1} \in \{u_0,u_1,u_2\}$ (note that $k \ge 6$ by Lemma 26).

(a) $u_0=u_{k-1}$. By Lemma 4, $\langle\{u_1,u_2,u_0,u_{k+1},u_k\}\rangle_G$ is a clique (not excluding the possibility that $u_{k+1} \in \{u_1,u_2\}$). Then $\langle\{u_1,\dots,u_k\}\rangle_G \in \mathcal{J}_4$ (since all edges between u_1,u_2 and u_{k-1},u_k are present and no other edges are possible by Lemma 2), and hence for $u_{k+1} \in \{u_1,u_2\}$ we have $\langle V(J)\rangle_G=\langle\{u_1,\dots,u_k\}\rangle_G \in \mathcal{J}_4$ and we are done, otherwise we have a contradiction with the minimality of J.

(b) $u_0 = u_k$. Then similarly, by Lemma 4, $\langle\{u_1, u_2, u_k, u_{k-1}, u_{k+1}\}\rangle_G$ is a clique and then, as before, for $u_{k+1} \in \{u_1, u_2\}$ we obtain $\langle V(J)\rangle_G = \langle\{u_1, \ldots, u_k\}\rangle_G \in \mathcal{J}_4$, and otherwise we have a contradiction with the minimality of J.

(c) $u_0 = u_{k+1}$. Then the only possible edges to be considered are the edges between u_1, u_2 and u_{k-1}, u_k. By Lemma 26, either $\{u_1u_k, u_1u_{k+1}, u_2u_k, u_2u_{k+1}\} \subset E(G)$, or $\{u_1u_k, u_1u_{k+1}, u_2u_k, u_2u_{k+1}\} \cap E(G) = \emptyset$. In the first case we have $\langle V(J) \setminus \{u_0\}\rangle_G = \langle\{u_1, \ldots, u_k\}\rangle_G \in \mathcal{J}_4$, contradicting the minimality of J, otherwise $\langle V(J)\rangle_G \in \mathcal{J}_3$.

Case 2. J is a path. By Lemma 26, either $\{u_1u_{k+1}, u_2u_{k+1}\} \subset E(G)$, or $\{u_1u_{k+1}, u_2u_{k+1}\} \cap E(G) = \emptyset$. In the first case, the walk $J - u_0 = u_{k+1}u_1u_2 \ldots u_ku_{k+1}$ is good in G, contradicting the minimality of J. Hence $u_1u_{k+1}, u_2u_{k+1} \notin E(G)$, and, symmetrically, $u_0u_{k-1}, u_0u_k \notin E(G)$.

It remains to consider the edges between u_1, u_2 and u_{k-1}, u_k. Again, by Lemma 26, either all of them or none of them are present. In the first case, the walk $J - \{u_0, u_{k+1}\} = u_ku_1u_2 \ldots u_{k-1}u_ku_1$ is good in G, contradicting the minimality of J; in the second case we have $\langle V(J)\rangle_G \in \mathcal{J}_1$ if $u_0u_{k+1} \notin E(G)$ and $\langle V(J)\rangle_G \in \mathcal{J}_2$ if $u_0u_{k+1} \in E(G)$. ∎

Proof of Lemma 7. If $3 \le s \le k-2$ or $3 \le t \le k'-2$, then the statement follows immediately by Lemma 2 (for $\{s,t\} \cap \{1,2\} \neq \emptyset$ we use the equality of neighborhoods of the vertices $u_3^1 = u_3^2$, and symmetrically for $s \in \{k-1, k\}$ or $t \in \{k'-1, k'\}$).

It remains to consider the cases when $s \in \{1, 2, k-1, k\}$ and $t \in \{1, 2, k'-1, k'\}$. By symmetry, it is sufficient to suppose $s, t \in \{1, 2\}$ (otherwise we relabel one or both walks).

1. Let $u_1^1 = u_2^2$. By Lemma 4, $\langle N_G(u_1^1)\rangle_G$ consists of a clique and an edge, while $\langle N_G(u_2^2)\rangle_G$ consists of a clique and a P_3, a contradiction. Hence $u_1^1 \neq u_2^2$ and, symmetrically, $u_1^2 \neq u_2^1$.
2. Suppose that $u_2^1 = u_2^2$. By Lemma 4, at most two vertices in $\langle N_G(u_2^i)\rangle_G$ can be of degree 1, namely, u_0^i and u_4^i, $i = 1, 2$. We distinguish two subcases.
   (a) $u_4^1 = u_4^2$. The only neighbor of u_4^i in $\langle N_G(u_2^i)\rangle_G$ is the vertex u_3^i, $i = 1, 2$; hence $u_3^1 = u_3^2$. By Lemma 23, u_1^i is the only neighbor of u_3^i in $\langle N_G(u_2^i)\rangle_G$, distinct from u_4^i, $i = 1, 2$, hence also $u_1^1 = u_1^2$. For $k = k' = 4$ we thus have $u_j^1 = u_j^2$, $j = 1, 2, 3, 4$; otherwise (i.e. if $k \ge 5$ or $k' \ge 5$) the statement follows from $u_3^1 = u_3^2$ by the beginning of the proof.
   (b) $u_0^1 = u_4^2$ (and hence $u_4^1 = u_0^2$). Similarly as in (a) we have $u_1^1 = u_3^2$. The vertex u_0^2 is of degree 1 in $\langle N_G(u_2^2)\rangle_G$ (since $u_0^2 = u_4^1$ and u_4^1 is of degree 1), hence $u_3^1 = u_1^2$. But then the vertices $u_3^1 = u_1^2$ and $u_4^1 = u_0^2$ have a common neighbor u_5^1 and $u_2^1u_5^1 \notin E(G)$, contradicting the fact that, by Lemma 4, $N_G[u_1^2] \setminus \{u_3^2\} = N_G[u_2^2] \setminus \{u_3^2, u_4^2\}$.
3. Finally, let $u_1^1 = u_1^2$. By Lemma 23, the only universal vertex in $\langle N_G(u_1^i)\rangle_G$ is u_2^i, $i = 1, 2$. Hence $u_2^1 = u_2^2$ and we are back in Case 2. ∎

Proposition 27. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle and let $J = u_0u_1 \ldots u_{k+1}$ be a maximal good walk in G. Then*

(i) *for every $a \in V(G)$, $p_a(G^*_{u_1u_k}) = p_a(G)$,*
(ii) *the graph $G^*_{u_1u_k}$ is Hamilton-connected if and only if G is Hamilton-connected.*

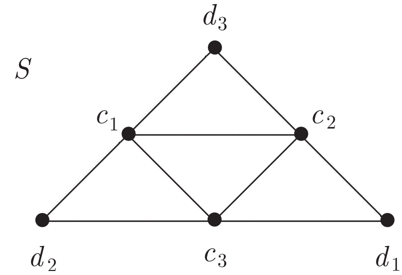

FIGURE 9. The graph S.

Proof. In the proof of Proposition 27 we will need the following result by Brandt et al. (see [6], Proposition 3.2). ■

Proposition Q (Brandt et al. [6]). *Let x be an eligible vertex of a claw-free graph G, G'_x the local completion of G at x, and a, b two distinct vertices of G. Then for every longest (a,b)-path $P'(a,b)$ in G'_x there is a path P in G such that $V(P)=V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an (a,b)-path $P(a,b)$ in G such that $V(P)=V(P')$ except perhaps in each of the following two situations* (*up to symmetry between a and b*):

(i) *There is an induced subgraph $H\subset G$ isomorphic to the graph S in Figure 9 such that both a and x are vertices of degree 4 in H. In this case G contains a path P_b such that b is an endvertex of P and $V(P_b)=V(P')$. If, moreover, $b\in V(H)$, then G contains also a path P_a with endvertex a and with $V(P_a)=V(P')$.*

(ii) *$x=a$ and $ab\in E(G)$. In this case there is always both a path P_a in G with endvertex a and with $V(P_a)=V(P')$ and a path P_b in G with endvertex b and with $V(P_b)=V(P')$.*

Let G and $J=u_o u_1 \dots u_{k+1}$ satisfy the assumptions of Proposition 27 and let S be the graph of Figure 9. For simplicity, set $G'=G^*_{u_1}$ and $G''=(G')^*_{u_k}=G^*_{u_1u_k}$. We show the following.

Claim 27.1. *There is no set $M\subset V(G)$ satisfying either of the following conditions*:

(i) $\langle M\rangle_G\simeq S$ and $d_{\langle M\rangle_G}(u_1)=4$ or $d_{\langle M\rangle_G}(u_2)=4$,

(ii) $\langle M\rangle_{G'}\simeq S$ and $d_{\langle M\rangle_{G'}}(u_k)=4$ or $d_{\langle M\rangle_{G'}}(u_{k-1})=4$.

Proof of Claim 27.1. Suppose there is such a set $M\subset V(G)$.

(i) If $d_{\langle M\rangle_G}(u_1)=4$, then $\langle N_{\langle M\rangle_G}(u_1)\rangle_G\simeq P_4$, but, by Lemma 4, $\langle N_G(u_1)\rangle_G$ consists of a clique and an edge, a contradiction.

Suppose that $d_{\langle M\rangle_G}(u_2)=4$, let e.g. $u_2=c_1$ (see Fig. 9). Then $\langle N_{\langle M\rangle_G}(u_2)\rangle_G$ is a P_4 with vertices d_2,c_1,c_2,d_1. By Lemma 4, the only possible induced P_4 in $\langle N_G(u_2)\rangle_G$ is $xu_1u_3u_4$, where $x\in N_G(u_2)\setminus\{u_3,u_4\}$, but then $u_1=c_1$ or $u_1=c_2$ and we are in the previous case.

(ii) Let first $J\notin\mathcal{J}_4$. Since $\langle N_G(u_k)\rangle_G=\langle N_{G'}(u_k)\rangle_{G'}$, and for $k\geq 5$ also $\langle N_G(u_{k-1})\rangle_G=\langle N_{G'}(u_{k-1})\rangle_{G'}$, the proof is symmetric to the proof in (i) in these cases. It remains to consider the case $d_{\langle M\rangle_{G'}}(u_{k-1})=4$ for $k=4$. Then $\langle N_{G'}(u_3)\rangle_{G'}$ can be covered by two cliques K_1, K_2, where $\{u_0,u_1,u_2\}\subset V(K_1)$ and $\{u_4,u_5\}\subset V(K_2)$, and hence the only possible induced P_4 is xu_2u_4y for $x\in V(K_1)$ and $y\in V(K_2)$. This again leads to the previous case. Secondly, if $J\in\mathcal{J}_4$, then $k\geq 8$, we have $\langle N_{G'}(u_k)\rangle_{G'}=\langle N_G(u_k)\cup\{u_3\}\rangle_{G'}$ and then, by Lemma 4 and by the definition of G', $\langle N_{G'}(u_k)\rangle_{G'}$ consists of a clique and an edge, a contradiction. ■

By Claim 27.1, the case (i) of Proposition Q is not possible. From this, again by Proposition Q, we conclude that:

- for any $a\in V(G)$, $p_a(G'')=p_a(G)$, i.e. statement (i) of Proposition 27 holds,
- if the statement (ii) of Proposition 27 fails, i.e. if $p_{ab}(G')\neq p_{ab}(G)$ or $p_{ab}(G'')\neq p_{ab}(G')$, then we have the situation described in case (ii) of Proposition Q, i.e. $ab\in E(G)$ and $x\in\{a,b\}$ (where $x=u_1$ or $x=u_k$, respectively).

Suppose that $p_{ab}(G')\neq p_{ab}(G)$. Then $u_1\in\{a,b\}$. Let $\tilde{G}$ denote the local completion of G at u_2. Since $N_G(u_1)\subset N_G(u_2)$ by Lemma 4, we have $E(G')\subset E(\tilde{G})$, and hence for any pair $a,b\in V(G)$ for which $p_{ab}(G')\neq p_{ab}(G)$ also $p_{ab}(\tilde{G})\neq p_{ab}(G)$. Thus, by Proposition Q, $u_2\in\{a,b\}$. Hence we conclude that if $p_{ab}(G')\neq p_{ab}(G)$, then $\{a,b\}=\{u_1,u_2\}$.

Symmetrically, if $p_{ab}(G'')\neq p_{ab}(G')$, then $\{a,b\}=\{u_{k-1},u_k\}$ (since the argument for u_1,u_2 used only the statements of Lemma 4 and of Proposition Q and these remain true also in G'). In the latter case (i.e. $\{a,b\}=\{u_{k-1},u_k\}$), we observe that $G''=G^*_{u_1u_k}=G^*_{u_ku_1}$. The proof for $G^*_{u_k}$ is then symmetric to the proof for G' and $\{a,b\}=\{u_1,u_2\}$, and the proof for $G^*_{u_ku_1}$ (i.e. for the local completion of $G^*_{u_k}$ at u_1) follows by Proposition Q. Hence it is sufficient to prove the statement for u_1,u_2.

Consider the following statements:

(a) G' is Hamilton-connected,
(b) G contains a hamiltonian (a,b)-path for all pairs $a,b\in V(G)$ except possibly $\{a,b\}=\{u_1,u_2\}$,
(c) G' contains a hamiltonian (u_2,u_3)-path,
(d) G contains a hamiltonian (u_1,u_2)-path,
(e) G is Hamilton-connected.

By the previous discussion, (a) $\Rightarrow$ (b). Obviously (a) $\Rightarrow$ (c) and (b) $\wedge$ (d) $\Rightarrow$ (e). Thus, in order to show that (a) $\Rightarrow$ (e) (i.e. to finish the proof of Proposition 27), it is sufficient to show that (c) $\Rightarrow$ (d).

Claim 27.2. *If G' contains a hamiltonian (u_2,u_3)-path, then G contains a hamiltonian (u_1,u_2)-path.*

Proof of Claim 27.2. Let P' be a hamiltonian (u_2,u_3)-path in G'. We first show that P' can be chosen such that $P'\subset G$.

By Lemma 4, every edge in $E(G')\setminus E(G)$ contains the vertex u_3. Thus, if P' contains an edge in $E(G')\setminus E(G)$, then this is the edge $u_3^-u_3$. If $u_2^+=u_1$, we set $P':=u_2u_3^-P'u_1u_3$ (since $u_2u_3^-\in E(G)$ by Lemma 4); for $u_2^+\neq u_1$ we replace in P' the path $u_1^-u_1u_1^+$ by the edge $u_1^-u_1^+$ and the edge $u_3^-u_3$ by the path $u_3^-u_1u_3$, i.e. we set $P':=u_2P'u_1^-u_1^+P'u_3^-u_1u_3$ (the edges we need are in G again by Lemma 4). Thus, in the rest of the proof we suppose that P' is a hamiltonian (u_2,u_3)-path in G and we construct a hamiltonian (u_1,u_2)-path P in G.

If $u_1=u_2^+$, then we set $P=u_1P'u_3u_2$, and if $u_1\notin\{u_2^+,u_3^-\}$, then we set $P=u_1u_3P'u_1^+u_1^-P'u_2$ (note that $u_1^-u_1^+\in E(G)$ by Lemma 4). Thus, we can suppose that $u_1=u_3^-$. For $u_2^+=u_4$ we then set $P=u_1P'u_4u_3u_2$, hence we can further suppose that $u_2^+\neq u_4$. Now, if $u_4u_5\in E(P')$ (which, by Lemma 2, necessarily occurs if $k\geq 6$), then for $u_5=u_4^+$ we set $P=u_1P'u_5u_3u_4P'u_2$ and for $u_4=u_5^+$ we set $P=u_1P'u_4u_3u_5P'u_2$.

Thus, it remains to consider the following situation: $u_1=u_3^-$, $u_2^+\neq u_4$, $u_4u_5\notin E(P')$ and $4\leq k\leq 5$.

If $k=4$, then u_3,u_4,u_5,u_4^-,u_4^+ are in a clique (by Lemma 4) and we replace $u_4u_4^+$ by $u_4u_3u_4^+$, i.e. we set $P=u_1P'u_4^+u_3u_4P'u_2$.

Finally, if $k=5$, then $u_4,u_5,u_4^-,u_4^+,u_5^-,u_5^+$ are in a clique (again by Lemma 4) and we set $P=u_1P'u_5^+u_5^-P'u_4^+u_5u_3u_4P'u_2$ if $P'=u_2P'u_4P'u_5P'u_1u_3$, and $P=u_1P'u_4u_3u_5u_4^-P'u_5^+u_5^-P'u_2$ if $P'=u_2P'u_5P'u_4P'u_1u_3$. ∎

Lemma 28. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, $J_1=u_0u_1\ldots u_{k+1}$, $J_2=v_0v_1\ldots v_{p+1}$ two maximal good walks in G, $\{u_1\ldots u_k\}\neq\{v_1\ldots v_p\}$, and let $G'=\mathrm{cl}_2(G^*_{v_1v_p})$. Then either $\langle V(J_1)\rangle_{G'}$ is a clique, or there are vertices w_0, w_{k+1} such that $w_0u_1\ldots u_kw_{k+1}$ is a maximal good walk in G'.*

If moreover $p\geq 6$, then also either $\langle V(J_2)\rangle_{G'}$ is a clique, or $v_1\ldots v_p$ is a maximal good walk in G'.

Proof. First note that, by Lemma 7, $\{u_1\ldots u_k\}\cap\{v_1\ldots v_p\}=\emptyset$. Let $G_0,G_1,\ldots,G_t$ be a sequence of graphs such that $G_0=G^*_{v_1v_p}$, $G_{i+1}=(G_i)^*_{z_i}$ for some z_i that is 2-eligible in G_i, $i=0,1,\ldots,t-1$, and $G_t=G'$. Set $J_1'=\{u_3,\ldots,u_{k-2}\}$, $J_2'=\{v_4,\ldots,v_{p-3}\}$ and let j be the smallest integer such that at least one of the following holds:

(i) there is a vertex $w\in J_1'\cup J_2'$ such that $d_{G_j}(w)>4$,
(ii) J_1 or $v_1\ldots v_p$ is not good in G_j.

Thus, there is an edge $e\in E(G_j)\setminus E(G_{j-1})$ such that either

(i′) e has one vertex at some $w\in J_1'\cup J_2'$, or
(ii′) e joins some vertices u_i,u_{i+p} or v_i,v_{i+p} for $3\leq p\leq 5$

(such an edge will be referred to as a *bad edge*).

If $j=0$, then a bad edge is obtained by local completion at v_1 or at v_p. Then clearly $v_1\ldots v_p$ remains good, and (i′) is not possible since neither v_1 nor v_p can be adjacent in G_{j-1} to any $w\in J_1'\cup J_2'$. Hence the bad edge has both vertices in $V(J_1)$. But, for v_1, all edges in $E((G_{j-1})^*_{v_1})\setminus E(G_{j-1})$ contain v_3, hence the existence of a bad edge implies $v_3\in V(J_1)$, contradicting Lemma 7. The argument for v_p is symmetric.

Hence $j\geq 1$, i.e. a bad edge is obtained by closing a 2-eligible vertex. We prove the statement for the case when the bad edge has at least one vertex w in $V(J_1)$; the proof for a bad edge with both vertices in $v_1\ldots v_p$ is the same.

We first verify the following two observations.

(*) *If $\langle V(J_1)\rangle_{G_t}$ is not a clique, then every vertex $w\in\{u_3,\ldots,u_{k-2}\}$ has in G_j no neighbors outside $V(J_1)$.*

Proof. Suppose $(*)$ fails and let $w=u_\alpha$ have a neighbor outside $V(J_1)$. Then w has in G_{j-1} a 2-eligible neighbor z, and, by the choice of j, $z\in\{u_{\alpha-2},u_{\alpha-1},u_{\alpha+1},u_{\alpha+2}\}$. Also by the choice of j, $z\notin\{u_3,\ldots,u_{k-2}\}$ (since $d_{G_{j-1}}(z)=4$ and any additional edge in $\langle N_{G_{j-1}}(z)\rangle_{G_{j-1}}$ would violate (ii). Thus, by symmetry, it remains to consider the cases $z\in\{u_1,u_2\}$.

If $k\geq 6$, then u_2 cannot be 2-eligible in G_{j-1} since u_4 is of degree 1 in $\langle N_{G_{j-1}}(u_2)\rangle_{G_{j-1}}$, and similarly with u_3 being of degree 1 in $\langle N_{G_{j-1}}(u_1)\rangle_{G_{j-1}}$ for $k\geq 5$. Since clearly $k\neq 4$ (otherwise there is nothing to do), it remains to consider the case $k=5$ and $z=u_2$. However, in this case, if u_2 happens to be 2-eligible, then it is easy to see that $\langle V(J_1)\rangle_{G_t}$ is a clique. ■

$(**)$ *If $\langle V(J_1)\rangle_{G_t}$ is not a clique, then no vertex u_i, $1\leq i\leq k$, is 2-eligible in G_{j-1}.*

Proof. We first consider the case $i\in\{1,2\}$. If u_1 is 2-eligible in G_{j-1} and $k=4$ or if u_2 is 2-eligible in G_{j-1} and $k\leq 5$, then, by Lemma 4, $\langle V(J_1)\rangle_{G_t}$ is a clique. In all remaining cases, by $(*)$ and by the choice of j, u_i has a neighbor of degree 1 in $\langle N_{G_{j-1}}(u_i)\rangle_{G_{j-1}}$, $i=1,2$, hence u_i cannot be 2-eligible. Symmetrically, $i\notin\{k-1,k\}$.

Hence $3\leq i\leq k-2$. Then $\langle N_{G_{j-1}}(u_i)\rangle_{G_{j-1}}$ contains a path P that is not in G. By the choice of j, P has no interior vertices, hence P is an edge. But then P is a bad edge in G_{j-1}, a contradiction. ■

By the assumption, there is an edge $xy\in E(G_j)\setminus E(G_{j-1})$ such that xy is a bad edge in G_j. By $(*)$ and $(**)$, there are the following two cases.

Case 1. $x\in\{u_1,u_2\}$, $y\in\{u_{k-1},u_k\}$ and xy is obtained by closing a vertex $z\notin V(J_1)$ that is 2-eligible in G_{j-1}. Then, by Lemma 4, $\{u_1,u_2,u_{k-1},u_k\}\subset N_{G_{j-1}}(z)$. Since closing at z creates a bad edge, $(k-1)-2\leq 4$, i.e. $k\leq 7$. But then, for any k, $4\leq k\leq 7$, $V(J_1)$ contains a vertex that is 2-eligible in G_j, implying $\langle V(J_1)\rangle_{G_t}$ is a clique.

Case 2. $k=4$, $x=u_0$, $y\in\{u_3,u_4\}$ or $k=5$, $x=u_0$, $y=u_4$ (or, symmetrically, $k=4$, $x=u_5$, $y\in\{u_1,u_2\}$ or $k=5$, $x=u_5$, $y=u_2$). Then, using Lemma 4, $\langle V(J_1)\rangle_{G_t}$ is again a clique. ■

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SURVEY

How Many Conjectures Can You Stand? A Survey

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Abstract We survey results and open problems in hamiltonian graph theory centered around two conjectures of the 1980s that are still open: every 4-connected claw-free graph (line graph) is hamiltonian. These conjectures have lead to a wealth of interesting concepts, techniques, results and equivalent conjectures.

Keywords Hamiltonian graph · Hamilton-connected · Claw-free graph · Line graph · Cubic graph · Dominating closed trail · Dominating cycle · Collapsible graph · Supereulerian graph · Snark · Cyclically 4-edge-connected · Essentially 4-edge-connected · Closure · Contractible graph

Mathematics Subject Classification (2000) 05C45 · 05C38 · 05C35

1 Introduction

Before we are going to introduce the necessary terminology for understanding the sequel, let us start by presenting the two conjectures that will play the main role throughout our exposition.

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Most of the results in this survey paper are inspired by the following two conjectures that were tossed in the 1980s, and later appeared in the cited papers. The first conjecture is due to Matthews and Sumner [50].

Conjecture 1 *Every 4-connected claw-free graph is hamiltonian.*

The second conjecture due to Thomassen was posed in [60], but was already mentioned in 1981 on page 12 of [6], and also appeared in [1].

Conjecture 2 *Every 4-connected line graph is hamiltonian.*

The above two highly related conjectures and their relationship to other open problems and results have been the subject of a number of specialized small scale workshops between 1996 and 2011 in Enschede, Nečtiny (twice), Hannover, Hájek and Domažlice (twice). In order to make the material available to a larger community we decided to compose this survey paper that contains most of the relevant material related to these intriguing open conjectures.

The presented material involves—apart from line graphs and claw-free graphs—cubic graphs, snarks, and concepts like Hamilton cycles, Hamilton-connectedness, dominating closed trails (circuits), and dominating cycles, and techniques involving closures, collapsible graphs, and edge-disjoint spanning trees.

The paper is organized as follows. We first continue in the next section by explaining the necessary terminology to understand the above statements and their relationship. Next we will introduce the tools that show that the two conjectures are in fact equivalent, and we analyze what the statement of the latter conjecture would mean for the root graph of the line graph. Then we will present a sequence of seemingly weaker but equivalent conjectures, and of seemingly stronger but equivalent conjectures. We finish with a survey of some of the existing partial solutions to the conjectures, and discuss how far we are from either proving or refuting the conjectures.

2 Basic Terminology and Concepts

All graphs in this survey are finite, undirected and loopless, and the majority is simple (in some results we allow multiple edges). We refer to [10] for standard terminology and notation.

We denote a (simple) graph G as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.

Adopting the terminology of [10], a graph is called *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all its vertices, i.e., a connected spanning 2-regular subgraph.

If H is a graph, then the *line graph of* H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two vertices in $L(H)$ are adjacent if and only if their corresponding edges in H share an end vertex (with a straightforward extension in case of multiple edges).

A graph G is a *line graph* if it is isomorphic to $L(H)$ for some graph H.

Which graphs are line graphs (of simple graphs) and which are not? This question was answered by a forbidden subgraph characterization due to Beineke [5].

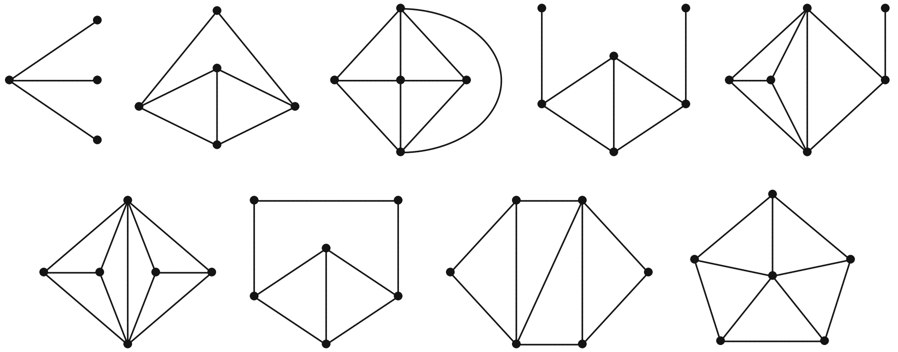

Fig. 1 The nine forbidden subgraphs for line graphs of simple graphs

Theorem 3 *A graph G is a line graph if and only if G does not contain a copy of any of the graphs of Fig. 1 as an* induced *subgraph.*

Let G be a graph and let S be a nonempty subset of $V(G)$. Then the subgraph of G *induced by* S, denoted by $G[S]$, is the graph with vertex set S, and all edges of G with both end vertices in S. H is an *induced subgraph* of G if it is induced in G by some subset of $V(G)$. G is H*-free* if H is not an induced subgraph of G. In particular, a graph G is *claw-free* if G does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph. Direct inspection of Beineke's result shows that every line graph is claw-free.

3 A Handful of Conjectures and More

Since line graphs are claw-free, Conjecture 1 is stronger than Conjecture 2. Or are they equivalent? (A question Herbert Fleischner posed during the EIDMA workshop on Hamiltonicity of 2-tough graphs, Hotel Hölterhof, Enschede, November 19-24, 1996 [8].)

To answer the question affirmatively, Zdeněk Ryjáček introduced a *closure concept* for claw-free graphs at the same workshop which was published in [53]. It is based on adding edges without destroying the (non)hamiltonicity (similar to the Bondy–Chvátal closure [9] for graphs with nonadjacent pairs with high degree sums).

The edges are added by looking at a vertex v and the subgraph of G induced by $N(v)$: the *neighborhood* of v.

If $G[N(v)]$ is connected and not a complete graph, all edges are added to turn $G[N(v)]$ into a complete graph.

This procedure is repeated in the new graph, etc., until it is impossible to add any more edges. By the following theorem due to Ryjáček [53], the closure $cl(G)$ we obtain this way is a well-defined graph.

Theorem 4 *Let G be a claw-free graph. Then*

– *the closure* $cl(G)$ *is uniquely determined,*
– $cl(G)$ *is hamiltonian if and only if G is hamiltonian,*
– $cl(G)$ *is the line graph of a triangle-free graph.*

The above theorem also shows that Conjectures 1 and 2 are equivalent. Moreover, it gives the opportunity to translate questions on hamiltonicity in claw-free graph to questions on hamiltonicity in line graphs, and results on line graphs to results on the more general class of claw-free graphs. We come back to this later when we discuss partial solutions to the two conjectures. Variants on the above closure technique and extensions are discussed in [18].

Here we follow the line of reasoning by turning our attention to what the statements of the conjectures entail for the *root graph* of the line graph.

Whenever we consider a line graph G, we can identify a graph H such that $G = L(H)$. If G is connected this H is unique, except for $G = K_3$: then H can be K_3 or $K_{1,3}$ (this is different for multigraphs, where we could also have three parallel edges, or two parallel edges and one additional incident edge; and there are other pairs of connected multigraphs with isomorphic line graphs). If we restrict ourselves to simple graphs and take $K_{1,3}$ in this exceptional case, we can talk of a unique graph H as the root graph of the connected line graph G isomorphic to $L(H)$. What is the counterpart in H of a Hamilton cycle in G? A *closed trail* (sometimes referred to as a *circuit* in the literature) is a connected *eulerian* subgraph, i.e., a connected subgraph in which all degrees are even. A *dominating closed trail* (DCT for short) is a closed trail T such that every edge has at least one end vertex on T. Note that this notion of domination is not equivalent to the usual notion of domination meaning that every vertex not on the trail has a neighbor on the trail; in our case of a DCT T in a graph H, the graph $H - V(T)$ is edgeless. Also note that a DCT might consist of only one vertex (in case the graph H is a star; then $L(H)$ is a complete graph).

There is an intimate relationship between DCTs in H and Hamilton cycles in $L(H)$, a result due to Harary and Nash-Williams [30] that is known since the 1960s.

Theorem 5 *Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.*

What is the counterpart in H of 4-connectivity in $L(H)$? Note that 4-edge-connectivity is not the right answer, because edge-cuts in H that consist of all edges incident to a single vertex v of H do not correspond to vertex-cuts in $L(H)$ if $H - v$ has at most one component containing edges. A graph H is *essentially* 4*-edge-connected* if it contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain an edge. It is not difficult to check that $L(H)$ is 4-connected if and only if H is essentially 4-edge-connected. The previous results and observations imply that the following conjecture is equivalent to Conjectures 1 and 2.

Conjecture 6 *Every essentially* 4*-edge-connected graph has a DCT.*

If H is *cubic*, i.e., 3-regular, then a DCT becomes a *dominating cycle* (abbreviated DC). H is *cyclically* 4*-edge-connected* if H contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain a cycle. It is not difficult to show that a cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected. Hence the following conjecture due to Ash and Jackson [2] is a specialization of Conjecture 6 to cubic graphs.

Conjecture 7 *Every cyclically* 4-*edge-connected cubic graph has a DC.*

Plummer [52] observed that Conjecture 7 is equivalent to the following two specializations of Conjecture 1.

Conjecture 8 *Every* 4-*connected* 4-*regular claw-free graph is hamiltonian.*

Conjecture 9 *Every* 4-*connected* 4-*regular claw-free graph in which each vertex lies on exactly two triangles is hamiltonian.*

Fleischner and Jackson [25] proved that Conjecture 7 is in fact also equivalent to the others. First note that one can transform an essentially 4-edge-connected graph into one with minimum degree at least three by first deleting the vertices with degree 1, and then replacing the paths with internal vertices with degree 2 by edges (*suppressing* vertices with degree 2). The main ingredient in their proof is a nice trick to replace vertices with degree more than 3 in the obtained graph by cycles without affecting the essentially 4-edge-connectivity.

Let H be an essentially 4-edge-connected graph of minimum degree $\delta(H) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Delete v and add a cycle on $d(v)$ new vertices, and join the new vertices to the original neighbors of v by a perfect matching. The resulting graph is called a *cubic inflation* of H at v. It is not unique, since it depends on the choice of the matching edges joining the new vertices to the original neighbors of v. Fleischner and Jackson [25] proved that by a suitable choice of these edges, some cubic inflation of H at v results in an essentially 4-edge-connected graph. By repeating this procedure, the resulting graph will eventually be cubic and still essentially (and hence cyclically) 4-edge-connected.

Before we continue with imposing further restrictions on the cubic graphs under consideration, we would like to mention the following two related conjectures that have been stated in [25] and are due to Jaeger and Bondy, respectively.

Conjecture 10 *Every cyclically* 4-*edge-connected cubic graph G has a cycle C such that $G - V(C)$ is acyclic.*

Conjecture 11 *Every cyclically* 4-*edge-connected cubic graph G on n vertices has a cycle of length at least cn, for some constant c with* $0 < c < 1$.

It is obvious that Conjecture 7 implies Conjecture 10, and it is not difficult to show that Conjecture 10 implies Conjecture 11. We are not aware of any attempts to establishing the equivalence of these conjectures, and we leave it as an open problem.

A further restriction to cyclically 4-edge-connected cubic graphs that are *not 3-edge-colorable*, is due to Fleischner [24] who posed the following conjecture.

Conjecture 12 *Every cyclically* 4-*edge-connected cubic graph that is not* 3-*edge-colorable has a DC.*

Kochol [39] proved that Conjecture 12 is equivalent to the others, by a constructive approach. By assuming a counterexample to Conjecture 7 and using this as a black box building block, he was able to construct a counterexample to Conjecture 12, using an auxiliary gadget that is almost cubic and not 3-edge-colorable. We skip the details.

For our final restriction on the cyclically 4-edge-connected cubic graphs under consideration, we now turn to snarks. In this paper a *snark* is defined as a cyclically 4-edge-connected cubic graph of girth at least 5 that is not 3-edge-colorable. Here the *girth* of a graph G is the length, i.e., the number of edges or vertices, of a shortest cycle in G. In the literature one can find several variants on this definition where either the restriction on the cyclically edge-connectivity or on the girth or on both are relaxed. Snarks turn up as the 'difficult' objects in many open problems in graph theory, including conjectures on double cycle covers and nowhere zero flows. These are beyond the scope of this survey. We refer to the books of Zhang [66] and [67] for more details and background.

The next conjecture has appeared independently at different places.

Conjecture 13 *Every snark has a dominating cycle.*

Conjecture 13 is also equivalent to the others, as shown in [13], using the constructive approach together with the concept of *contractible subgraphs*. We will explain some of the key ingredients here but refer to [13] for more details. The first step in the proof of the equivalence is based on a refinement of a technique introduced in [56].

In [56], the notion of A*-contractible* graphs is introduced. For a graph H and a subgraph F of H, $H|_F$ denotes the graph obtained from H by contracting F to a single vertex and adding some new vertices and edges in order to keep the same number of edges. This is done by identifying the vertices of F as one new vertex v_F, replacing the edges between vertices of F and vertices of $V(H) \setminus V(F)$ by the same number of edges between v_F and the adjacent vertices of $V(H) \setminus V(F)$, and by replacing the created loops (i.e., one for each edge of F) by *pendant* edges, i.e., edges incident with v_F and one other newly added incident vertex of degree 1. Note that $H|_F$ may contain multiple edges but has the same number of edges as H. A vertex of F is a vertex of *attachment* if it has a neighbor in $V(H) \setminus V(F)$. The set of vertices of attachment of F with respect to H is denoted by $A_H(F)$.

For a subset $X \subset V(H)$, and a partition $\mathscr{A}$ of X into subsets, $E(\mathscr{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1, a_2 are in the same element (i.e., the same equivalence class) of $\mathscr{A}$. Now $H^{\mathscr{A}}$ denotes the graph with vertex set $V(H^{\mathscr{A}}) = V(H)$ and edge set $E(H^{\mathscr{A}}) = E(H) \cup E(\mathscr{A})$ (where $E(H)$ and $E(\mathscr{A})$ are considered to be disjoint, so if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathscr{A})$, then e_1 and e_2 are parallel edges in $H^{\mathscr{A}}$).

Let F be a graph and $A \subset V(F)$. Then F is A*-contractible*, if for every even subset $X \subset A$ (i.e., with $|X|$ even) and for every partition $\mathscr{A}$ of X into two-element subsets, the graph $F^{\mathscr{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathscr{A})$. Note that the case $X = \emptyset$ implies that an A-contractible graph has a DCT containing all vertices of A.

The importance of A-contractible graphs lies in the fact proved in [56] that a connected graph F is A-contractible if and only if, for any H such that $F \subset H$ and $A_H(F) = A$, H has a DCT if and only if $H|_F$ has a DCT. In fact, the authors of [56] proved the stronger result that the (extended) contraction (as defined above) of an A-contractible subgraph of a graph H does not affect the maximum number of edges dominated by a closed trail in H. Note that this number corresponds to the length of a longest cycle in $L(H)$.

In [13], the following slightly weaker notion of a *weakly A-contractible* graph plays an essential role. The difference with the above notion is that only nonempty even subsets $X \subset A$ are required to have the above property. This means that a weakly A-contractible graph is not required to have a DCT containing all vertices of A. Using this weaker notion, one of the key auxiliary results proved in [13] yields that for a 2-connected cubic graph H with a weakly $A_H(F)$-contractible subgraph F of H, H has a DC if and only if $H|_F$ has a DCT. This obviously imposes structural restrictions on possible minimal counterexamples to the conjectures on the existence of a DC in certain cubic graphs. This is combined in [13] with a second step in which it is shown that replacing a subgraph of a cubic graph does not affect the (non)existence of a DC if certain *compatible mappings* are respected. Without going into the technical details of explaining what these mappings entail, this enables the replacement of 4-cycles in a possible counterexample to Conjecture 7 in order to construct a counterexample with girth at least 5 (Note that the only cyclically 4-edge-connected cubic graph with triangles is K_4). This is then further combined in [13] with techniques that were previously used in [39] in order to construct a snark without a DC under the assumption of a counter example to Conjecture 7.

We like to bring the following two conjectures that were posed in [13] to the reader's attention. The first of these two conjectures was shown to be equivalent to the other conjectures.

Conjecture 14 *Every cyclically* 4*-edge-connected cubic graph contains a weakly contractible subgraph* F *with* $\delta(F) = 2$.

The following statement, also posed as a conjecture in [13], implies the above, but we do not know whether it is equivalent to the above conjecture.

Conjecture 15 *Every cyclically* 4*-edge-connected cubic graph* G *contains a weakly contractible subgraph* F *with* $|A_G(F)| \geq 4$.

To date Conjecture 13 is the seemingly weakest conjecture on the existence of a DC in certain cubic graphs that is equivalent to Conjectures 1 and 2. All snarks up to 36 vertices were tested for the existence of a DC by Brinkmann et al. [11]. Due to the role snarks play in other areas we would like to pose the following two open questions.

– Is there a link to conjectures on *Double Cycle Covers*?
– Is there a link to conjectures on *Nowhere-Zero Flows*?

Taking a slightly different approach, we continue with presenting some other seemingly weaker conjectures. Kochol [40] proved equivalence with seemingly weaker versions, using a concept called *sublinear defect*. As an example, he proved that Conjecture 2 is equivalent to the following conjecture.

Conjecture 16 *There are sublinear functions* $f_1(n)$ *and* $f_2(n)$ *such that every* 4*-connected line graph* G *of order* n *contains* $\leq f_1(n)$ *paths that cover* $\geq n - f_2(n)$ *vertices of* G.

Similar techniques were introduced and applied in [3] to obtain equivalent versions of the 2-tough conjecture, and in [4] successfully applied with suitable small gadgets to obtain counterexamples to the 2-tough conjecture. Although the 2-tough conjecture restricted to claw-free graphs is equivalent to Conjecture 1, it is beyond the scope of this survey. We refer the reader to [12] for more details. Inspired by these techniques, independently of [39] it has been shown in [14] that Conjectures 1 and 2 are equivalent to seemingly weaker conjectures in which the conclusion is replaced by a conclusion similar to the one in Conjecture 16. We use the term *r-path-factor* for a spanning subgraph consisting of at most r paths. A *2-factor* is a set of vertex-disjoint cycles that together contain all the vertices of the graph, i.e., a 2-regular spanning subgraph.

Theorem 17 *Let $k \geq 2$ be an integer, and let $f(n)$ be a function of n with the property that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Then the following statements are equivalent.*

(1) Every k-connected claw-free graph is hamiltonian.
(2) Every k-connected claw-free graph on n vertices has an $f(n)$-path-factor.
(3) Every k-connected claw-free graph on n vertices has a 2-factor with at most $f(n)$ components.
(4) Every k-connected claw-free graph on n vertices has a spanning tree with at most $f(n)$ vertices of degree one.
(5) Every k-connected claw-free graph on n vertices has a path of length at least $n - f(n)$.

The key ingredient for proving the above equivalences is the auxiliary result proved in [14] that the existence of a k-connected nonhamiltonian claw-free graph G on n vertices implies the existence of such a graph G^* on at most $2n - 2$ vertices that contains a *k-clique*, i.e., a set of k mutually adjacent vertices. This result enables the construction of k-connected claw-free graphs on at most $(2r + 1)(2n - 2)$ vertices without an r-path-factor, assuming that there is a k-connected nonhamiltonian claw-free graph G on n vertices, by simply taking $2r + 1$ vertex-disjoint copies of G^* and adding all edges between the k-clique vertices of all the copies.

By results in [32], where it has been shown that a claw-free graph G has an r-path-factor if and only if $cl(G)$ has an r-path-factor, and in [55], where it has been shown that a claw-free graph G has a 2-factor with at most r components if and only if $cl(G)$ has such a 2-factor, the equivalence of statements (1), (2) and (3) in the above theorem also holds for line graphs.

In this section we have presented a sequence of gradually seemingly weaker conjectures that turned out to be equivalent. In the next section we are going to present some seemingly stronger conjectures.

4 Seemingly Stronger Versions for Cubic Graphs

Fouquet and Thuillier [27] considered a seemingly stronger version than the Ash-Jackson-Conjecture (Conjecture 7). Although the next conjecture is equivalent to Conjecture 7, the conclusion is stronger in the sense that it requires a DC containing any two given disjoint edges, as follows.

Conjecture 18 *In a cyclically 4-edge-connected cubic graph any two disjoint edges are on a DC.*

Establishing equivalent conjectures with stronger conclusions might help in an attempt to refute the conjectures. The above equivalence was extended by Fleischner and Kochol [26] by requiring a DC through any two given edges.

Conjecture 19 *In a cyclically* 4*-edge-connected cubic graph any two edges are on a DC.*

Brinkmann et al. [11] have verified Conjecture 19 for all not 3-edge-colorable cyclically 4-edge-connected cubic graphs with girth at least 4 up to 34 vertices, and for all snarks on 36 vertices.

There are several further equivalent versions involving other subgraphs of cubic graphs, like Conjecture 14. We present two others here without going too much into the technical details. Interested readers are invited to consult the sources [43] and [45], respectively. We need some additional terminology. Let H be a graph with minimum degree $\delta(H) = 2$ and suppose that the set $V_2(H)$ of all vertices with degree 2 in H has four elements. We say that H is *$V_2(H)$-dominated* if the graph $H + \{e_1, e_2\}$ arising from H after adding two new edges $e_1 = xy$ and $e_2 = wz$ (possibly creating multiple edges) such that $\{x, y, w, z\} = V_2(H)$ has a dominating closed trail containing e_1 and e_2. We say that H is *strongly $V_2(H)$-dominated* if H is $V_2(H)$-dominated and moreover the graph $H+e$ obtained from H by adding the new edge e has a dominating closed trail containing e for any newly added edge $e = uv$ for $\{u, v\} \subset V_2(H)$.

The following two conjectures appeared in [43] and [45], respectively.

Conjecture 20 *Any subgraph H of an essentially* 4*-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated.*

Conjecture 21 *Any subgraph H of an essentially* 4*-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is strongly $V_2(H)$-dominated.*

We now turn to seemingly stronger versions than Conjecture 2 for line graphs. Adopting the terminology of [10], a graph is called *Hamilton-connected* (sometimes called hamiltonian-connected in the literature) if it admits a Hamilton path between any two distinct given vertices. It is easy to check that any Hamilton-connected graph on at least 4 vertices is necessarily 3-connected.

Kužel and Xiong [46] established the equivalence of Conjecture 2 with the following conjecture.

Conjecture 22 *Every 4-connected line graph of a multigraph is Hamilton-connected.*

Ryjáček and Vrána [58] further extended the equivalence to claw-free graphs by proving that the following conjecture is equivalent to Conjecture 22.

Conjecture 23 *Every 4-connected claw-free graph is Hamilton-connected.*

One of the key ingredients in their equivalence proof is a result from [57] that extends the closure technique used in [53] to establish the equivalence of Conjectures 1 and 2. In this new version of the closure technique, the *2-closure*, edges are added to a noncomplete neighborhood in a claw-free graph G if this neighborhood

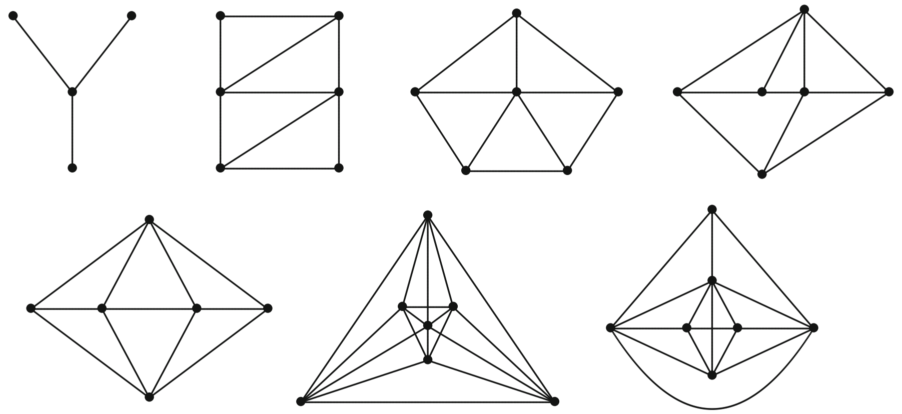

Fig. 2 The seven forbidden subgraphs for line graphs of multigraphs

induces a 2-connected subgraph instead of just a connected one. Then it is proved that the new graph G' is Hamilton-connected if and only if G is Hamilton-connected. We note here that it is not always true that a Hamilton path between two vertices u and v exists in G if and only if it exists in G'. Successively adding edges to a claw-free graph G according to this new version yields a unique graph denoted $cl_2(G)$. One of the serious difficulties in this approach is that the successive application of this new closure operation to 2-connected neighborhoods does not always result in a line graph (of a multigraph). One of the structures that can appear in $cl_2(G)$ is the *square* of a cycle, i.e., the graph obtained from a cycle by adding edges between nonadjacent vertices that have a common neighbor. The closure operation defined in [58] deals with these squares of cycles separately (by adding all the edges to turn them into complete graphs on the same vertex set) and defines an additional closure operation on *good walks* in the graph $cl_2(G)$ if it is not the square of a cycle. We will not explain the details involved in the handling of these good walks, but we conclude here with the statement that this extension guarantees that the resulting *multigraph closure* is a unique graph $cl_M(G)$, and that it is the line graph of a multigraph. Moreover, this new graph $cl_M(G)$ is Hamilton-connected if and only if the original graph G is Hamilton-connected. For convenience, we add the counterpart of Fig. 1 which shows the forbidden induced subgraphs of line graphs of multigraphs. These are illustrated in Fig. 2.

5 A Link to the P Versus NP Problem

At present the seemingly strongest version of the conjectures for line graphs is by Kužel, Ryjáček and Vrána [45].

Adopting the terminology of [45], a graph G is called *1-Hamilton-connected* if for any vertex x of G there is a Hamilton path in $G - x$ between any two vertices, and G is called *2-edge-Hamilton-connected* if the graph $G + X$ has a Hamilton cycle containing all edges of X for any $X \subseteq \{xy | x, y \in V(G)\}$ with $1 \leq |X| \leq 2$. It is easy to check that for both properties 4-connectedness is a necessary condition (except for complete graphs on at most 4 vertices).

Using the equivalence of Conjecture 2 and Conjecture 21, in [45] it is proved that the following conjecture is equivalent to Conjecture 2.

Conjecture 24 *Every* 4*-connected line graph of a multigraph is* 1*-Hamilton-connected (2-edge-Hamilton-connected).*

This version strongly suggests that Conjecture 2 (and all equivalent versions) might fail, for the following reasons. If the above conjecture is true, it implies that a line graph is 1-Hamilton-connected (2-edge-Hamilton-connected) if and only if it is 4-connected. It is well-known that the connectivity of a (line) graph can be determined in polynomial time. It is an NP-complete problem to decide whether a line graph is hamiltonian (see, e.g., [7]). It is not difficult to show that deciding whether a given graph is 1-Hamilton-connected is also NP-complete. It seems not unlikely that deciding whether a given graph is 1-Hamilton-connected remains NP-complete when restricted to line graphs. If one would be able to show this, however, it would imply that Conjecture 2 cannot be true, unless P=NP. In other words, the validity of Conjecture 2 would imply polynomiality of both 1-Hamilton-connectedness and 2-edge-Hamilton-connectedness in line graphs.

We add here as a side remark that, on the other hand, it is an easy exercise to show that a result of Sanders (see [59, p. 342]) implies that every 4-connected planar graph is 1-Hamilton-connected. Thus for a given planar graph one can decide in polynomial time whether it is 1-Hamilton-connected or not, whereas deciding whether a planar graph is hamiltonian is an NP-complete problem.

6 One Step Beyond

Very recently, the closure techniques of [57,58] have been strengthened and adapted to work for the stronger notion of 1-Hamilton-connectivity. In [44], the concept of multigraph closure is further strengthened in such a way that this adapted closure of a claw-free graph is the line graph of a multigraph with at most two triangles or at most one double edge. In [54], this is used to obtain a closure that turns a claw-free graph into a line graph of a multigraph while preserving the property of (not) being 1-Hamilton-connected. This yields the following currently seemingly strongest version of the conjectures.

Conjecture 25 *Every 4-connected claw-free graph is 1-Hamilton-connected.*

7 Positive Results Related to the Conjectures

The gap between the conjecture(s) and the positive results is narrowing, in the following sense. If we look at the connectivity conditions in Conjectures 1 and 2, then the first natural question is whether one can prove a theorem on hamiltonicity of claw-free graphs or line graphs if one imposes a stronger connectivity condition. The earliest result in this direction is due to Zhan [65] (and was independently proved by Jackson [33]).

Theorem 26 *Every 7-connected line graph (of a multigraph) is hamiltonian.*

In fact, Zhan proved the stronger result that such graphs are Hamilton-connected. For this purpose, he slightly generalized Theorem 5 to formulate an equivalent result on the existence of dominating trails between pairs of edges in the root graph H of a line graph $G = L(H)$ such that each edge of H is dominated by an internal vertex of the trail. He then used an approach that is typical for most of the results in this section. We will present some of the ingredients here, starting with a classic result on the existence of k edge-disjoint spanning trees due to Nash-Williams [51] and Tutte [61].

Theorem 27 *A graph G has k edge-disjoint spanning trees if and only if for every partition $\mathscr{P}$ of $V(G)$ we have $\varepsilon(\mathscr{P}) \geq k(|\mathscr{P}| - 1)$, where $\varepsilon(\mathscr{P})$ counts the number of edges of G joining distinct parts of $\mathscr{P}$.*

Kundu [42] observed that Theorem 27 has the following consequence.

Theorem 28 *Every k-edge-connected graph has at least $\lceil (k-1)/2 \rceil$ edge-disjoint spanning trees.*

The use of the existence of two edge-disjoint spanning trees for obtaining a spanning eulerian subgraph was observed by several researchers independently, and appeared in a paper by Jaeger [36].

Theorem 29 *Every graph with two edge-disjoint spanning trees has a spanning eulerian subgraph.*

The intuition behind this result is that the vertices of odd degree in one of the trees can be paired and connected by edge-disjoint paths in the other tree to form a spanning eulerian subgraph (a spanning closed trail).

Combining the above results, we immediately obtain the next corollary.

Corollary 30 *(i) Every 4-edge-connected graph has a spanning eulerian subgraph.*
(ii) Every 4-edge-connected graph has a hamiltonian line graph.

On the other hand, we know that Conjecture 2 is equivalent to the conjecture (see Conjecture 6) that every essentially 4-edge-connected graph has a hamiltonian line graph. At first sight the gap between Corollary 30(i) and Conjecture 6 does not look that large. Moreover in Corollary 30(i) we obtain a spanning eulerian subgraph, whereas we would only need a DCT, i.e., a dominating eulerian subgraph in order to prove Conjecture 6. Nevertheless Conjecture 6 and all the equivalent conjectures seem to be very hard. As a side remark and a possible approach to solving the conjectures, we would like to present another conjecture, that would clearly imply Conjecture 6, and was put up by Jackson [34]. It resembles the way one can prove that 4-connected planar graphs are hamiltonian by proving assertions on the existence of certain cycles (paths) in 2-connected planar graphs.

Conjecture 31 *Every 2-edge-connected graph G has an eulerian subgraph H with at least three edges such that each component of $G - V(H)$ is linked by at most three edges to H.*

Vrána [63] recently observed that Conjecture 31 is equivalent to Conjecture 2.

We continue with sketching the approach to proving Theorem 26 and similar results. Similarly to the way we have been proving the equivalence of many of the conjectures mentioned earlier, the first step is to consider the root graph of the line graph, and the equivalent property one has to establish, e.g., the existence of a DCT or of a trail between two given edges that internally dominates all edges of the root graph. In the next step the root graph is usually reduced by deleting the vertices with degree one (or with only one neighbor in the case of multigraphs) and suppressing the vertices with degree two. In the third step the degree and connectivity properties of the reduced graph are used to establish the existence of a spanning eulerian subgraph (or trail between two given edges). In this step the existence of two disjoint spanning trees (or something slightly more sophisticated) is usually the intermediate goal.

Theorem 26 has been extended to results on 6-connected line graphs with some additional conditions. The proof in [65] together with Theorem 4 immediately implies that every 6-connected claw-free graph G with $\delta(G) \geq 7$ is hamiltonian. More careful considerations show that the condition $\delta(G) \geq 7$ can be weakened to 'at most 33 vertices have degree 6' (Li [8]) or 'the vertices of degree 6 are independent' (Fan [8]). Further extensions to 6-connected line graphs with some additional conditions and the conclusion Hamilton-connected, but following basically the same method as in [65], can be found in [31]. Even further extensions can be found in [64], but they still need an additional condition bounding the number of vertices with degree 6 to at most 74 or the structure they induce to at most 8 disjoint K_4s (for 6-connected claw-free graphs to be hamiltonian) or bounding the number of vertices with degree 6 to at most 54 or the structure they induce to at most 5 disjoint K_4s (for 6-connected line graphs to be Hamilton-connected). The proofs in [64] use a similar approach as in the above sketch, but combined with a powerful reduction technique based on *collapsible* graphs introduced by Catlin [19]. Since this technique and its refinements play an important role in obtaining results on the existence of spanning closed trails and DCTs, we will give a brief outline of the basics involved. Before doing so, we first present the currently best connectivity result related to Conjectures 1 and 2 due to Kaiser and Vrána [37].

Theorem 32 *Every* 5*-connected claw-free graph with minimum degree at least* 6 *is Hamilton-connected.*

The proof of Theorem 32 is very technical and too complicated and long to present here. Basically, the proof is along the same lines as the proofs of the other results in this section. However, instead of finding two edge-disjoint spanning trees the authors use a far more sophisticated approach to find *quasitrees* with *tight complements* in *hypergraphs* associated with the root graphs. They apply this to prove that an essentially 5-edge-connected graph in which every edge has at least 6 neighboring edges contains a connected eulerian subgraph spanning all the vertices of degree at least 4. This suffices to prove Theorem 32 for line graphs and with the conclusion hamiltonian. Refinements of the techniques then show the validity of the more general statement. The authors state in their concluding section of [37] that it is conceivable that a further refinement in some parts of their analysis might improve the result a bit, perhaps even to all 5-connected line (claw-free) graphs. On the other hand, they believe that the 4-connected case would require major new ideas. For instance, the root graph H of a

4-connected line graph may be cubic, in which case it is not clear how to associate a suitable hypergraph with H in the first place.

To finish this section we give the basic definitions and results related to the technique of collapsible graphs. We refer to [20] for a survey on applications of the technique.

A graph is called *supereulerian* if it contains a spanning eulerian subgraph. A graph H is *collapsible* if for every even subset X of $V(H)$, H has a subgraph H_X such that $H - E(H_X)$ is connected and X is the set of odd degree vertices of H_X. As examples, it is easy to see that a cycle of length 3 (or an edge of multiplicity 2 in a multigraph) is a collapsible graph and it is not difficult to show that a graph containing two edge-disjoint spanning trees is a collapsible graph. But also many graphs that are only a few edges short of having two edge-disjoint spanning trees are collapsible (see, e.g., [21]). The importance of collapsible graphs is immediate from the following result proved by Catlin [19].

Theorem 33 *If H is a collapsible subgraph of a graph G, then G is supereulerian (collapsible) if and only if G/H is supereulerian (collapsible).*

Here G/H is the graph obtained from G by contracting all edges of H and removing all loops. The theorem gives a powerful reduction method for studying supereulerian graphs because one can contract any collapsible subgraph without affecting this property. It was shown in [19] that any (multi)graph G has a unique collection of maximal collapsible subgraphs, so contracting them yields a well-defined unique graph called the *reduction* of G. Apart from applications in the area of our survey, there are many applications of the above reduction method in the study of cycle double covers, nowhere-zero 4-flows, etc. These are beyond the scope of this survey.

Motivated by the idea to modify the above technique to the study of DCTs instead of spanning closed trails, Veldman [62] refined Catlin's technique by handling vertices of degree 1 and 2 in a special way (since degree 1 vertices cannot occur on any closed trail, and the two neighbors of a degree 2 vertex are on any DCT). This refinement can be described in the following way. For a simple graph H, let $D(H) = \{v \in V(H) | d_H(v) = 1, 2\}$. For an independent set X of $D(H)$, let $I_X(H)$ be the graph obtained from H by contracting one edge incident with each vertex of X. Veldman then defined H as X*-collapsible* if $I_X(H)$ is collapsible in the Catlin sense. Also this refined reduction technique is a powerful tool for studying hamiltonicity of line graphs, in particular for dense graphs. However, the main drawback of Catlin's and Veldman's techniques is that the search for maximal collapsible subgraphs is very difficult. In this context, a natural question is whether the claw-free closure concept can be strengthened by using line graph techniques or by combining them with closure techniques. A first attempt in this direction was done in [17], but the major work was done in [56], where it was shown that the reduction techniques of Catlin and Veldman can be reformulated in terms of a closure technique for line graphs. This closure technique might be more convenient to use since it avoids the necessity of a search for maximal collapsible subgraphs. It is based on the concept of A-contractible graphs that was introduced earlier. We refer to [56] for more details and to [18] for a survey on closure techniques (this survey does not contain the work of [56]).

8 Related Results with a Weaker Conclusion

First of all, if we drop the connectivity condition of the 2-regular spanning subgraph, we move from a Hamilton cycle to a 2-factor. Enomoto et al. [22] proved that every 2-*tough* graph contains a 2-factor. Since $2k$-connected claw-free graphs are k-tough by a result in [50], this implies the following.

Theorem 34 *Every 4-connected claw-free graph has a 2-factor.*

It does not seem easy to use this as a starting point to show that there is a 2-factor with only one component, although there are some results that give upper bounds on the number of components (see, e.g., [15,16,29]). These results are beyond the scope of this paper.

By Theorem 3.1 in Jackson and Wormald [35], every connected claw-free graph has a 2-*walk*, i.e., a closed walk which passes every vertex at most twice. Clearly, the edges of a 2-walk induce a connected factor with maximum degree at most 4. In [14] the following related result is proved.

Theorem 35 *Every* 4*-connected claw-free graph contains a connected factor which has degree two or four at each vertex.*

By the results of Kriesell [41] it is possible to prove the related result that between every pair of distinct vertices in a 4-connected line graph there exists a spanning trail which passes every vertex at most twice. As with the 2-factor result these results do not seem to help in finding a way to prove Conjectures 1 and 2, although they supply some supporting evidence in favor of the conjectures.

9 Related Results with Additional Conditions

We have already presented some results in which a connectivity condition is accompanied by another condition, e.g., Theorem 32. Another way of obtaining positive results related to the conjectures is by relaxing the 4-connectedness and adding something else. Many such results involve degree conditions and other neighborhood conditions. Such results have been surveyed in several papers (see, e.g., [12,23,28]). We do not want to discuss such conditions in this survey, but here is a connectivity-only result.

If we add an 'essentially connectivity' condition there is this result due to Lai et al. [48].

Theorem 36 *Every* 3*-connected, essentially* 11*-connected claw-free (line) graph is hamiltonian.*

The proof of Theorem 36 is based on the technique of collapsible graphs by Catlin applied to the graph obtained from the root graph of the line graph by deleting vertices with degree 1 and suppressing vertices with degree 2. We omit the details.

Recently, Kaiser and Vrána [38] were able to decrease the 11 to 9 in the above theorem. In their proof they use a slight modification of their proof approach to Theorem 32 in [37]. The proof is again based on quasitrees with tight complements in hypergraphs, but in the proof they have to work around quasitrees which contain *bad type leaves*. This can be done by suitably choosing the hyperedges of the associated hypergraph. We refer to [37] for the details.

Perhaps the 11 in Theorem 36 can be replaced by 5, which would be best possible (by the line graph of the Petersen graph in which the edges of a perfect matching are subdivided exactly once). An open question is how far we can decrease the 11 (or 9) by raising the 3 to a 4 in the theorem.

10 Restrictions on the Root Graph

Using the technique of collapsible graphs, Lai [47] proved the following partial affirmative answer to Conjecture 2 by restricting the root graph to the class of *planar* graphs, i.e., graphs that can be embedded in the plane in such a way that the edges only intersect in incident vertices.

Theorem 37 *Every 4-connected line graph of a planar graph is hamiltonian.*

Kriesell [41] proved a similar result on line graphs of claw-free (multi)graphs with the stronger conclusion of Hamilton-connectedness. In fact, he proved the following more general result.

Theorem 38 *Let G be a graph such that $L(G)$ is 4-connected and every vertex of degree 3 in G is on an edge of multiplicity at least 2 or on a triangle of G. Then $L(G)$ is Hamilton-connected.*

Lai, Shao and Zhan [49] did something similar for *quasi claw-free* graphs, i.e., in which every pair of vertices u and v at distance 2 has a common neighbor w the neighbors of which are in $N(u) \cup N(v) \cup \{u, v\}$.

Theorem 39 *Every 4-connected line graph of a quasi claw-free graph is Hamilton-connected.*

11 Conclusion

We presented many conjectures, most of which have been shown to be equivalent to the conjecture that 4-connected claw-free graphs are hamiltonian. We also presented several results that supply supporting evidence in favor of the conjectures, including the most recent result that 5-connected claw-free graphs with minimum degree at least 6 are Hamilton-connected. There are many other results on hamiltonian properties of sufficiently connected claw-free graphs, including many that have not been listed here. In most of the proofs of the results that are closely related to the open conjectures, closure techniques are used to restrict the statements to line graphs. Then the root graphs are considered and the aim is to find a (closed or open) trail (internally) dominating all edges. A common approach is the following. First the degree 1 vertices are deleted, then the degree 2 vertices are suppressed, and now one tries to show that the reduced graph has a suitable spanning (closed) trail. This is usually accomplished by applying the technique of finding two edge-disjoint spanning trees (or similar structures that yield suitable trails), or by the technique of collapsible subgraphs, or by advanced closure concepts. It seems that none of these techniques is capable of tackling the open conjectures. Does the latter conclusion suggest that the conjectures are all false? We now tend to believe that there might exist nonhamiltonian 4-connected claw-free graphs, but we have no strong opinion. It is our sincere hope that this survey will inspire new research into this intriguing and challenging field.

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Thomassen's Conjecture Implies Polynomiality of 1-Hamilton-Connectedness in Line Graphs

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Abstract: A graph G is 1-Hamilton-connected if $G-x$ is Hamilton-connected for every $x \in V(G)$, and G is 2-edge-Hamilton-connected if the graph $G+X$ has a hamiltonian cycle containing all edges of X for any $X \subset E^{+}(G) = \{xy \mid x, y \in V(G)\}$ with $1 \leq |X| \leq 2$. We prove that Thomassen's conjecture (every 4-connected line graph is hamiltonian, or, equivalently, every snark has a dominating cycle) is equivalent to the statements that every 4-connected line graph is 1-Hamilton-connected and/or 2-edge-Hamilton-connected. As a corollary, we obtain that Thomassen's conjecture implies polynomiality of both 1-Hamilton-connectedness and 2-edge-Hamilton-connectedness in line graphs. Consequently, proving that

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1-Hamilton-connectedness is NP-complete in line graphs would disprove Thomassen's conjecture, unless $P=NP$. J Graph Theory 69: 241–250, 2012

Keywords: *line graph; 4-connected; Hamiltonian; Hamilton-connected; dominating cycle; Thomassen's conjecture; snark*

1. INTRODUCTION

By a *graph* we mean a finite undirected loopless graph $G=(V(G),E(G))$ allowing multiple edges. We follow the most common graph-theoretical notation and for notation and concepts not defined here we refer the reader e.g. to [2].

A graph G is said to be *hamiltonian* if G has a *hamiltonian cycle*, i.e. a cycle of length $|V(G)|$, and *Hamilton-connected* if, for any $x,y \in V(G)$, G has a *hamiltonian* (x,y)*-path*, i.e. an (x,y)-path P with $V(P)=V(G)$. Obviously, a hamiltonian graph must be 2-connected and a Hamilton-connected graph must be 3-connected. A graph G is k*-Hamilton-connected* if, for any $X \subset V(G)$ with $|X|=k$, the graph $G-X$ is Hamilton-connected. It is easy to see that a k-Hamilton-connected graph must be $(k+3)$-connected.

We will use $L(H)$ for the line graph of a graph H. Recall that every line graph is *claw-free*, i.e. does not contain an induced subgraph isomorphic to the claw $K_{1,3}$, and that a line graph $G=L(H)$ is k-connected if and only if H is *essentially* k*-edge-connected*, i.e. H has no edge-cutset $X \subset E(H)$ such that $|X|<k$ and at least two components of $G-X$ contain at least one edge (such an X will be referred to as an *essential edge-cutset*). Also recall that if an edge in a graph H is *pendant* (i.e. one of its vertices has degree 1), then the corresponding vertex in $G=L(H)$ is *simplicial*, i.e. its neighborhood induces a complete graph.

If a graph H has no edge-cutset $X \subset E(H)$ such that $|X|<k$ and at least two components of $G-X$ contain at least one cycle, we say that H is *cyclically* k*-edge-connected*. It is a well-known fact (see e.g. [5]) that a *cubic* (i.e. 3-regular) graph H is cyclically 4-edge-connected if and only if H is essentially 4-edge-connected. A cyclically 4-edge-connected cubic graph H of girth (length of shortest cycle) $g(H) \geq 5$ that is not 3-edge-colorable is called a *snark*.

A closed trail (i.e. an Eulerian subgraph) T in a graph H is said to be *dominating* if every edge of H has at least one vertex on T. It is a well-known fact (see [9]) that if G is a line graph of order at least 3 and $G=L(H)$, then G is hamiltonian if and only if H contains a dominating closed trail. For $a,b \in E(H)$, a trail T is said to be an (a,b)*-trail* if a is the first and b is the last edge of T. A trail T in a graph H is *internally dominating* if every edge of H has at least one vertex in the set of internal vertices of T. Let $G=L(H)$, $a,b \in V(G)$, and let $\bar{a},\bar{b} \in E(H)$ be the edges of H that correspond to a,b. Analogously to [9] (see e.g. [14]), a line graph G of order at least 3 has a hamiltonian (a,b)-path if and only if H has an internally dominating $(\bar{a},\bar{b})$-trail.

Thomassen [17] posed the following conjecture.

Conjecture A (Thomassen [17]). *Every* 4*-connected line graph is Hamiltonian.*

Since then, many statements that are seemingly stronger or weaker than Conjecture A have been proved to be equivalent to it. Below we list some of them. The reference always refers to the paper in which the equivalence with Conjecture A was established.

Theorem B. *The following statements are equivalent with Conjecture A.*

(i) [15] *Every* 4*-connected claw-free graph is Hamiltonian.*
(ii) [5] *Every essentially* 4*-edge-connected graph has a dominating closed trail.*
(iii) [5] *Every cyclically* 4*-edge-connected cubic graph has a dominating cycle.*
(iv) [11] *Every cyclically* 4*-edge-connected cubic graph that is not* 3*-edge-colorable has a dominating cycle.*
(v) [3] *Every snark has a dominating cycle.*

Statement (iii) of Theorem B was strengthened as follows.

Theorem C. *The following statements are equivalent with Conjecture A.*

(i) [7] *Any two independent edges of a cyclically* 4*-edge-connected cubic graph are contained in a dominating cycle.*
(ii) [6] *Any two edges of a cyclically* 4*-edge-connected cubic graph are contained in a dominating cycle.*

On the positive side, the strongest known results related to Conjecture A are the following.

Theorem D.

(i) [10] *Every* 5*-connected claw-free graph G with minimum degree $\delta(G)\geq 6$ is Hamiltonian.*
(ii) [16] *Every* 6*-connected claw-free graph with at most* 29 *vertices of degree* 6 *is Hamilton-connected.*

2. MAIN RESULT

Set $E^+(G)=\{xy|\ x,y\in V(G)\}$, and for $X\subset E^+(G)$ set $G+X=(V(G),E(G)\cup X)$ (note that we admit $E(G)\cap X\neq\emptyset$). A graph G is said to be *k-edge-Hamilton-connected* if, for any $X\subset E^+(G)$ such that $|X|\leq k$ and X determines a path system, the graph $G+X$ has a hamiltonian cycle containing all edges of X (note that by a *path system* we mean a forest each component of which is a path).

The following facts are easy to observe.

Proposition 1. *Let G be a graph. Then*

(i) *G is* 1*-edge-Hamilton-connected if and only if G is Hamilton-connected,*
(ii) *G is* 2*-edge-Hamilton-connected if and only if*

  (α) *G is* 1*-Hamilton-connected, and*
  (β) *for any four distinct vertices $x_1,x_2,x_3,x_4\in V(G)$, G has a path factor consisting of two paths P_1,P_2 such that both P_1 and P_2 have one endvertex in $\{x_1,x_2\}$ and one endvertex in $\{x_3,x_4\}$,*

(iii) *if G is k-edge-Hamilton-connected, then G is $(k+2)$-connected.*

Proof. Parts (i) and (ii) follow immediately from the definitions. Let G be k-edge-Hamilton-connected and let $\{a_1,\dots,a_\ell\}\subset V(G)$, $\ell\le k+1$, be a cutset of G. Then for $X=\{a_1a_2,a_2a_3,\dots,a_{\ell-1}a_\ell\}$ the graph G has no Hamiltonian cycle containing all edges of X. This contradiction proves part (iii). ■

Our main result, Theorem 2, shows that Conjecture A is equivalent to the statement(s) that every 4-connected line graph has any of the above mentioned properties. Note that the equivalence of (i) and (ii) was originally established in the unpublished paper [13].

Theorem 2. *The following statements are equivalent.*

(i) *Every* 4*-connected line graph is Hamiltonian.*
(ii) *Every* 4*-connected line graph is Hamilton-connected.*
(iii) *Every* 4*-connected line graph is* 1*-Hamilton-connected.*
(iv) *Every* 4*-connected line graph is* 2*-edge-Hamilton-connected.*

Proof of Theorem 2 is postponed to Section 3.

We will now discuss complexity aspects of Theorem 2.

The problem to decide whether a given graph G has a hamiltonian (a,b)-path for given vertices a,b is one of the classical NP-complete problems (see [8]), and the hamiltonian problem remains NP-complete even when restricted to line graphs (see e.g. [1] for the hamiltonian path problem). The problem to decide whether G is Hamilton-connected is also known to be NP-complete [4]. The complexity of the corresponding Hamilton-connectedness problem in line graphs is not known, however, it is usually supposed to be NP-complete. We now consider the next step (we include the easy proof here since we are not aware of its being published).

1-HC
Instance: *A graph* G.
Question: *Is* G 1*-Hamilton-connected*?

Theorem 3. 1*-HC is NP-complete.*

Proof. Obviously 1-HC $\in$ NP. We transform the Hamilton-connectedness problem to 1-HC. Given a graph G, take a vertex $w\notin V(G)$ and set $G'=(V(G)\cup\{w\},E(G)\cup\{wx|\ x\in V(G)\})$. We show that G' is 1-Hamilton-connected if and only if G is Hamilton-connected. Suppose first that G is Hamilton-connected. We show that for any $x,y,u\in V(G')$, $G'-u$ has a hamiltonian (x,y)-path. Let P be a hamiltonian (x,y)-path in G. If $u\neq w$, then $P'=xPu^-wu^+Py$ is a hamiltonian (x,y)-path in $G'-u$, and for $u=w$ we simply set $P'=P$. Conversely, if G' is 1-Hamilton-connected, then $G=G'-w$ is Hamilton-connected by definition. ■

Thus, we can analogously define the following problems.

1-HCL
Instance: *A line graph* G.
Question: *Is* G 1*-Hamilton-connected*?

2-E-HCL
Instance: *A line graph* G.
Question: *Is* G 2*-edge-Hamilton-connected*?

Note that, with respect to the above mentioned facts, a common expectation would probably be that both these problems are NP-complete.

If Conjecture A is true, then, by Theorem 2, we have that every 4-connected line graph is 2-edge-Hamilton-connected (hence also 1-Hamilton-connected). Conversely, by Proposition 1(iii), every 2-edge-Hamilton-connected graph is 4-connected and, similarly, every 1-Hamilton-connected graph is 4-connected. From this we observe that if Conjecture A is true, then

(i) a line graph G is 1-Hamilton-connected if and only if G is 4-connected,
(ii) a line graph G is 2-edge-Hamilton-connected if and only if G is 4-connected.

Consequently, Conjecture A, if true, would imply polynomiality of both 1-HCL and 2-E-HCL. We thus have the following consequence.

Theorem 4. *At least one of the followings is true*:

(i) *Both* 1*-HCL and* 2*-E-HCL are polynomial.*
(ii) *Conjecture A fails.*

Remark. Note that Theorem 4 means that proving NP-completeness of 1-HCL or 2-E-HCL would imply the existence of a 4-connected non-hamiltonian line graph (and also, e.g. the existence of a snark with no dominating cycle, etc.), unless P = NP.

3. PROOF OF THEOREM 2

We first mention several results that will be needed for our proof.

Set $V_i(H)=\{x\in V(H)|d_H(x)=i\}$ and let H be a graph with $\delta(H)=2$ and $|V_2(H)|=4$. Then H is said to be $V_2(H)$*-dominated* if for any two edges $e_1=u_1v_1, e_2=u_2v_2\in E^+(H)$ with $\{u_1,v_1,u_2,v_2\}=V_2(H)$ the graph $H+\{e_1,e_2\}$ has a dominating closed trail containing e_1 and e_2, and H is said to be *strongly* $V_2(H)$*-dominated* if H is $V_2(H)$-dominated and for any $e=uv\in E^+(H)$ with $u,v\in V_2(H)$, the graph $H+\{e\}$ has a dominating closed trail containing e. Note that in the special case of a cubic graph a dominating closed trail becomes a dominating cycle.

The following was proved in [12].

Theorem E (Kužel [12])**.** *Conjecture A is equivalent to the statement that any subgraph H of an essentially* 4*-edge-connected cubic graph with $\delta(H)=2$ and $|V_2(H)|=4$ is $V_2(H)$-dominated.*

We will need the following slight strengthening of Theorem E.

Theorem 5. *Conjecture A is equivalent to the statement that any subgraph H of an essentially* 4*-edge-connected cubic graph with $\delta(H)=2$ and $|V_2(H)|=4$ is strongly $V_2(H)$-dominated.*

Proof. Suppose that Conjecture A is true, let H be a subgraph of an essentially 4-edge-connected cubic graph with $\delta(H)=2$ and $|V_2(H)|=4$, let $V_2(H)=\{a,b,c,d\}$, set $e=ab$ and suppose that $H+\{e\}$ has no dominating cycle containing e.

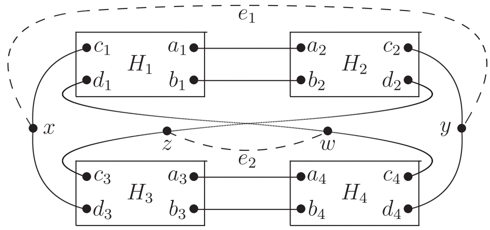

FIGURE 1. The graph F.

Let H_i, $i=1,2,3,4$ be four vertex-disjoint copies of H, denote $V_2(H_i)=\{a_i,b_i,c_i,d_i\}$, $i=1,2,3,4$, and let F' be the graph with $V(F')=\bigcup_{i=1}^4 V(H_i)$ and $E(F')=(\bigcup_{i=1}^4 E(H_i))\cup\{a_1a_2,b_1b_2,a_3a_4,b_3b_4,c_1d_3,c_2d_4,d_1c_4,d_2c_3\}$. Finally, let F be the graph obtained from F' by subdividing the following edges with new vertices: c_1d_3 with a vertex x, c_2d_4 with a vertex y, c_3d_2 with a vertex z and c_4d_1 with a vertex w, and set $e_1=xy$ and $e_2=zw$ (Fig. 1).

By Theorem E, the graph $F+\{e_1,e_2\}$ has a dominating cycle C with $e_1,e_2\in E(C)$. As $\{w,x,y,z\}$ separates $H_1\cup H_2$ from $H_3\cup H_4$, both e_1 and e_2 must be incident to edges on C to both $H_1\cup H_2$ and $H_3\cup H_4$. But no matter how we pick these edges, two of w,x,y,z are adjacent on C to some c_i, d_i, contradicting that $H_j+a_jb_j$ has no dominating cycle containing a_jb_j for $j\in\{1,2,3,4\}\cap\{3-i,7-i\}$.

Conversely, if every subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H)=2$ and $|V_2(H)|=4$ is strongly $V_2(H)$-dominated, then clearly every such H is $V_2(H)$-dominated and Conjecture A is true by Theorem E. ■

We will also need the following operation (see [5]). Let H be a graph, $z\in V(H)$ a vertex of degree $d\geq 4$, and let $u_1,u_2,\ldots,u_d$ be an ordering of neighbors of z (we allow repetition in case of parallel edges). Then the graph H_z, obtained from the disjoint union of $G-z$ and the cycle $C_z=z_1,z_2,\ldots,z_dz_1$ by adding the edges u_iz_i, $i=1,\ldots,d$, is called an *inflation of H at z*. If $\delta(H)\geq 3$, then, by successively taking an inflation at each vertex of degree greater than 3 we can obtain a cubic graph H^I, called a *cubic inflation of H*. The inflation of a graph at a vertex is not unique (since it depends on the ordering of neighbors of z) and it is possible that the operation decreases the edge-connectivity of the graph. However, the following was proved in [5].

Lemma F (Fleischner and Jackson [5])**.** *Let H be an essentially 4-edge-connected graph with minimum degree $\delta(H)\geq 3$. Then some cubic inflation of H is essentially 4-edge-connected.*

Let H' be a cubic inflation of a graph H and for any $z\in V(H)$ set $I(z)=V(C_z)$ if $d_H(z)>3$ and $I(z)=\{z\}$ otherwise. Observing that a dominating cycle in H' must contain at least one vertex in $I(z)$ for each $z\in V(H)$ with $d_H(z)\geq 4$, we immediately have the following fact (which is implicit in [5]).

Lemma G (Fleischner and Jackson [5])**.** *Let H be a graph with $\delta(H)\geq 3$ and let H^I be a cubic inflation of H. Let C be a dominating cycle in H^I. Then H has a dominating closed trail T such that*

(i) *T contains all vertices of degree at least* 4,
(ii) *if $uv\in E(C)$ and $u\in I(x)$, $v\in I(y)$ for some $x,y\in V(H)$, $x\neq y$, then $xy\in E(T)$.*

Proof of Theorem 2. It is sufficient to prove that (i) implies (iv). Thus, suppose that Conjecture A is true and let G be a minimum counterexample to the statement (iv) of Theorem 2, i.e. G is a 4-connected line graph that is not 2-edge-Hamilton-connected but every 4-connected line graph G' with $|V(G')|<|V(G)|$ is 2-edge-Hamilton-connected. Let $Y \subset E^+(G)$ be such that $|Y| \leq 2$ and $G+Y$ has no hamiltonian cycle containing all edges of Y.

If $|Y|=1$, then denote $Y=\{e_1\}$, choose an arbitrary $e_2 \in E(G)$ such that e_1, e_2 have no vertex in common, and set $X=\{e_1,e_2\}$. If $|Y|=2$, then denote $Y=\{e_1,e_2\}$ and set $X=Y$. Denote $e_1=ab$, $e_2=cd$, and choose the notation such that possibly $b=d$. With a slight abuse of notation, we will use X also for the subgraph determined by e_1, e_2. To reach a contradiction, it is sufficient to show that $G+X$ has a hamiltonian cycle containing all edges of X.

Claim 1. *None of the vertices a, b, c, d is simplicial.*

Proof of Claim 1. Suppose that $u \in \{a,b,c,d\}$ is simplicial.
Case 1: $d_X(u)=1$. Without loss of generality suppose $u=a$, and set $G'=G-u$. Then G' is a 4-connected line graph with $|V(G')|<|V(G)|$, hence G' is 2-edge-Hamilton-connected. Choose $a' \in N_G(u)$ such that $a' \notin \{b,c,d\}$ (this is always possible since $d_G(u) \geq 4$) and set $e_1'=a'b$ and $X'=\{e_1', e_2\}$. Let C' be a hamiltonian cycle in $G'+X'$ containing e_1' and e_2. Then $C=a'ae_1bC'a'$ is a hamiltonian cycle in G containing e_1 and e_2, a contradiction.
Case 2: $d_X(u)=2$. Then, by the choice of notation, $u=b=d$. Similarly as before, $G'=G-u$ is 2-edge-Hamilton-connected. Set $e'=ac$, $X'=\{e'\}$ and let C' be a hamiltonian cycle in G' containing X'. Then $C=aucC'a$ is a hamiltonian cycle in G containing X, a contradiction. ■

Let now H be a graph such that $L(H)=G$, and let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the edges corresponding to the vertices $a,b,c,d \in V(G)$, respectively. By Claim 1, none of the edges $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ is pendant.

We now distinguish two cases.
Case 1: $\{a,b\} \cap \{c,d\} = \emptyset$. We define a graph H_4 by the following construction.

- H' is a graph obtained from H by subdividing each of the edges $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ with a new vertex a', b', c', d', respectively,
- H_1 is a graph obtained from H' by adding a new vertex u and edges ua', ub', uc', ud',
- H_2 is obtained from H_1 by removing vertices of degree 1 and suppressing vertices of degree 2.

Then H_2 is essentially 4-edge-connected with minimum degree $\delta(H_2) \geq 3$ and, by Lemma F, H_2 has an essentially 4-edge-connected cubic inflation H_3. Finally, let H_4 be obtained from H_3 by removing $I(u)$ (i.e. the vertices of the cycle that corresponds to the vertex u of H_2).

Then H_4 satisfies the assumptions of Theorem 5, hence $H_4+\{a'b', c'd'\}$ has a dominating cycle containing $a'b'$ and $c'd'$.

By Lemma G, $(H_2-u)+\{a'b', c'd'\}$ has a dominating closed trail T containing the edges $a'b'$, $c'd'$ and all vertices of degree at least 4. The graph H is essentially 4-edge-connected, hence for every vertex of H of degree 1 or 2, all its neighbors are of degree at least 4. Thus, T is a dominating closed trail also in $H'+\{a'b', c'd'\}$. Since T contains

the edges $a'b'$ and $c'd'$, $G+X$ has a hamiltonian cycle containing the edges e_1 and e_2, a contradiction.
Case 2: $\{a,b\}\cap\{c,d\}\neq\emptyset$. By the choice of notation, we have $b=d$ and the vertices a,b,c are distinct. By the assumption, G is not 2-edge-Hamilton-connected, hence $G-b$ has no hamiltonian (a,c)-path, implying that $H-\bar{b}$ has no internally dominating $(\bar{a},\bar{c})$-trail.

Claim 2. *Neither $\bar{a}$ and $\bar{b}$ nor $\bar{b}$ and $\bar{c}$ share a vertex of degree* 2.

Proof of Claim 2. By symmetry, suppose that $\bar{a}$ and $\bar{b}$ share a vertex v of degree 2. Then $ab\in E(G)$. Let K denote the subgraph of G induced by $N_G(a)\setminus\{b,c\}$. Since $d_H(v)=2$, K is a clique of order at least 2.

Let H' be obtained from H by suppressing the vertex v, i.e. $\bar{a}$ and $\bar{b}$ coincide in H' into an edge $\bar{w}$. Set $G'=L(H')$. Then G' is obtained from G by contraction of the edge ab into a vertex w. Clearly, G' is 4-connected, hence, by the minimality of G, G' is 2-edge-Hamilton-connected. Let a_1 be an arbitrary vertex in K, set $e_1'=wa_1$ and $e_2'=wc$, and let C' be a hamiltonian cycle in $G'+\{e_1',e_2'\}$ containing e_1' and e_2'. Then $C=a_1abcC'a_1$ is a hamiltonian cycle in $G+X$ containing e_1 and e_2, a contradiction. ∎

Let H_1 be the graph obtained from H by removing vertices of degree 1 and suppressing vertices of degree 2. Then H_1 is essentially 4-edge-connected. Let a^*,b^*,c^* denote the edges of H_1 that correspond to the edges $\bar{a},\bar{b},\bar{c}$ of H. Note that possibly $a^*=c^*$ (if $\bar{a}$ and $\bar{c}$ share a vertex of degree 2), but, by Claim 2, $a^*\neq b^*$ and $b^*\neq c^*$.

Let H_2 be an essentially 4-edge-connected cubic inflation of H_1 and, with a slight abuse of notation, let a^*,b^*,c^* denote the edges of H_2 that correspond to these edges of H_1. Set $a^*=a_1a_2$, $b^*=b_1b_2$, $c^*=c_1c_2$.

Claim 3. *The edges a^*,b^*,c^* (and hence also the edges $\bar{a},\bar{b},\bar{c}$) do not share a vertex of degree* 3.

Proof of Claim 3. Let, to the contrary, $w=a_1=b_1=c_1$ be of degree 3. If $\bar{a}=wa_1'$ for some $a_1'\neq a_2$, then, by the construction of H_1, a_1' is of degree 2 in H and $\{a_1'a_2,b_2w,c_2w\}$ is an essential edge-cutset separating the edge $a_1'w$ from the rest of H, a contradiction. Hence $a^*=\bar{a}$ and, similarly, $b^*=\bar{b}$ and $c^*=\bar{c}$.

By Theorem C(ii), H_2 has a dominating cycle C containing a^* and c^*. Since w is of degree 3, C does not contain b^*. By Lemma G and since H is essentially 4-edge-connected, H has a dominating closed trail T containing $\bar{a}$ and $\bar{c}$ and not containing $\bar{b}$. But then T is an internally dominating $(\bar{a},\bar{c})$-trail in $H-\bar{b}$, a contradiction. ∎

By Claim 3, we either have $a^*=c^*$, or either a^*,c^* or a^*,b^* have no common vertex. Let H_3 and H_4 be the graphs obtained from H_2 as follows:

(i) if a^*,c^* have no vertex in common, then H_3 is obtained from H_2 by subdividing each of the edges a^*,c^* with a new vertex a',c', respectively, and by adding the edge $a'c'$, and H_4 is obtained from H_3 by deleting the edges $a'c'$ and b^* (but keeping the vertices a',c',b_1,b_2);
(ii) if $a^*=c^*$, then $H_3=H_2$ and H_4 is obtained from H_3 by deleting the edges a^*, b^* (but keeping the vertices a_1,a_2,b_1,b_2), and, for consistence, by relabeling $a_1:=a'$ and $a_2:=c'$;

(iii) if a^*, b^* have no vertex in common, then H_3 is obtained from H_2 by subdividing a^* and b^* with a new vertex a' and b' and adding the edge $a'b'$ and then subdividing $a'b'$ and c^* with a new vertex d' and c' and adding the edge $d'c'$, and H_4 is obtained from H_3 by deleting the vertices b' and d'.

It is an easy observation that an essentially 4-edge-connected cubic graph remains essentially 4-edge-connected if we subdivide two independent edges and connect the new vertices with a new edge. Hence, in all three cases, the graph H_3 is essentially 4-edge-connected. Since H_4 is a subgraph of H_3 with $\delta(H_4) = 2$ and $|V_2(H_4)| = 4$, H_4 satisfies the assumptions of Theorem 5. Then the graph $H_4 + \{a'c'\}$ has a dominating cycle containing the edge $a'c'$, implying that $H - \bar{b}$ has an internally dominating $(\bar{a}, \bar{c})$-trail, a contradiction. ■

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Hamilton cycles in 5-connected line graphs

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ABSTRACT

A conjecture of Carsten Thomassen states that every 4-connected line graph is hamiltonian. It is known that the conjecture is true for 7-connected line graphs. We improve this by showing that any 5-connected line graph of minimum degree at least 6 is hamiltonian. The result extends to claw-free graphs and to Hamilton-connectedness.

1. Introduction

Is there a positive constant C such that every C-connected graph is hamiltonian? Certainly not, as shown by the complete bipartite graphs $K_{n,n+1}$, where n is large. The situation may change, however, if the problem is restricted to graphs not containing a specified forbidden induced subgraph. For instance, for the class of *claw-free* graphs (those not containing an induced $K_{1,3}$), Matthews and Sumner [18] conjectured the following in 1984.

Conjecture 1 (*Matthews and Sumner*)**.** *Every 4-connected claw-free graph is hamiltonian.*

The class of claw-free graphs includes all line graphs. Thus, Conjecture 1 would in particular imply that every 4-connected line graph is hamiltonian. This was stated at about the same time as a separate conjecture by Thomassen [23].

Conjecture 2 (*Thomassen*)**.** *Every 4-connected line graph is hamiltonian.*

Although formally weaker, Conjecture 2 was shown to be equivalent to Conjecture 1 by Ryjáček [21]. Several other statements are known to be equivalent to these conjectures, including the Dominating Cycle Conjecture [5,6]; for more work related to these equivalences, see also [2,11,12].

Conjectures 1 and 2 remain open. The best general result to date in the direction of Conjecture 2 is due to Zhan [26] and Jackson (unpublished).

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Theorem 3 (*Zhan; Jackson*)**.** *Every 7-connected line graph is hamiltonian.*

In fact, the result of [26] shows that any 7-connected line graph G is *Hamilton-connected* — it contains a Hamilton path from u to v for each choice of distinct vertices u, v of G.

For 6-connected line graphs, hamiltonicity has been proved only for restricted classes of graphs [9,25]. Many papers investigate the Hamiltonian properties of other special types of line graphs; see, e.g., [15,16] and the references given therein.

The main result of the present paper is the following improvement of Theorem 3.

Theorem 4. *Every 5-connected line graph with minimum degree at least 6 is hamiltonian.*

This provides a partial result towards Conjecture 2. Furthermore, the theorem can be strengthened in two directions: it extends to claw-free graphs by a standard application of the results of [21], and it remains valid if 'hamiltonian' is replaced by 'Hamilton-connected'.

One of the ingredients of our method is an idea used (in a simpler form) in [10] to give a short proof of the characterization of graphs with k disjoint spanning trees due to Tutte [24] and Nash-Williams [19] (the 'tree-packing theorem'). It may be helpful to consult [10] as a companion to Section 5 of the present paper.

The paper is organized as follows. In Section 2, we recall the necessary preliminary definitions concerning graphs and hypergraphs. Section 3 introduces several notions related to quasigraphs, a central concept of this paper. Here, we also state our main result on quasitrees with tight complement (Theorem 5). Sections 4–7 elaborate the theory needed for the proof of this theorem, which is finally given in Section 8. Sections 9 and 10 explain why quasitrees with tight complement are important for us, by exhibiting their relation to connected eulerian subgraphs of a graph. This relation is used in Section 10 to prove the main result of this paper, which is Theorem 4 and its corollary for claw-free graphs. In Section 11, we outline a way to further strengthen this result by showing that graphs satisfying the assumptions of Theorem 4 are in fact Hamilton-connected. Closing remarks are given in Section 12.

The end of each proof is marked by □. In proofs consisting of several claims, the end of the proof of each claim is marked by △.

2. Preliminaries

All the graphs considered in this paper are finite and may contain parallel edges but no loops. The vertex set and the edge (multi)set of a graph G is denoted by $V(G)$ and $E(G)$, respectively. For background on graph theory and any terminology which is not explicitly introduced, we refer the reader to [4].

A *hypergraph* H consists of a vertex set $V(H)$ and a (multi)set $E(H)$ of subsets of $V(H)$ that are called the *hyperedges* of H. We will be dealing exclusively with *3-hypergraphs*, that is, hypergraphs each of whose hyperedges has cardinality 2 or 3. Multiple copies of the same hyperedge are allowed. Throughout this paper, any hypergraph is assumed to be a 3-hypergraph unless stated otherwise. Furthermore, the symbol H will always refer to a 3-hypergraph with vertex set V. For $k \in \{2, 3\}$, a *k-hyperedge* is a hyperedge of cardinality k.

To picture a 3-hypergraph, we will represent a vertex by a solid dot, a 2-hyperedge by a line as usual for graphs, and a 3-hyperedge e by three lines joining each vertex of e to a point which is not a solid dot (see Fig. 1).

In our argument, 3-hypergraphs are naturally obtained from graphs by replacing each vertex of degree 3 by a hyperedge consisting of its neighbours. Conversely, we may turn a 3-hypergraph H into a graph $Gr(H)$: for each 3-hyperedge e of H, we add a vertex v_e and replace e by three edges joining v_e to each vertex of e.

As in the case of graphs, the hypergraph H is *connected* if for every nonempty proper subset $X \subseteq V$, there is a hyperedge of H intersecting both X and $V - X$. If H is connected, then an *edge-cut* in H is any inclusionwise minimal set of hyperedges F such that $H - F$ is disconnected. For any integer k, the hypergraph H is *k-edge-connected* if it is connected and contains no edge-cuts of cardinality less than k. The *degree* of a vertex v is the number of hyperedges incident with v.

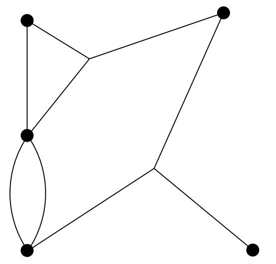


Fig. 1. A 3-hypergraph H with three 2-hyperedges and two 3-hyperedges.

To extend the notion of induced subgraph to hypergraphs, we adopt the following definition. For $X \subseteq V$, we define $H[X]$ (the *induced subhypergraph of H on X*) as the hypergraph with vertex set X and hyperedge set

$$E(H[X]) = \{e \cap X : e \in E(H) \text{ and } |e \cap X| \geq 2\}.$$

If $e \cap X = f \cap X$ for distinct hyperedges e, f, we include this hyperedge in multiple copies. Furthermore, we assume a canonical assignment of hyperedges of H to hyperedges of $H[X]$. To stress this fact, we always write the hyperedges of $H[X]$ as $e \cap X$, where $e \in E(H)$.

Let $\mathcal{P}$ be a partition of a set X. $\mathcal{P}$ is *trivial* if $\mathcal{P} = \{X\}$. A set $Y \subseteq X$ is *$\mathcal{P}$-crossing* (or: *Y crosses $\mathcal{P}$*) if it intersects at least two classes of $\mathcal{P}$.

As usual, another partition $\mathcal{R}$ of X *refines* $\mathcal{P}$ (written as $\mathcal{R} \leq \mathcal{P}$) if every class of $\mathcal{R}$ is contained in a class of $\mathcal{P}$. In this case, we also say that $\mathcal{R}$ is *finer* than $\mathcal{P}$ or that $\mathcal{P}$ is *coarser*. If $\mathcal{R} \leq \mathcal{P}$ and $\mathcal{R} \neq \mathcal{P}$, then we write $\mathcal{R} < \mathcal{P}$ and say that $\mathcal{R}$ is *strictly finer* (and $\mathcal{P}$ is *strictly coarser*). It is well known that the order $\leq$ on partitions of X is a lattice; the infimum of any two partitions $\mathcal{P}$, $\mathcal{R}$ (i.e., the unique coarsest partition that refines both $\mathcal{P}$ and $\mathcal{R}$) is denoted by $\mathcal{P} \wedge \mathcal{R}$.

If $Y \subseteq X$, then the partition *induced on Y by $\mathcal{P}$* is

$$\mathcal{P}[Y] = \{P \cap Y : P \in \mathcal{P} \text{ and } P \cap Y \neq \emptyset\}.$$

3. Quasigraphs

A basic notion in this paper is that of a quasigraph. It is a generalization of tree representations and forest representations used, e.g., in [7].

Recall from Section 2 that H is a 3-hypergraph on vertex set V. A *quasigraph* in H is a pair (H, π), where π is a function assigning to each hyperedge e of H a set $\pi(e) \subseteq e$ which is either empty or has cardinality 2. The value $\pi(e)$ is called the *representation* of e under π. Usually, the underlying hypergraph is clear from the context, and we simply speak about a quasigraph π. Quasigraphs will be denoted by lowercase Greek letters.

In this section, π will be a quasigraph in H. Considering all the nonempty sets $\pi(e)$ as graph edges, we obtain a graph π^* on V. The hyperedges e with $\pi(e) \neq \emptyset$ are said to be *used* by π. The set of all such hyperedges of H is denoted by $E(\pi)$. The edges of the graph π^*, in contrast, are denoted by $E(\pi^*)$ as expected. We emphasize that, by definition, π^* spans all the vertices in V.

To picture π, we use a bold line to connect the vertices of $\pi(e)$ for each hyperedge e used by π. An example of a quasigraph is shown in Fig. 2.

The quasigraph π is a *acyclic* (or a *quasiforest*) if π^* is a forest; π is a *quasitree* if π^* is a tree. Furthermore, we define π to be a *quasicycle* if π^* is the union of a cycle and a (possibly empty) set of isolated vertices. The hypergraph H is *acyclic* if there exists no quasicycle in H.

If e is a hyperedge of H, then $\pi - e$ is the quasigraph obtained from π by changing the value at e to $\emptyset$. The *complement* $\overline{\pi}$ of π is the spanning subhypergraph of H comprised of all the hyperedges of H not used by π. Since π includes the information about its underlying hypergraph H, it makes sense to speak about its complement without specifying H (although we sometimes do specify it for emphasis). Note that $\overline{\pi}$ is not a quasigraph.

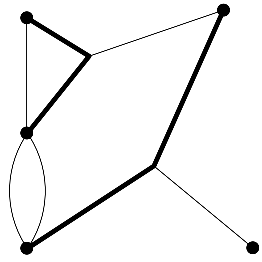
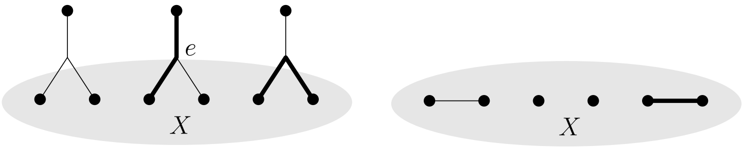


Fig. 2. A quasigraph ρ in the hypergraph of Fig. 1.



(a) Possible types of 3-hyperedges e with $|e \cap X| = 2$ with respect to the quasigraph π.

(b) The corresponding 2-hyperedges of the induced quasigraph. Note that e does not have a corresponding hyperedge.

Fig. 3. An illustration to the definition of the π-section at X.

How to define an analogue of the induced subgraph for quasigraphs? Let $X \subseteq V$. At first sight, a natural choice for the underlying hypergraph of a quasigraph induced by π on X is $H[X]$. It is clear how to define the value of the quasigraph on a hyperedge $e \cap X$, except if $|e| = 3$ and $|e \cap X| = 2$ (see Fig. 3(a)). In particular, if $\pi(e)$ intersects both X and $V - X$, then $e \cap X$ will not be used by the induced quasigraph; furthermore, it is (at least for our purposes) not desirable to include $e \cap X$ in the complement of the induced quasigraph either. This brings us to the following replacement for $H[X]$ (cf. Fig. 3(b)).

The π*-section* of H at X is the hypergraph $H[X]^\pi$ defined as follows:

- $H[X]^\pi$ has vertex set X,
- its hyperedges are the sets $e \cap X$, where e is a hyperedge of H such that $|e \cap X| \geq 2$ and $\pi(e) \subseteq X$.

The quasigraph π in H naturally determines a quasigraph $\pi[X]$ in $H[X]^\pi$, defined by

$$(\pi[X])(e \cap X) = \pi(e),$$

where $e \in E(H)$ and $e \cap X$ is any hyperedge of $H[X]^\pi$. We refer to $\pi[X]$ as the quasigraph *induced* by π on X. Let us stress that whenever we speak about the complement of $\pi[X]$, it is – in accordance with the definition – its complement in $H[X]^\pi$.

The ideal quasigraphs for our purposes in the later sections of this paper would be quasitrees with connected complement. It turns out, however, that this requirement is too strong, and that the following weaker property will suffice. The quasigraph π *has tight complement* (in H) if one of the following holds:

(a) $\overline{\pi}$ is connected, or
(b) there is a partition $V = X_1 \cup X_2$ such that for $i = 1, 2$, X_i is nonempty and $\pi[X_i]$ has tight complement (in $H[X_i]^\pi$); furthermore, there is a hyperedge $e \in E(\pi)$ such that $\pi(e) \subseteq X_1$ and $e \cap X_2 \neq \emptyset$.

The definition is illustrated in Fig. 4.

Our main result regarding quasitrees in hypergraphs is the following.

Theorem 5. *Let H be a 4-edge-connected 3-hypergraph. If no 3-hyperedge in H is included in any edge-cut of size 4, then H contains a quasitree with tight complement.*

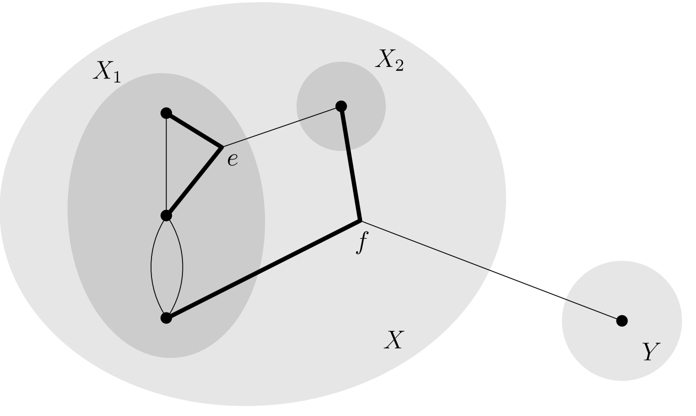


Fig. 4. The quasigraph ρ of Fig. 2 has tight complement in H. The ovals show the subsets of V relevant to the definition of tight complement. For $i = 1, 2$, $\rho[X_i]$ has connected complement in $H[X_i]^\rho$, so $\rho[X]$ has tight complement in $H[X]^\rho$ 'thanks to' the hyperedge e. Similarly, f makes the complement of ρ in H tight.

Theorem 5 will be proved in Section 8.

An equivalent definition of quasigraphs with tight complement is based on the following concept. Let us say that a partition $\mathcal{P}$ of V is π*-narrow* if for every $\mathcal{P}$-crossing hyperedge e of H, $\pi(e)$ is also $\mathcal{P}$-crossing. (We call $\mathcal{P}$ 'narrow' since none of these sets $\pi(e)$ fits into a class of $\mathcal{P}$.) For instance, the partition shown in Fig. 5(b) below is π-narrow. Observe that the trivial partition is π-narrow for any π.

Lemma 6. *A quasigraph π in H has tight complement if and only if there is no nontrivial π-narrow partition of V.*

Proof. We prove the 'only if' part by induction on the number of vertices of H. If $|V| = 1$, the assertion is trivial. Assume that $|V| > 1$ and that $\mathcal{P}$ is a nontrivial partition of V; we aim to prove that $\mathcal{P}$ is not π-narrow. Consider the two cases in the definition of tight complement. If $\overline{\pi}$ is connected (Case (a)), then there is a $\mathcal{P}$-crossing hyperedge e of $\overline{\pi}$. Since $\pi(e) = \emptyset$ is not $\mathcal{P}$-crossing, $\mathcal{P}$ is not π-narrow.

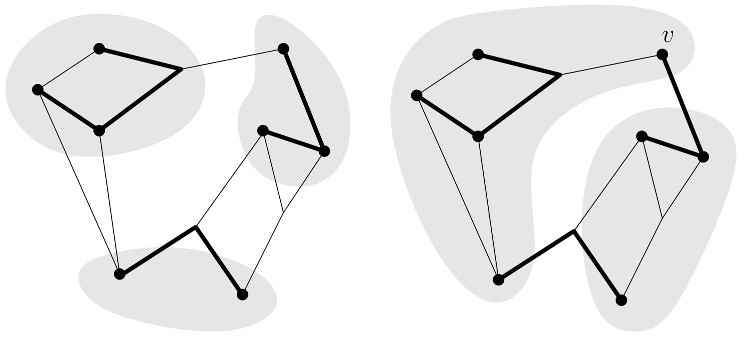
In Case (b), there is a partition $V = X_1 \cup X_2$ into nonempty sets such that each $\pi[X_i]$ has tight complement in $H[X_i]^\pi$. Suppose that $\mathcal{P}[X_1]$ is nontrivial. By the induction hypothesis, it is not $\pi[X_1]$-narrow. Consequently, there is a hyperedge f of $H[X_1]^\pi$ contained in $\pi[X_1]$ and such that $\pi(f) \subseteq P \cap X_1$, where $P \in \mathcal{P}$. It follows that $\mathcal{P}$ is not π-narrow as claimed.

By symmetry, we may assume that both $\mathcal{P}[X_1]$ and $\mathcal{P}[X_2]$ are trivial. Since $\mathcal{P}$ is nontrivial, it must be that $\mathcal{P} = \{X_1, X_2\}$. Case (b) of the definition of tight complement ensures that there is a hyperedge $e \in E(\pi)$ such that $\pi(e) \subseteq X_1$ and $e \cap X_2 \neq \emptyset$. Since e is $\mathcal{P}$-crossing and $\pi(e)$ is not, $\mathcal{P}$ is not π-narrow. This finishes the proof of the 'only if' part.

The 'if' direction will be proved by contradiction. Suppose that V admits no nontrivial π-narrow partition, but π does not have tight complement in H. Let $\mathcal{R}$ be a coarsest possible partition of V such that each $\pi[X]$, where $X \in \mathcal{R}$, has tight complement in $H[X]^\pi$. (To see that at least one partition with this property exists, consider the partition of V into singletons.) Since $\mathcal{R}$ is nontrivial by assumption, there is an $\mathcal{R}$-crossing hyperedge e of H with $\pi(e) \subseteq R_1$, where R_1 is some class of $\mathcal{R}$. Since e is $\mathcal{R}$-crossing, it intersects another class R_2 of $\mathcal{R}$. By the definition, $\pi[R_1 \cup R_2]$ has tight complement in $H[R_1 \cup R_2]^\pi$, which contradicts the maximality of $\mathcal{R}$. $\square$

4. Narrow and wide partitions

We begin this section by modifying the definition of a π-narrow partition of V. If π is a quasigraph in H, then a partition $\mathcal{P}$ of V is π*-wide* if for every hyperedge e of H, $\pi(e)$ is a subset of a class of $\mathcal{P}$. (In particular, $\pi(e)$ is not $\mathcal{P}$-crossing for any $\mathcal{P}$-crossing hyperedge e.) An example of a π-wide partition



(a) A quasigraph τ in H and the positive τ-parts of H (the grey regions).

(b) The negative τ-part of H. Note that the vertex v belongs to a larger negative τ-part, although it forms a component of $\overline{\tau}$ on its own.

Fig. 5. Positive and negative parts.

is shown in Fig. 5(a) below. Again, the trivial partition is π-wide for any π. Lemma 6 has the following easier analogue.

Lemma 7. *If π is a quasigraph in H, then π^* is connected if and only if there is no nontrivial π-wide partition of V.*

Proof. We begin with the 'only if' direction. Suppose that $\mathcal{P}$ is a nontrivial partition of V. Since π^* is a connected graph with vertex set V, there is an edge $\pi(e)$ of π^* crossing $\mathcal{P}$. This shows that $\mathcal{P}$ is not π-wide.

Conversely, suppose that π^* is disconnected, and let $\mathcal{P}$ be the partition of V whose classes are the vertex sets of components of π^*. Let e be a hyperedge of H. We claim that $\pi(e)$ is not $\mathcal{P}$-crossing. This is certainly true if $e \notin E(\pi)$. In the other case, $\pi(e)$ is an edge of π^* and both of its endvertices must be contained in the same component of π^*, which proves the claim. We conclude that $\mathcal{P}$ is a (nontrivial) π-wide partition of V. $\square$

It is interesting that both the class of π-narrow partitions and the class of π-wide partitions are closed with respect to meets in the lattice of partitions:

Observation 8. *If π is a quasigraph in H and $\mathcal{P}$ and $\mathcal{R}$ are π-narrow partitions, then $\mathcal{P} \wedge \mathcal{R}$ is π-narrow. Similarly, if $\mathcal{P}$ and $\mathcal{R}$ are π-wide, then $\mathcal{P} \wedge \mathcal{R}$ is π-wide.*

By Observation 8, for any quasigraph π in H, there is a unique finest π-narrow partition of V, which will be denoted by $\mathcal{A}_-(\pi; H)$. Similarly, there is a unique finest π-wide partition of V, denoted by $\mathcal{A}_+(\pi; H)$. If the hypergraph is clear from the context, we write just $\mathcal{A}_+(\pi)$ or $\mathcal{A}_-(\pi)$. Lemmas 6 and 7 provide us with a useful interpretation of $\mathcal{A}_+(\pi)$ and $\mathcal{A}_-(\pi)$. It is not hard to show from the latter lemma that the classes of $\mathcal{A}_+(\pi)$ are exactly the vertex sets of components of π^*. Similarly, by Lemma 6, the classes of $\mathcal{A}_-(\pi)$ are all maximal subsets X of V such that $\pi[X]$ has tight complement in $H[X]^\pi$.

We call the classes of $\mathcal{A}_+(\pi)$ the *positive* π-parts of H and the classes of $\mathcal{A}_-(\pi)$ the *negative* π-parts of H. (See Fig. 5 for an illustration.) The terms 'positive' and 'negative' are chosen with regard to the terminology of photography, with 'positive' used for π and 'negative' for its complement, in accordance with the above discussion.

We note the following simple corollary of Lemma 6.

Lemma 9. *Let π be a quasigraph in H. For $i = 1, 2$, let $X_i \subseteq V$ be such that $\pi[X_i]$ has tight complement in $H[X_i]^\pi$. Then the following holds:*

(i) *each X_i is contained in a class of $\mathcal{A}_-(\pi)$ (as a subset), and*
(ii) *if H contains a hyperedge e such that e intersects each X_i and $\pi(e) \subseteq X_1$ (we allow $e \notin E(\pi)$), then $X_1 \cup X_2$ is contained in a class of $\mathcal{A}_-(\pi)$.*

Proof. (i) Clearly, if $\mathcal{P}$ is a π-narrow partition of V, then $\mathcal{P}[X_1]$ is $\pi[X_1]$-narrow; it follows that $\mathcal{A}_-(\pi)[X_1] \geq \mathcal{A}_-(\pi[X_1])$. By Lemma 6, $\mathcal{A}_-(\pi[X_1])$ is trivial. Hence $\mathcal{A}_-(\pi)[X_1]$ is also trivial. A symmetric argument works for X_2.

(ii) It suffices to prove that $\pi[X_1 \cup X_2]$ has tight complement in $H[X_1 \cup X_2]^\pi$. If not, let $\mathcal{P}$ be a nontrivial $\pi[X_1 \cup X_2]$-narrow partition of $X_1 \cup X_2$. By the assumption, each $\mathcal{P}[X_i]$ has to be trivial as it is $\pi[X_i]$-narrow. Thus, $\mathcal{P} = \{X_1, X_2\}$. However, since $\pi(e) \subseteq X_1$, this is not a $\pi[X_1 \cup X_2]$-narrow partition — a contradiction. □

We use the partitions $\mathcal{A}_+(\pi)$ and $\mathcal{A}_-(\pi)$ to introduce an order on quasigraphs. If π and σ are quasigraphs in H, then we write

$$\pi \trianglelefteq \sigma \quad \text{if } \mathcal{A}_+(\pi) \leq \mathcal{A}_+(\sigma) \text{ and } \mathcal{A}_-(\pi) \leq \mathcal{A}_-(\sigma).$$

Clearly, $\trianglelefteq$ is a partial order.

For a set $X \subseteq V$, let us say that two quasigraphs π and σ in H are *X-similar* if the following holds for every hyperedge e of H:

(1) $\pi(e) \subseteq X$ if and only if $\sigma(e) \subseteq X$, and
(2) if $\pi(e) \not\subseteq X$, then $\pi(e) = \sigma(e)$.

Let us collect several easy observations about X-similar quasigraphs.

Observation 10. *If $X \subseteq V$ and quasigraphs π and σ are X-similar, then the following holds:*

(i) $H[X]^\pi = H[X]^\sigma$,
(ii) *if $X \in \mathcal{A}_+(\pi)$, then $\mathcal{A}_+(\sigma) \leq \mathcal{A}_+(\pi)$,*
(iii) *if $X \in \mathcal{A}_-(\pi)$, then $\mathcal{A}_-(\sigma) \leq \mathcal{A}_-(\pi)$.*

The following lemma is an important tool which facilitates the use of induction in our argument.

Lemma 11. *Let $X \subseteq V$ and let π and σ be X-similar quasigraphs in H. Then the following holds:*

$$\text{if } \pi[X] \trianglelefteq \sigma[X], \quad \text{then } \pi \trianglelefteq \sigma.$$

Proof. Note that by Observation 10(i), $H[X]^\pi = H[X]^\sigma$. We need to prove that

$$\text{if } \mathcal{A}_-(\pi[X]) \leq \mathcal{A}_-(\sigma[X]), \quad \text{then } \mathcal{A}_-(\pi) \leq \mathcal{A}_-(\sigma), \tag{1}$$

and an analogous assertion (1^+) with all occurrences of '−' replaced by '+'.

We prove (1). By the definition of $\mathcal{A}_-(\sigma)$, (1) is equivalent to the statement that

if every $\sigma[X]$-narrow partition of X is $\pi[X]$-narrow (in $H[X]^\pi$),
then every σ-narrow partition of V is π-narrow (in H).

Assume thus that every $\sigma[X]$-narrow partition is $\pi[X]$-narrow and that $\mathcal{P}$ is a σ-narrow partition of V. For contradiction, suppose that $\mathcal{P}$ is not π-narrow.

We claim that $\mathcal{P}[X]$ is $\sigma[X]$-narrow in $H[X]^\sigma$. Let $e \cap X$ be a $\mathcal{P}[X]$-crossing hyperedge of $H[X]^\sigma$ (where $e \in E(H)$). Then e is $\mathcal{P}$-crossing, and since $\mathcal{P}$ is σ-narrow, $\sigma(e)$ is $\mathcal{P}$-crossing. By the definition of $H[X]^\sigma$, $\sigma(e) \subseteq X$ and thus $\sigma(e) = \sigma[X](e \cap X)$ is $\mathcal{P}[X]$-crossing. This proves the claim.

Since every $\sigma[X]$-narrow partition of X is assumed to be $\pi[X]$-narrow, $\mathcal{P}[X]$ is $\pi[X]$-narrow.

On the other hand, $\mathcal{P}$ is not π-narrow, so there is a $\mathcal{P}$-crossing hyperedge f of H such that $\pi(f)$ is not $\mathcal{P}$-crossing. However, $\sigma(f)$ is $\mathcal{P}$-crossing as $\mathcal{P}$ is σ-narrow. Thus, $\pi(f) \neq \sigma(f)$, and since π and σ are X-similar, both $\pi(f)$ and $\sigma(f)$ are subsets of X. It follows that $\sigma(f)$, and therefore also the hyperedge $f \cap X$ of $H[X]^\sigma = H[X]^\pi$, is $\mathcal{P}[X]$-crossing. We have seen that $\mathcal{P}[X]$ is $\pi[X]$-narrow, and this observation implies that $\pi(f)$ is $\mathcal{P}[X]$-crossing and therefore $\mathcal{P}$-crossing. This contradicts the choice of f.

The proof of (1^+) is similar to the above but simpler. The details are omitted. □

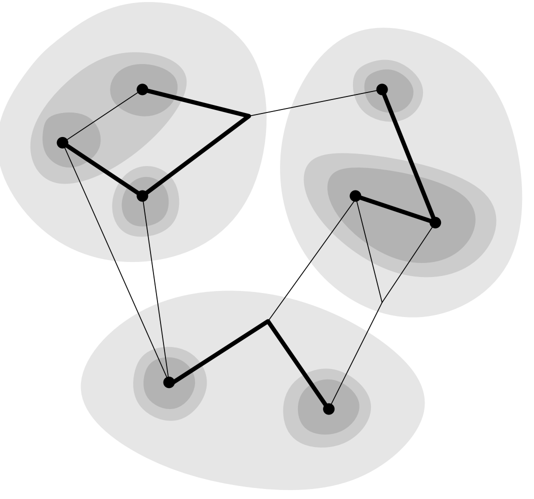


Fig. 6. The partition sequence of the quasigraph τ from Fig. 5. Partitions $\mathcal{P}_0^\tau$, $\mathcal{P}_1^\tau$ and $\mathcal{P}_2^\tau$ are shown in different grey shades from light to dark. Note that the classes of $\mathcal{P}_2^\tau$ are τ-solid.

5. Partition sequences

Besides the order $\trianglelefteq$ introduced in Section 4, we will need another derived order $\preceq$ on quasigraphs, one that is used in the basic optimization strategy in our proof. Let π be a quasigraph in H. Similarly as in [10], we associate with π a sequence of partitions of V, where each partition is a refinement of the preceding one. Since H is finite, the partitions 'converge' to a limit partition whose classes have a certain favourable property.

Recall from Section 4 that there is a uniquely defined partition of V into positive π-parts; we will let this partition be denoted by $\mathcal{P}_0^\pi$. The *partition sequence* of π is the sequence

$$\mathbb{P}^\pi = (\mathcal{P}_0^\pi, \mathcal{P}_1^\pi, \dots),$$

where for even (odd) $i \geq 1$, $\mathcal{P}_i^\pi$ is obtained as the union of partitions of X into positive (negative, respectively) $\pi[X]$-parts of $H[X]^\pi$ as X ranges over classes of $\mathcal{P}_{i-1}^\pi$. (See Fig. 6.) Thus, for instance, for even $i \geq 2$ we can formally write

$$\mathcal{P}_i^\pi = \bigcup_{X \in \mathcal{P}_{i-1}^\pi} \mathcal{A}_+(\pi[X]).$$

Since H is finite, we have $\mathcal{P}_k^\pi = \mathcal{P}_{k+2}^\pi$ for large enough k, and we set $\mathcal{P}_\infty^\pi = \mathcal{P}_k^\pi$.

Let us call a set $X \subseteq V$ *π-solid* (in H) if $\pi[X]$ is a quasitree with tight complement in $H[X]^\pi$. By the construction, any class of $\mathcal{P}_\infty^\pi$ is π-solid.

Let us define a lexicographic order on sequences of partitions: if $(\mathcal{A}_0, \mathcal{A}_1, \dots)$ and $(\mathcal{B}_0, \mathcal{B}_1, \dots)$ are sequences of partitions of V, write

$$(\mathcal{A}_0, \mathcal{A}_1, \dots) \preceq_L (\mathcal{B}_0, \mathcal{B}_1, \dots)$$

if there exists some i such that for $j < i$, $\mathcal{A}_j = \mathcal{B}_j$, while $\mathcal{A}_i$ strictly refines $\mathcal{B}_i$.

We can now define the order $\preceq$ on quasigraphs as promised. Let π and σ be quasigraphs in H. Define

$$\pi \preceq \sigma \quad \text{if } \pi \trianglelefteq \sigma \text{ and } \mathbb{P}^\pi \preceq_L \mathbb{P}^\sigma.$$

If $\pi \preceq \sigma$ but $\sigma \not\preceq \pi$, we write $\pi \prec \sigma$.

From Lemma 11, we can deduce a similar observation regarding the order $\preceq$ (in which the implication is actually replaced by equivalence).

Lemma 12. *Let $X \subseteq V$ and assume that either X is a positive π-part of H, or $\mathcal{P}_0^\pi$ is trivial and X is a negative π-part of H. Let π and σ be X-similar quasigraphs in H. Then the following holds:*

$$\pi[X] \preceq \sigma[X] \quad \text{if and only if } \pi \preceq \sigma.$$

Proof. We consider two cases depending on whether X is a positive or negative π-part of H.

Case 1: X is a positive π-part of H.

Since π and σ are X-similar, we have

$$\begin{aligned} \mathbb{P}^\pi &= (\mathcal{P}_0^\pi, \mathcal{P}_1^{\pi[X]} \cup \mathcal{P}_1^\pi[V-X], \mathcal{P}_2^{\pi[X]} \cup \mathcal{P}_2^\pi[V-X], \dots) \quad \text{and} \\ \mathbb{P}^\sigma &= (\mathcal{P}_0^\sigma, \mathcal{P}_1^{\sigma[X]} \cup \mathcal{P}_1^\pi[V-X], \mathcal{P}_2^{\sigma[X]} \cup \mathcal{P}_2^\pi[V-X], \dots). \end{aligned} \tag{2}$$

Assume first that $\pi[X] \preceq \sigma[X]$. Eqs. (2) imply that for each $i \geq 1$, $\mathcal{P}_i^\pi \leq \mathcal{P}_i^\sigma$. Furthermore, $\pi[X] \trianglelefteq \sigma[X]$ and Lemma 11 imply that $\pi \trianglelefteq \sigma$. In particular,

$$\mathcal{P}_0^\pi = \mathcal{A}_+(\pi) \leq \mathcal{A}_+(\sigma) = \mathcal{P}_0^\sigma$$

so $\mathbb{P}^\pi \preceq_L \mathbb{P}^\sigma$ and therefore also $\pi \preceq \sigma$.

Conversely, assume that $\pi \preceq \sigma$. The fact that $\mathbb{P}^\pi \preceq_L \mathbb{P}^\sigma$ together with (2) implies that for $i \geq 1$, $\mathcal{P}_i^{\pi[X]} \leq \mathcal{P}_i^{\sigma[X]}$. Recall that X is a positive π-part of H. We claim that X is also a positive σ-part of H; indeed, this follows from the fact that $\mathcal{P}_0^\pi \leq \mathcal{P}_0^\sigma$ and that π and σ are X-similar. This claim implies

$$\mathcal{P}_0^{\pi[X]} = X = \mathcal{P}_0^{\sigma[X]} \tag{3}$$

and, consequently, $\mathbb{P}^{\pi[X]} \preceq_L \mathbb{P}^{\sigma[X]}$. It remains to verify that $\pi[X] \trianglelefteq \sigma[X]$. This follows from (3) and the observation that $\mathcal{P}_1^{\pi[X]} \leq \mathcal{P}_1^{\sigma[X]}$. (Here we use the fact that if $\mathcal{P}_0^\pi$ is trivial, then $\mathcal{P}_1^\pi = \mathcal{A}_-(\pi)$).

Case 2: $\mathcal{P}_0^\pi$ is trivial and X is a negative π-part of H.

In this case, Eqs. (2) are replaced by

$$\begin{aligned} \mathbb{P}^\pi &= \Big(\{V\}, \mathcal{A}_-(\pi[X]) \cup \mathcal{P}_1^\pi[V-X], \\ &\quad \mathcal{P}_0^{\pi[X]} \cup \mathcal{P}_2^\pi[V-X], \mathcal{P}_1^{\pi[X]} \cup \mathcal{P}_3^\pi[V-X], \dots\Big) \quad \text{and} \\ \mathbb{P}^\sigma &= \Big(\{V\}, \mathcal{A}_-(\sigma[X]) \cup \mathcal{P}_1^\pi[V-X], \\ &\quad \mathcal{P}_0^{\sigma[X]} \cup \mathcal{P}_2^\pi[V-X], \mathcal{P}_1^{\sigma[X]} \cup \mathcal{P}_3^\pi[V-X], \dots\Big). \end{aligned} \tag{4}$$

Assume first that $\pi \preceq \sigma$. Since X is a positive π-part of H, the partition $\mathcal{A}_-(\pi[X])$ appearing in the second term of $\mathbb{P}^\pi$ is trivial. A similar observation holds for σ in place of π. Hence, $\mathbb{P}^\pi$ and $\mathbb{P}^\sigma$ are equal in their first two terms and (4) directly implies that $\mathbb{P}^{\pi[X]} \preceq_L \mathbb{P}^{\sigma[X]}$. Moreover, $\pi[X] \trianglelefteq \sigma[X]$ is implied by (4) as well. We conclude that $\pi[X] \preceq \sigma[X]$.

The converse implication follows from (4) without any further effort. The proof is complete. □

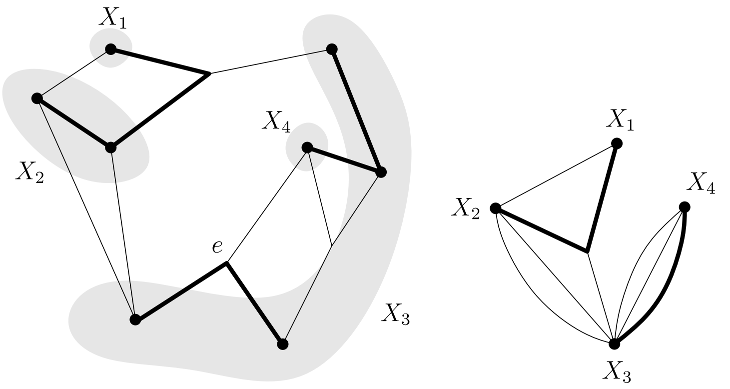
Corollary 13. *Let π and σ be X-similar quasigraphs in H, where $X \in \mathcal{P}_i^\pi$ for some i. Then the following holds:*

$$\pi[X] \preceq \sigma[X] \quad \text{if and only if} \quad \pi \preceq \sigma.$$

Proof. Follows from Lemma 12 by easy induction. □

We conclude this section by a lemma that suggests a relation between $\preceq$-maximal and acyclic quasigraphs. If π and σ are quasigraphs in H, then let us call σ a *restriction* of π if for every hyperedge e of H, $\sigma(e)$ equals either $\pi(e)$ or $\emptyset$.

Lemma 14. *Let π be a quasigraph in H and $i \geq 0$. If $\pi[X]$ is acyclic for each $X \in \mathcal{P}_i^\pi$, but π itself is not acyclic, then there exists an acyclic restriction σ of π such that $\sigma \succ \pi$.*



(a) A quasigraph π and a partition $\mathcal{P}$ of V.

(b) The contracted quasigraph $\pi/\mathcal{P}$ in $H/\mathcal{P}$. Observe that although e is used by π, $e/\mathcal{P}$ is a hyperedge of $\overline{\pi/\mathcal{P}}$.

Fig. 7. An example of contraction.

Proof. Suppose that γ is a quasicycle in H such that $E(\gamma) \subseteq E(\pi)$. By the assumption, not all of the edges of γ^* are contained in the same class of $\mathcal{P}_i^\pi$; in other words, γ^* contains a $\mathcal{P}_i^\pi$-crossing edge. Let $k \geq 0$ be the least integer such that γ^* contains a $\mathcal{P}_k^\pi$-crossing edge $\gamma(e)$ (where $e \in E(H)$).

Since $\mathcal{P}_0^\pi$ is a partition of V into positive π-parts and γ is a restriction of π, there are no $\mathcal{P}_0^\pi$-crossing edges in γ^*. Thus, $k \geq 1$. Similarly, if $j \geq 2$ is even and $X \in \mathcal{P}_{j-1}^\pi$, then $H[X]^\pi$ contains no $\mathcal{P}_j^\pi[X]$-crossing edges. It follows that k is odd. Let Y be the class of $\mathcal{P}_{k-1}^\pi$ containing all edges of γ^* as subsets.

Set $\rho = \pi - e$. Observe that $(\rho[Y])^*$ is a connected graph spanning Y, since $(\pi[Y])^*$ has this property, and the removal of the edge $\pi(e)$ cannot disconnect $(\pi[Y])^*$ as $\pi(e)$ is contained in a cycle in π^*. Thus, $\mathcal{P}_0^\rho = \{Y\}$.

Assume that $\pi(e) = z_1z_2$ and let Z_i ($i = 1, 2$) be the class of $\mathcal{P}_k^\pi$ containing z_i. Since each Z_i is a class of $\mathcal{A}_-(\pi[Y])$, $\rho[Z_i]$ has tight complement in $H[Z_i]^\rho$. Now the hyperedge $e \cap Y$ containing z_1 and z_2 is not used by ρ. By Lemma 9(ii), $Z_1 \cup Z_2$ is contained in a class of $\mathcal{A}_-(\rho[Y])$. Consequently,

$$\mathcal{A}_-(\rho[Y]) > \mathcal{A}_-(\pi[Y])$$

and therefore $\rho[Y] \succ \pi[Y]$. By Corollary 13, $\rho \succ \pi$.

If ρ is not acyclic, we repeat the previous step. Since H is finite, we will arrive at an acyclic restriction $\sigma \succ \pi$ of π after finitely many steps. □

6. Contraction and substitution

In this section, we introduce two concepts related to partitions: contraction and substitution.

Let $\mathcal{P}$ be a partition of V. The *contraction* of $\mathcal{P}$ is the operation whose result is the hypergraph $H/\mathcal{P}$ defined as follows. For $A \subseteq V$, define $A/\mathcal{P}$ as the subset of $\mathcal{P}$ consisting of all the classes $P \in \mathcal{P}$ such that $A \cap P \neq \emptyset$. The hypergraph $H/\mathcal{P}$ has vertex set $\mathcal{P}$ and it hyperedges are all the sets of the form $e/\mathcal{P}$, where e ranges over all $\mathcal{P}$-crossing hyperedges. Thus, $H/\mathcal{P}$ is a 3-hypergraph, possibly with multiple hyperedges. As in the case of induced subhypergraphs, each hyperedge f of $H/\mathcal{P}$ is understood to have an assigned corresponding hyperedge e of H such that $f = e/\mathcal{P}$.

If π is a quasigraph in H, we define $\pi/\mathcal{P}$ as the quasigraph in $H/\mathcal{P}$ consisting of the hyperedges $e/\mathcal{P}$ such that $\pi(e)$ is $\mathcal{P}$-crossing; the representation is defined by

$$(\pi/\mathcal{P})(e/\mathcal{P}) = \pi(e)/\mathcal{P}.$$

(Contraction is illustrated in Fig. 7.) In keeping with our notation, the complement of $\pi/\mathcal{P}$ in $H/\mathcal{P}$ is denoted by $\overline{\pi/\mathcal{P}}$. Observe that if $e \in E(H)$, then $e/\mathcal{P}$ is an edge of $\overline{\pi/\mathcal{P}}$ if and only if e is $\mathcal{P}$-crossing and $\pi(e)$ is not. The following lemma will be useful.

Lemma 15. *Let $\mathcal{R} \leq \mathcal{P}$ be partitions of V and π be a quasigraph in H. If $\gamma/\mathcal{R}$ is a quasicycle in $\overline{\pi/\mathcal{R}}$, then one of the following holds:*

(a) *for some $X \in \mathcal{P}$, $\gamma[X]/\mathcal{R}[X]$ is a quasicycle in the complement of $\pi[X]/\mathcal{R}[X]$ in $H[X]^{\pi}/\mathcal{R}[X]$,*
(b) *$\gamma/\mathcal{P}$ is a nonempty quasigraph in $\overline{\pi/\mathcal{P}}$ such that $(\gamma/\mathcal{P})^*$ is an eulerian graph (a graph with all vertex degrees even).*

Proof. We will use two formal equalities whose proof is left to the kind reader as a slightly tedious exercise: for $X \in \mathcal{P}$ and any quasigraph σ in H,

$$\sigma[X]/\mathcal{R}[X] = (\sigma/\mathcal{R})[\mathcal{R}[X]], \tag{5}$$

$$H[X]^{\pi}/\mathcal{R}[X] = (H/\mathcal{R})[\mathcal{R}[X]]^{\pi/\mathcal{R}}. \tag{6}$$

Let $\gamma/\mathcal{R}$ be a quasicycle in $\overline{\pi/\mathcal{R}}$. Suppose that there is $X \in \mathcal{P}$ such that every edge of $(\gamma/\mathcal{R})^*$ is a subset of $\mathcal{R}[X]$. Let $\tilde{\gamma} = (\gamma/\mathcal{R})[\mathcal{R}[X]]$. Thus, $\tilde{\gamma}$ is a quasicycle in $(H/\mathcal{R})[\mathcal{R}[X]]$ and $E(\tilde{\gamma})$ is disjoint from $E((\pi/\mathcal{R})[\mathcal{R}[X]])$. We infer that $\tilde{\gamma}$ is a quasigraph in $(H/\mathcal{R})[\mathcal{R}[X]]^{\pi/\mathcal{R}}$. Using (6), we find that $\tilde{\gamma}$ is a quasigraph in $H[X]^{\pi}/\mathcal{R}[X]$. Finally, we use (5) twice (for γ and π) and conclude that condition (a) holds.

Thus, we may assume that the endvertices Y_1, Y_2 of some edge $\gamma(e)$ of $(\gamma/\mathcal{R})^*$ are classes of $\mathcal{R}$ contained in different classes of $\mathcal{P}$ (say, X_1 and X_2, respectively). Thus, $\gamma/\mathcal{P}$ is a nonempty quasigraph in $H/\mathcal{P}$. Furthermore, $E(\gamma/\mathcal{P})$ is clearly disjoint from $E(\pi/\mathcal{P})$. To verify (b), it remains to prove that $(\gamma/\mathcal{P})^*$ is eulerian. This is immediate from the fact that $(\gamma/\mathcal{P})^*$ can be obtained from the graph $(\gamma/\mathcal{R})^*$ (which consists of a cycle and isolated vertices) by identifying certain sets of vertices (namely those contained in the same class of $\mathcal{P}$). $\square$

If $X \subseteq V$ and σ is a quasigraph in $H[X]^{\pi}$, we define the *substitution* of σ into π as the operation which produces the following quasigraph $\pi|\sigma$ in H:

$$(\pi|\sigma)(e) = \begin{cases} \pi(e) & \text{if } e \cap X \notin E(H[X]^{\pi}), \\ \sigma(e \cap X) & \text{otherwise.} \end{cases}$$

This yields a well-defined represented subhypergraph of H (see Fig. 8). More generally, let $\mathcal{P}$ be a family of disjoint subsets of V and for each $X \in \mathcal{P}$, let σ_X be a quasigraph in $H[X]^{\pi}$. Assume we substitute each σ_X into π in any order. For distinct $X \in \mathcal{P}$, the hyperedge sets of the hypergraphs $H[X]^{\pi}$ are pairwise disjoint, since $e \in E(H[X]^{\pi})$ only if $|e \cap X| \geq 2$. It follows easily that the resulting hypergraph σ in H is independent of the chosen order. This hypergraph will be denoted by

$$\sigma = \pi | \{\sigma_X : X \in \mathcal{P}\}.$$

Substitution behaves well with respect to taking induced quasigraphs and contraction.

Lemma 16. *Let π be a quasigraph in H and $\mathcal{P}$ a partition of V. Suppose that for each $X \in \mathcal{P}$, σ_X is a quasigraph in $H[X]^{\pi}$, and define*

$$\sigma = \pi | \{\sigma_X : X \in \mathcal{P}\}.$$

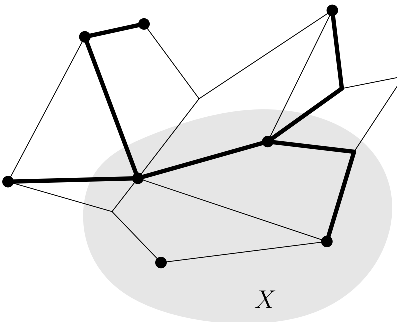
Then the following holds for every $Y \subseteq X \in \mathcal{P}$:

(i) $H[Y]^{\sigma} = (H[X]^{\pi})[Y]^{\sigma_X}$,
(ii) $\sigma[Y] = \sigma_X[Y]$.

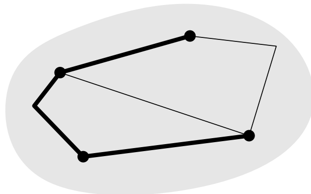
Furthermore,

(iii) $\sigma/\mathcal{P} = \pi/\mathcal{P}$.

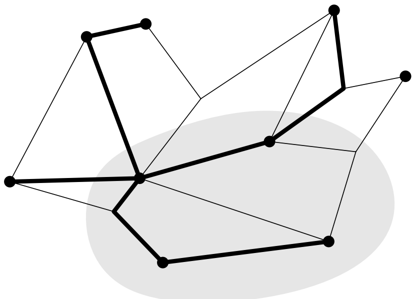
Proof. (i) Using the definition of $H[Y]^{\sigma}$ and the definition of substitution, it is not hard to verify that $e_0 \subseteq V$ is a hyperedge of $H[Y]^{\sigma}$ if and only if $e_0 = e \cap Y$, where e is a hyperedge of H such that $|e \cap Y| \geq 2$, $\pi(e) \subseteq X$ and $\sigma_X(e \cap X) \subseteq Y$. If we expand the right hand side of the equality in (i) according to these definitions, we arrive at precisely the same set of conditions.



(a) A quasigraph π in H and a set $X \subseteq V$.



(b) A quasigraph σ in $H[X]^\pi$.



(c) The quasigraph $\pi|\sigma$.

Fig. 8. An example of substitution.

(ii) Both sides of the equation are quasigraphs in $H[Y]^\sigma$. We will check that they assign the same value to a hyperedge $e \cap Y$ of $H[Y]^\sigma$. For such hyperedges, we have

$$\sigma[Y](e \cap Y) = \sigma(e) = \sigma_X(e \cap X) \tag{7}$$

where the second equality follows from the definition of substitution. On the other hand, by part (i), $e \cap Y$ is a hyperedge of $(H[X]^\pi)[Y]^{\sigma_X}$, and thus

$$\sigma_X[Y](e \cap Y) = \sigma_X(e \cap X). \tag{8}$$

The assertion follows by comparing (7) and (8).

(iii) Both $\sigma/\mathcal{P}$ and $\pi/\mathcal{P}$ are quasigraphs in $H/\mathcal{P}$. Let $e/\mathcal{P}$ be a hyperedge of $H/\mathcal{P}$, where $e \in E(H)$. Using the definitions of substitution and contraction, one can check that

$$(\sigma/\mathcal{P})(e/\mathcal{P}) = \begin{cases} \pi(e)/\mathcal{P} & \text{if } e \cap X \notin E(H[X]^\pi) \text{ and } \pi(e) \text{ is } \mathcal{P}\text{-crossing,} \\ \sigma_X(e)/\mathcal{P} & \text{if } e \cap X \in E(H[X]^\pi) \text{ and } \sigma_X(e) \text{ is } \mathcal{P}\text{-crossing,} \\ \emptyset & \text{otherwise.} \end{cases}$$

However, the middle case can never occur since $\sigma_X(e) \subseteq X$ and $\sigma_X(e)$ is therefore not $\mathcal{P}$-crossing. It follows easily that $(\sigma/\mathcal{P})(e/\mathcal{P}) = (\pi/\mathcal{P})(e/\mathcal{P})$. $\square$

7. The Skeletal Lemma

In this section, we prove a lemma which is a crucial piece of our method. It leads directly to an inductive argument for the existence of a quasitree with tight complement under suitable assumptions, which will be given in Section 8.

If π is a quasigraph in H, then a partition $\mathcal{P}$ of V is said to be π-*skeletal* if every $X \in \mathcal{P}$ is π-solid and the complement of $\pi/\mathcal{P}$ in $H/\mathcal{P}$ is acyclic.

Lemma 17 (*Skeletal Lemma*)**.** *Let π be an acyclic quasigraph in H. Then there is an acyclic quasigraph σ in H such that $\sigma \succeq \pi$ and σ satisfies one of the following:*

(a) $\sigma \succ \pi$, *or*
(b) *there is a σ-skeletal partition $\mathcal{S}$.*

Proof. We proceed by contradiction. Let the pair (π, H) be a counterexample such that H has minimal number of vertices; thus, no acyclic quasigraph $\sigma \succeq \pi$ in H satisfies any of (a) and (b). Note that π is not a quasitree with tight complement (which includes the case $|V| = 1$), for otherwise $\sigma = \pi$ would satisfy condition (b) with $\mathcal{S} = \{V\}$.

Claim 1. *$\mathcal{P}_0^\pi$ is nontrivial.*

Suppose the contrary and note that $\mathcal{P} := \mathcal{A}_-(\pi)$ is nontrivial. Consider a set $Y \in \mathcal{P}$ and the acyclic quasigraph $\pi[Y]$. By the minimality of H, there is a quasigraph $\sigma_Y \succeq \pi[Y]$ in $H[Y]^\pi$ satisfying condition (a) or (b) (with respect to $\pi[Y]$ and $H[Y]^\pi$). Define

$$\sigma = \pi \,|\, \{\sigma_Y : Y \in \mathcal{P}\}.$$

By Lemmas 14 and 16(ii), we may assume that σ is acyclic.

Assume first that for some $Y \in \mathcal{P}$, $\sigma_Y \succ \pi[Y]$ (case (a) of the lemma). Since $\sigma[Y] = \sigma_Y$ (Lemma 16(ii)), Lemma 12 implies that $\sigma \succ \pi$, a contradiction with the choice of π.

We conclude that case (b) holds for each $Y \in \mathcal{P}$, namely that there exists a partition $\mathcal{S}_Y$ which is σ_Y-skeletal in $H[Y]^\pi$. Set

$$\mathcal{S} = \bigcup_{Y \in \mathcal{P}} \mathcal{S}_Y.$$

We claim that $\mathcal{S}$ is σ-skeletal. Let $Z \in \mathcal{S}$ and assume that $Z \subseteq Y \in \mathcal{P}$. Since Z is σ_Y-solid, and since $\sigma[Z] = \sigma_Y[Z]$ and $H[Y]^\sigma = (H[Y]^\pi)[Z]^{\sigma_Y}$ by Lemma 16(i)–(ii), Z is σ-solid.

Suppose that $\overline{\sigma/\mathcal{S}}$ is not acyclic and choose a quasigraph γ in H such that $\gamma/\mathcal{S}$ is a quasicycle in $\overline{\sigma/\mathcal{S}}$. By Lemma 15, $\gamma/\mathcal{P}$ is a nonempty quasigraph in the complement $\overline{\pi/\mathcal{P}}$ of $\pi/\mathcal{P}$ in $H/\mathcal{P}$. However, by the definition of $\mathcal{A}_-(\pi)$, every $\mathcal{P}$-crossing hyperedge of H belongs to $\pi/\mathcal{P}$ and thus cannot be used by $\gamma/\mathcal{P}$, a contradiction. It follows that $\overline{\sigma/\mathcal{S}}$ is indeed acyclic and $\mathcal{S}$ is σ-skeletal. This contradiction with the choice of π concludes the proof of the claim. $\triangle$

For each $X \in \mathcal{P}_0^\pi$, $H[X]^\pi$ has fewer vertices than H. By the minimality of H, there is an acyclic quasigraph $\rho_X \succeq \pi[X]$ in $H[X]^\pi$. Define

$$\rho = \pi \,|\, \left\{\rho_X : X \in \mathcal{P}_0^\pi\right\}.$$

By Lemma 12, $\rho \succeq \pi$. Note that since $\mathcal{P}_0^\pi$ is π-wide, ρ^* is the disjoint union of the graphs ρ_X^* ($X \in \mathcal{P}_0^\pi$). Therefore, ρ is acyclic.

If $\rho_X \succ \pi[X]$ for some $X \in \mathcal{P}_0^\pi$, then by Lemmas 16(ii) and 12, $\rho \succ \pi$ and we have a contradiction. Consequently, for each $X \in \mathcal{P}_0^\pi$, there is a ρ_X-skeletal partition $\mathcal{R}_X$ (with respect to the hypergraph $H[X]^\pi$). We define a partition $\mathcal{R}$ of V by

$$\mathcal{R} = \bigcup_{X \in \mathcal{P}_0^\pi} \mathcal{R}_X. \tag{9}$$

Similarly as in the proof of Claim 1, each $Y \in \mathcal{R}$ is easily shown to be ρ-solid. An important difference in the present situation, however, is that $\mathcal{R}$ may not be ρ-skeletal as there may be quasicycles in $\overline{\rho/\mathcal{R}}$. Any such quasicycle γ' can be represented by a quasigraph γ in H such that $\gamma' = \gamma/\mathcal{R}$.

Thus, let γ be a quasigraph in H such that $\gamma/\mathcal{R}$ is a quasicycle in $\overline{\rho/\mathcal{R}}$. By Lemma 15, there are two possibilities:

(a) for some $X \in \mathcal{P}_0^\pi$, $\gamma[X]/\mathcal{R}_X$ is a quasicycle in the complement of $\rho[X]/\mathcal{R}_X$ in $H[X]^\rho/\mathcal{R}_X$, or
(b) $\gamma/\mathcal{P}_0^\pi$ is a nonempty quasigraph in the complement of $\rho/\mathcal{P}_0^\pi$ in $H/\mathcal{P}$ such that $(\gamma/\mathcal{P}_0^\pi)^*$ is an eulerian graph.

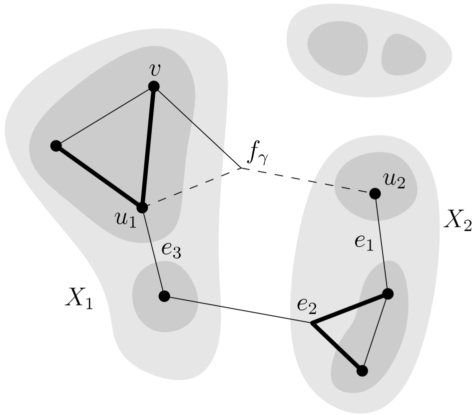


Fig. 9. An illustration to the proof of Claim 2. Some hyperedges are omitted. The light grey regions are the classes of $\mathcal{P}_0^\pi$, the darker ones are the classes of $\mathcal{R}$. Bold lines indicate the quasigraph ρ. The set $\{f_\gamma, e_1, e_2, e_3\}$ corresponds to a quasicycle γ in $H/\mathcal{R}$. The quasigraph σ is obtained by including f_γ in $E(\rho)$, with the representation given by dashed lines. Note that v is contained in the same negative σ-part as u_1.

Since $\rho[X] = \rho_X$ (Lemma 16(ii)) and $\mathcal{R}_X$ is ρ_X-skeletal, case (a) is ruled out. Thus, we can choose a hyperedge f_γ of H such that $\gamma(f_\gamma)$ is $\mathcal{P}_0^\pi$-crossing. As $\gamma/\mathcal{R}$ is a quasicycle in $\overline{\rho/\mathcal{R}}$, $\rho(f_\gamma)$ is contained in a class of $\mathcal{R}$. If f_γ is used by ρ, then this class will be denoted by Y_γ and we will say that the chosen hyperedge f_γ is a *connector for* Y_γ.

Claim 2. *For each quasicycle* $\gamma/\mathcal{R}$ in $\overline{\rho/\mathcal{R}}$, *the hyperedge* f_γ *is used by* ρ.

Suppose to the contrary that $\gamma(f_\gamma) = u_1u_2$, where each u_i $(i = 1, 2)$ is contained in a different class X_i of $\mathcal{P}_0^\pi$. By Lemma 11 and Observation 10(ii), $\mathcal{P}_0^\pi = \mathcal{P}_0^\rho$. Let σ be the quasigraph in H defined by

$$\sigma(e) = \begin{cases} \pi(e) & \text{if } e \neq f_\gamma, \\ u_1u_2 & \text{otherwise} \end{cases}$$

(see Fig. 9). Considering the role of the hyperedge e, we see that

$$\mathcal{P}_0^\rho < \mathcal{P}_0^\sigma. \tag{10}$$

Next, we would like to prove that

$$\mathcal{A}_-(\rho) \leq \mathcal{A}_-(\sigma). \tag{11}$$

First of all, we claim that u_1 and u_2 are contained in the same class of $\mathcal{A}_-(\sigma)$. Let the vertices on the unique cycle in $(\gamma/\mathcal{R})^*$ be $T_1, \ldots, T_k$ in this order, where each T_i is a class of $\mathcal{R}$, $u_1 \in T_1$ and $u_2 \in T_k$. By symmetry, we may assume that $|f_\gamma \cap T_k| = 1$ (i.e., T_1 is the only class of $\mathcal{R}$ which may contain more than one vertex of f_γ).

By Lemma 16(i)–(ii), together with the fact that each $Y \in \mathcal{R}$ is ρ_X-solid (where $Y \subseteq X \subseteq \mathcal{P}_0^\pi$), each T_i $(i = 1, \ldots, k)$ is ρ-solid. Thus, T_i is also σ-solid for $i \geq 2$. Let T_1' be the negative $\sigma[T_1]$-part of $H[T_1]^\sigma$ containing u_1.

For $i = 1, \ldots, k-1$, let e_i be the hyperedge of $E(\gamma)$ such that $\gamma(e_i)/\mathcal{R} = T_iT_{i+1}$ (choosing $e_1 \neq f_\gamma$ if $k = 2$). Let $T = T_1' \cup T_2 \cup \cdots \cup T_k$. Using the minimality of H and Lemma 9(ii), it is easy to prove that T is a subset of a class, say Q, of $\mathcal{A}_-(\sigma)$. Note that Q contains u_1 and u_2 as claimed.

If (11) is false, then the unique vertex of $f_\gamma - \{u_1, u_2\}$ is necessarily contained in a class of $\mathcal{A}_-(\sigma)$ distinct from Q. In that case, however, $\mathcal{A}_-(\sigma)$ is not σ-narrow as $\sigma(f_\gamma) \subseteq Q$. This contradiction with the definition proves (11).

By (10) and (11), $\pi \preceq \rho \prec \sigma$. Moreover, σ is acyclic, since ρ is acyclic and $\sigma(f_\gamma)$ has endvertices in distinct components of ρ^*. Thus, σ satisfies condition (a) in the statement of the lemma, contradicting the choice of π. $\triangle$

For any $Y \in \mathcal{R}$, let $conn(Y)$ be the set of all connectors for Y, and write

$$conn_2(Y) = \{f \cap Y : f \in conn(Y)\}.$$

Note that for any connector f for Y, $f \cap Y$ is a 2-hyperedge of $\rho[Y]$.

Let us describe our strategy in the next step in intuitive terms (see Fig. 10 for an illustration). We want to modify ρ within the classes of $\mathcal{R}$ and 'free' one of the hyperedges f_γ from ρ, which would enable us to apply the argument from the proof of Claim 2 and reach a contradiction. If no such modification works, we obtain a quasigraph σ and a partition $\mathcal{S}$ which refines $\mathcal{R}$. The effect of the refinement is to 'destroy' all quasicycles $\gamma/\mathcal{R}$ in $\overline{\rho/\mathcal{R}}$ by making the representation $\rho(f_\gamma)$ of each associated connector f_γ $\mathcal{S}$-crossing. Thanks to this, it will turn out that $\mathcal{S}$ is σ-skeletal as required to satisfy condition (b).

Thus, let $Y \in \mathcal{R}$ and set

$$\tilde{H}_Y = H[Y]^\rho - conn_2(Y),$$
$$\tilde{\rho}_Y = \rho[Y] - conn_2(Y)$$

(we allow $conn_2(Y) = \emptyset$) and observe that $\tilde{\rho}_Y$ is an acyclic quasigraph in $\tilde{H}_Y$. Let σ_Y be a $\preceq$-maximal acyclic quasigraph in $\tilde{H}_Y$ such that $\sigma_Y \succeq \tilde{\rho}_Y$. We define a quasigraph τ_Y in $H[Y]^\rho$ by

$$\tau_Y(e) = \begin{cases} e & \text{if } e \in conn_2(Y), \\ \sigma_Y(e) & \text{otherwise.} \end{cases}$$

Claim 3. *For all* $Y \in \mathcal{R}$,

$$\mathcal{A}_+(\sigma_Y; \tilde{H}_Y) = \mathcal{A}_+(\tilde{\rho}_Y; \tilde{H}_Y).$$

From $\sigma_Y \succeq \tilde{\rho}_Y$, we know that the left hand side in the statement of the claim is coarser than (or equal to) the right hand side. Suppose that for some $Y \in \mathcal{R}$, $\mathcal{A}_+(\sigma_Y; \tilde{H}_Y)$ is strictly coarser than $\mathcal{A}_+(\tilde{\rho}_Y; \tilde{H}_Y)$. Then we can choose vertices $u_1, u_2 \in Y$ which are contained in different classes U_1, U_2, respectively, of $\mathcal{A}_+(\tilde{\rho}_Y; \tilde{H}_Y)$, but in the same class U of $\mathcal{A}_+(\sigma_Y; \tilde{H}_Y)$. Since Y is ρ-solid, the graph $\rho[Y]^*$ contains a path P joining u_1 to u_2. The choice of u_1 and u_2 implies the following:

(A1) P contains the edge $f_\gamma \cap Y \in conn_2(Y)$ for some quasicycle γ, and
(A2) all the edges of $E(P) \cap conn_2(Y)$ are contained in a cycle in $(\rho|\sigma_Y)^*$.

We choose a quasicycle γ satisfying (A1) and let τ be the quasigraph in H obtained as

$$\tau = (\rho|\tau_Y) - f_\gamma \cap Y.$$

By (A2) and the fact that $\rho[Y]$ is connected, $\tau[Y]$ is connected as well. Since σ_Y has tight complement in $\tilde{H}_Y$, $\tau[Y]$ has tight complement in $H[Y]^\rho$ (the two complements coincide). Thus, Y is τ-solid. By Corollary 13, $\tau \succeq \rho$. By Lemma 14 and the fact that $\rho \succeq \pi$, we may assume that τ is acyclic.

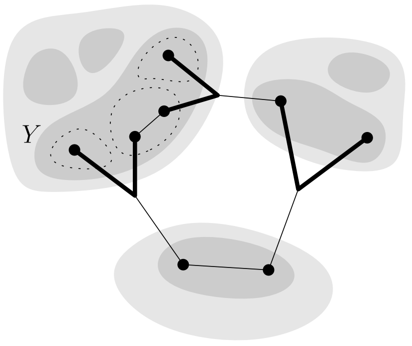
Since ρ and τ are Y-similar, we have

$$\overline{\rho/\mathcal{R}} = \overline{\tau/\mathcal{R}}.$$

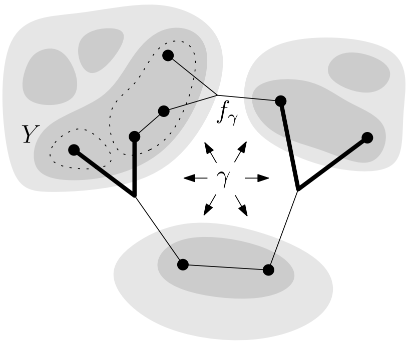
In particular, the quasicycle γ in $\overline{\rho/\mathcal{R}}$ (associated with f_γ) is also a quasicycle in $\overline{\tau/\mathcal{R}}$. As f_γ is not used by τ (and $\tau \succeq \rho$), we can repeat the argument used in the proof of Claim 2, namely add f_γ (with a suitable representation) to τ and reach a contradiction with the choice of π. $\triangle$

We will now construct a σ-skeletal partition of V. Let $Y \in \mathcal{R}$. By the choice of H and the maximality of σ_Y, there is a σ_Y-skeletal partition $\mathcal{S}_Y$ of Y (in $\tilde{H}_Y$). We define a quasigraph σ in H and a partition $\mathcal{S}$ of V by

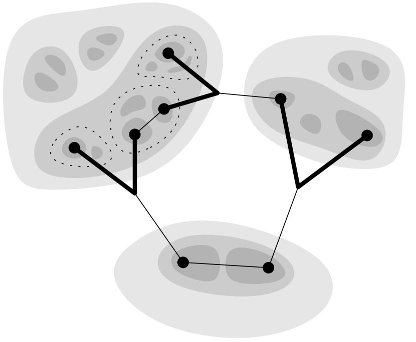
$$\sigma = \rho|\{\tau_Y : Y \in \mathcal{R}\},$$
$$\mathcal{S} = \bigcup_{Y \in \mathcal{R}} \mathcal{S}_Y.$$



(a) Bold lines show the quasigraph ρ, the dotted regions are the positive $\tilde{\rho}_Y$-parts of $\tilde{H}_Y$.



(b) The dotted regions here are the positive σ_Y-parts of $\tilde{H}_Y$. If the partition is strictly coarser than in (a), we can 'free' a suitable connector f_γ and use it as before.



(c) Otherwise, we obtain a finer partition $\mathcal{S}$ (darkest grey regions) such that $\rho(f_\gamma)$ is $\mathcal{S}$-crossing for each γ.

Fig. 10. An illustration to the proof of Claim 3 and the following part of the proof. We use similar conventions as in Fig. 9.

We aim to show that $\mathcal{S}$ is σ-skeletal. Let $Z \in \mathcal{S}$ and suppose that $Z \subseteq Y \subseteq X$, where $X \in \mathcal{P}_0^\pi$ and $Y \in \mathcal{R}$. Since $\sigma[Z] = \sigma_Y[Z]$ and $\mathcal{S}_Y$ is σ_Y-skeletal, $\sigma[Z]$ is a quasitree.

To show that the complement of $\sigma[Z]$ in $H[Z]^\sigma$ is tight, we use Lemma 16(i):

$$H[Z]^\sigma = (H[Y]^\rho)[Z]^{\tau_Y} = \tilde{H}_Y[Z]^{\tau_Y} = \tilde{H}_Y[Z]^{\sigma_Y}. \tag{12}$$

Here, the second and the third equality follows from Claim 3 which implies that any connector for Y intersects two classes of $\mathcal{A}_+(\sigma_Y; \tilde{H}_Y)$. From (12) and the fact that $\sigma_Y[Z]$ has tight complement in $\tilde{H}_Y[Z]^{\sigma_Y}$, it follows that $\sigma[Z]$ has tight complement as well.

It remains to prove that $\overline{\sigma/\mathcal{S}}$ is acyclic. Suppose, for the sake of a contradiction, that γ is a quasigraph in H such that $\gamma/\mathcal{S}$ is a quasicycle in $\overline{\sigma/\mathcal{S}}$. Note that the complement of $\tau_Y/\mathcal{S}_Y$ in $H[Y]^\rho$ is the same as the complement of $\sigma_Y/\mathcal{S}_Y$ in $\tilde{H}_Y$, and hence acyclic. By Lemma 15, $\gamma/\mathcal{R}$ is a nonempty quasigraph in $\overline{\rho/\mathcal{R}}$ with $(\gamma/\mathcal{R})^*$ eulerian.

Let δ be a restriction of γ such that $\delta/\mathcal{R}$ is a quasicycle in $\overline{\rho/\mathcal{R}}$. Every such quasicycle has an associated hyperedge f_δ which is a connector for a class $Y_\delta \in \mathcal{R}$ (Claim 2). In particular, f_δ is used by ρ. By the fact that f_δ intersects two classes of $\mathcal{A}_+(\sigma_{Y_\delta}; \tilde{H}_{Y_\delta})$, $\rho(f_\delta)$ is $\mathcal{S}$-crossing. This implies that $\sigma(f_\delta)$ is $\mathcal{S}$-crossing, which contradicts the assumption that $\gamma/\mathcal{S}$ is a quasicycle in $\overline{\sigma/\mathcal{S}}$. The proof is complete. $\square$

8. Proof of Theorem 5

We can now prove our main result regarding spanning trees in hypergraphs, announced in Section 3 as Theorem 5:

Theorem. *Let H be a 4-edge-connected 3-hypergraph. If no 3-hyperedge of H is included in any edge-cut of size 4, then H contains a quasitree with tight complement.*

Proof. Let π be a $\preceq$-maximal acyclic quasigraph in H. By the Skeletal Lemma (Lemma 17), there exists a π-skeletal partition $\mathcal{P}$ of V. For the sake of a contradiction, suppose that π is not a quasitree with tight complement. In particular, $\mathcal{P}$ is nontrivial.

Assume that $H/\mathcal{P}$ has n vertices (that is, $|\mathcal{P}| = n$) and m hyperedges. For $k \in \{2, 3\}$, let m_k be the number of k-hyperedges of $\pi/\mathcal{P}$. Similarly, let $\overline{m_k}$ be the number of k-hyperedges of $\overline{\pi/\mathcal{P}}$. Thus, $m = m_2 + m_3 + \overline{m_2} + \overline{m_3}$.

Since $\overline{\pi/\mathcal{P}}$ is acyclic, the graph $Gr(\overline{\pi/\mathcal{P}})$ (defined in Section 2) is a forest. As $Gr(\overline{\pi/\mathcal{P}})$ has $n + \overline{m_3}$ vertices and $\overline{m_2} + 3\overline{m_3}$ edges, we find that

$$\overline{m_2} + 2\overline{m_3} \leq n - 1. \tag{13}$$

Since $\mathcal{P}$ is π-solid and π is an acyclic quasigraph, we know that $m_2 + m_3 \leq n - 1$. Moreover, by the assumption that π is not a quasitree with a tight complement, either this inequality or (13) is strict. Summing the two, we obtain

$$m + \overline{m_3} \leq 2n - 3. \tag{14}$$

We let n_4 be the number of vertices of $H/\mathcal{P}$ of degree 4, and n_{5+} be the number of the other vertices. Since $n \geq 2$ and H is 4-edge-connected, we have $n = n_4 + n_{5+}$. By double counting,

$$4n_4 + 5n_{5+} \leq 2(m_2 + \overline{m_2}) + 3(m_3 + \overline{m_3}) = 2m + m_3 + \overline{m_3}. \tag{15}$$

The left hand side equals $4n + n_{5+}$. Using (14), we find that

$$4n + n_{5+} \geq 2m + 2\overline{m_3} + n_{5+} + 6.$$

Combining with (15), we obtain

$$m_3 \geq \overline{m_3} + n_{5+} + 6. \tag{16}$$

We show that $m_3 \leq n_{5+}$. Let $T' = (\pi/\mathcal{P})^*$ be the forest on $\mathcal{P}$ which represents $\pi/\mathcal{P}$. In each component of T', choose a root and direct the edges of T' away from it. To each 3-hyperedge $e \in E(\pi/\mathcal{P})$, assign the head $h(e)$ of the arc $\pi(e)$. By the assumptions of the theorem, no edge-cut of size 4 contains a 3-hyperedge, so $h(e)$ is a vertex of degree at least 5. At the same time, since each vertex is the head of at most one arc in the directed forest, it gets assigned to at most one hyperedge. The inequality $m_3 \leq n_{5+}$ follows. This contradiction to inequality (16) proves that π is a quasitree with tight complement. $\square$

9. Even quasitrees

In the preceding sections, we were busy looking for quasitrees with tight complement in hypergraphs. In this and the following section, we will explain the significance of such quasitrees for the task of finding a Hamilton cycle in the line graph of a given graph.

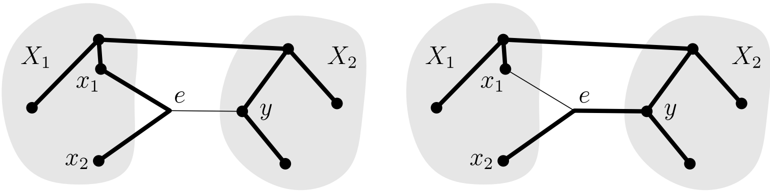
Let π be a quasitree in H. For a set $X \subseteq V$, we define a number $\Phi_\pi(X) \in \{0, 1\}$ by

$$\Phi_\pi(X) \equiv \sum_{v \in X} d_{\pi^*}(v) \pmod 2.$$

Observe that $\Phi_\pi(X) = 0$ if and only if X contains an even number of vertices whose degree in the tree π^* is odd.

For $X \subseteq V$, we say that π is *even on* X if for every component K of $\overline{\pi}$ whose vertex set is a subset of X, it holds that $\Phi_\pi(V(K)) = 0$. If π is even on V, then we just say π is *even*.

The main result of this section is the following:



(a) The quasitree π.

(b) The quasitree π', resulting from the switch.

Fig. 11. The case $\Phi_\pi(X_1) = 1$ in the proof of Lemma 19. The grey regions are the sets X_1 and X_2. Note how the switch of the representation of e changes the parity of exactly one vertex degree in X_1.

Lemma 18. *If π is a quasitree in H with tight complement, then there is a quasigraph ρ in H such that $E(\rho) = E(\pi)$ and ρ is an even quasitree in H.*

Lemma 18 is a direct consequence of the following more technical statement (to derive Lemma 18, set $X = V$):

Lemma 19. *Let π be a quasitree in H and $X \subseteq V$. Assume that $\Phi_\pi(X) = 0$ and π has tight complement in $H[X]^\pi$. Then there is a quasitree ρ in H such that π and ρ are X-similar, and ρ is even on X.*

Proof. We proceed by induction on $|X|$. We may assume that $|X| \geq 2$, since otherwise the claim is trivially true. Similarly, if $\overline{\pi}[X]$ is connected, then the assumption $\Phi_\pi(X) = 0$ implies that π is even on X. Thus, we assume that $\overline{\pi}[X]$ is disconnected.

The definition implies that there is a partition $X = X_1 \cup X_2$ such that:

(B1) for each $i = 1, 2$, $\pi[X_i]$ has tight complement in $H[X_i]^\pi$,
(B2) there is a hyperedge e intersecting X_2 with $\pi(e) \subseteq X_1$, and
(B3) for any hyperedge f intersecting both X_1 and X_2, we have $f \in E(\pi)$.

If $\Phi_\pi(X_1) = 0$, then we may use the induction hypothesis with X_1 playing the role of X. The result is a quasitree ρ_1 in H which is even on X_1 and X_1-similar to ρ. In particular, $\Phi_{\rho_1}(X_1) = 0$ and hence also $\Phi_{\rho_1}(X_2) = 0$. Using the induction hypothesis for X_2, we obtain a quasitree ρ_2 in H which is even on X_2; furthermore, being X_2-similar to ρ_1, it is even on X_1 as well. By (B3), the vertex set of every component K of $\overline{\pi}$ with $V(K) \subseteq X$ is a subset of X_1 or X_2. Thus, $\rho := \rho_2$ is even on X, and clearly X-similar to π.

It remains to consider the case that $\Phi_\pi(X_1) = 1$, illustrated in Fig. 11. Here we need to 'switch' the representation of e (the hyperedge from (B2)) as follows. Let $e = x_1x_2y$, with $\pi(e) = x_1x_2$. The removal of the edge x_1x_2 from π^* splits π^* into two components, each containing one of x_1 and x_2. By symmetry, we may assume that y is contained in the component containing x_1. We define a new quasigraph π' in H by

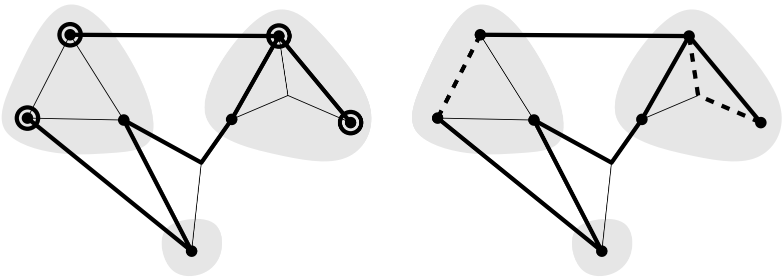
$$\pi'(e) = \begin{cases} x_2y & \text{if } f = e, \\ \pi(f) & \text{otherwise.} \end{cases}$$

Note that π' is a quasitree and $\Phi_{\pi'}(X_1) = 0$. Consequently, we can proceed as before, apply the induction hypothesis and eventually obtain a representation ρ which satisfies the assertions of the lemma. $\square$

10. Hamilton cycles in line graphs and claw-free graphs

We recall two standard results which interpret the connectivity and the hamiltonicity of a line graph in terms of its preimage. The first result is a folklore observation, the second is due to Harary and Nash-Williams [8]. We combine them into one theorem, but before we state them, we recall some necessary terminology.

Let G be a graph. An edge-cut C in G is *trivial* if it consists of all the edges incident with some vertex v of G. The graph G is *essentially k-edge-connected* ($k \geq 1$) if every edge-cut in G of size less than k is trivial. A subgraph D of G is *dominating* if $G - V(D)$ has no edges.



(a) The vertices of odd degree in π^* are circled.

(b) Since each component of $\overline{\pi}$ contains an even number of circled vertices, we can complete π^* to an eulerian graph (the added edges are shown as dashed bold lines).

Fig. 12. An illustration to Lemma 22. The grey regions are the components of $\overline{\pi}$, where π is the quasigraph shown by solid bold lines.

Theorem 20. *For any graph G and $k \geq 1$, the following holds:*

(i) *$L(G)$ is k-connected if and only if G is essentially k-edge-connected,*
(ii) *$L(G)$ is hamiltonian if and only if G contains a dominating connected eulerian subgraph C.*

In a similar spirit, the minimum degree of $L(G)$ equals the minimum edge weight of G, where the *weight* of an edge e is defined as the number of edges incident with e and distinct from it.

Given a set X of vertices of G, an *X-join* in G is a subgraph G' of G such that a vertex of G is in X if and only if its degree in G' is odd. (In particular, $\emptyset$-joins are eulerian subgraphs).

We will need a lemma which has been used a number of times before, either explicitly or implicitly. For completeness, we sketch a quick proof.

Lemma 21. *If T is a tree and X is a set of vertices of T of even cardinality, then T contains an X-join.*

Proof. By induction on the order of T. If $|V(T)| = 1$, the assertion is trivial. Otherwise, choose an edge $e = v_1v_2$ and let T_1 and T_2 be components of $T - e$, T_1 being the one which contains v_1. Let X_1 be $X \cap V(T_1)$ if the size of this set is even; otherwise, set $X_1 = (X \cap V(T_1)) \oplus \{v_1\}$, where $\oplus$ stands for the symmetric difference. The induction yields an X_1-join T_1' in T_1. A set X_2 and an X_2-join T_2' in T_2 is obtained in a symmetric way. It is easy to check that the union of T_1' and T_2', with e added if $|X \cap V(T_1)|$ is odd, is an X-join. $\square$

If G_1 and G_2 are two graphs, then $G_1 + G_2$ denotes the graph whose vertex set is the (not necessarily disjoint) union of vertex sets of G_1 and G_2, and whose multiset of edges is the multiset union of $E(G_1)$ and $E(G_2)$.

As the following lemma shows, an even quasitree in H allows one to find a connected spanning eulerian subgraph of $Gr(H)$ (see Fig. 12 for an illustration):

Lemma 22. *If π is an even quasitree in H, then there is a quasigraph τ in H such that $E(\pi)$ and $E(\tau)$ are disjoint, and $\pi^* + \tau^*$ is a connected eulerian subgraph of the graph $Gr(H)$ spanning all vertices in V.*

Proof. Let K be a component of $\overline{\pi}$, and let X be the set of vertices of K whose degree in π^* is odd. Since π is even, $|X|$ is even. Choose a spanning tree T of the (connected) graph $Gr(K)$. Using Lemma 21, choose a subforest T' of T such that for every vertex w of $Gr(K)$, $d_{T'}(w)$ is odd if and only if $w \in X$. In $\pi^* + T'$, all the vertices of K have even degrees. In fact, the same holds for any vertex v_e of $Gr(K)$, where e is a hyperedge of H of size 3: if e is used by π, then $d_{\pi^*+T'}(v_e) = 2$, and otherwise we have

$$d_{\pi^*+T'}(v_e) = d_{T'}(v_e),$$

which is even since $v_e \notin X$. In particular, there is a quasigraph τ_K in H such that $\tau_K^* = T'$.

We apply the above procedure repeatedly, one component of $\overline{\pi}$ at a time. For this, we need to be sure that a 3-hyperedge e will not be used by τ_{K_1} as well as τ_{K_2}, where K_1 and K_2 are distinct components of $\overline{\pi}$. This is clear, however, since e can only be used by τ_K if $|e \cap V(K)| \geq 2$. Thus, the components of $\overline{\pi}$ can be treated independently, and we eventually obtain an eulerian subgraph S of $Gr(H)$. Since it contains the tree π^*, S spans all of V, and since each of the trees $(\tau_K)^*$ contains an edge incident with a vertex in V (unless $(\tau_K)^*$ is edgeless), it follows that S is connected. □

Using Theorem 20, it will be easy to derive our main result (Theorem 4) as a consequence of the following proposition. Let us remark that the proposition is closely related to a conjecture made by Jackson (see [1, Conjecture 4.48]) and implies one of its three versions.

Proposition 23. *If G is an essentially 5-edge-connected graph with minimum edge weight at least 6, then G contains a connected eulerian subgraph spanning all the vertices of degree at least 4 in G.*

Proof. For the sake of a contradiction, let G be a counterexample with as few vertices as possible. Since the claim is trivially true for a one-vertex graph, we may assume $|V(G)| \geq 2$. For brevity, a *good subgraph* in a graph G' will be a connected eulerian subgraph spanning all the vertices of degree at least 4 in G'.

Claim 1. *The minimum degree of G is at least 3.*

Suppose first that G contains a vertex v of degree 2 with distinct neighbours w_1 and w_2. If we suppress v, the resulting graph G' will be essentially 5-edge-connected. Furthermore, the minimum edge weight of G' is at least 6 unless G is the triangle vw_1w_2 with the edge w_1w_2 of multiplicity 5, which is however not a counterexample to the proposition. By the minimality assumption, G' contains a good subgraph C'. It is easy to see that the corresponding subgraph of G is also good.

Suppose then that G contains a vertex u of degree 1 or 2 with a single neighbour z. Let U be the set of all the vertices of degree 1 or 2 in G whose only neighbour is z. If $V(G) = U \cup \{z\}$, then the Eulerian subgraph consisting of just the vertex z shows that G is not a counterexample to the proposition. Thus, z has a neighbour x outside U. In fact, since G is essentially 5-edge-connected, z is incident with at least 5 edges whose other endvertex is not in U. Let e be an edge with endvertices z and x. Since the degree of x is at least 3, the edge weight of e in $G - U$ is at least 6. This implies that the minimum edge weight of $G - U$ is at least 6. Since the removal of U does not create any new minimal essential edge-cut, $G - U$ is essentially 5-edge-connected. Since the degree of z in $G - U$ is at least 5, any good subgraph in $G - U$ is a good subgraph in G. Thus, $G - U$ is a smaller counterexample than G, contradicting the minimality of G. △

Claim 2. *No vertex of degree 3 in G is incident with a pair of parallel edges.*

Suppose that v is a vertex of degree 3 incident with parallel edges e_1, e_2. If v has only one neighbour, then any good subgraph of $G - v$ is good in G. By the minimality of G, v must have exactly two neighbours, say w and z, where w is incident with e_1 and e_2. Let G' be obtained from G by removing v and adding the edge e_0 with endvertices w and z.

It is easy to see that G' is essentially 5-edge-connected, and that any good subgraph of G' can be modified to a good subgraph of G (as $d_G(w) \geq 6$). We show that the minimal edge weight in G' is at least 6.

Suppose the contrary and let e be an edge of G' of weight less than 6. We have $e \neq e_0$ as the assumptions imply that $d_G(w) \geq 6$ and $d_G(z) \geq 5$, so the weight of any edge with endvertices w and z in G' is at least 8. Thus, e is an edge of G.

It must be incident with w, for otherwise its weight in G' would be the same as in G. Let u be the endvertex of e distinct from w. Since $d_G(w) \geq 6$, w is incident in G' with at least 3 edges of G' distinct from e_0 and e. By the weight assumption, u must be incident with only at most one edge of G' other than e, contradicting Claim 1. △

Let H be the 3-hypergraph whose vertex set V is the set of all vertices of G whose degree is at least 4; the hyperedges of H are of two kinds:

- the edges of G with both endvertices in V,
- 3-hyperedges consisting of the neighbours of any vertex of degree 3 in G.

Note that H is well-defined, for any neighbour of a vertex of degree 3 in G must have degree at least 4 (otherwise they would be separated from the rest of the graph by an essential edge-cut of size at most 4). Furthermore, by Claim 2, any vertex of degree 3 does indeed have three distinct neighbours in V.

In the following two claims, we show that H satisfies the hypotheses of Theorem 5.

Claim 3. *The hypergraph H is 4-edge-connected.*

Suppose that this is not the case and F is an inclusionwise minimal edge-cut in H with $|F| \leq 3$. Let A be the vertex set of a component of $H - F$.

Let $e \in F$. By the minimality of G, $|e - A| \geq 1$. We assign to e an edge e' of G, defined as follows:

- if $|e| = 2$, then $e' = e$,
- if $|e| = 3$ and $e \cap A = \{u\}$, then $e' = uv_e$,
- if $|e| = 3$, $|e \cap A| = 2$ and $e - A = \{u\}$, then $e' = uv_e$.

Observe that $F' := \left\{e' : e \in F\right\}$ is an edge-cut in G. Since G is 5-edge-connected, F' must be a trivial edge-cut. This means that a vertex $v \in V$ has degree 3 in H, a contradiction as v has degree at least 4 in G and therefore also in H. $\triangle$

The other claim regards edge-cuts of size 4 in H:

Claim 4. *No 3-hyperedge of H is included in an edge-cut of size 4 in H.*

Let F be an edge-cut of size 4 in H. As in the proof of Claim 3, we consider the corresponding edge-cut F' in G. Since G is essentially 5-edge-connected, one component of $G - F'$ consists of a single vertex w whose degree in G is 4. Assuming that F includes a 3-hyperedge e, we find that in G, w has a neighbour v of degree 3. Since the weight of the edge vw is 5, we obtain a contradiction with our assumptions about G. $\triangle$

Since the assumptions of Theorem 5 are satisfied, we can use it to find a quasitree π with tight complement in H. By Lemmas 18 and 22, $Gr(H) = G$ admits a connected eulerian subgraph spanning the set V. This is what we wanted to find. $\square$

We can now prove our main theorem, stated as Theorem 4 in Section 1:

Theorem. *Every 5-connected line graph of minimum degree at least 6 is hamiltonian.*

Proof. Let $L(G)$ be a 5-connected line graph of minimum degree at least 6. By Theorem 20(i), G is essentially 5-edge-connected. Furthermore, the minimum edge weight of G is at least 6. By Proposition 23, G contains a connected eulerian subgraph C spanning all the vertices of degree at least 4. By Theorem 20(ii), it is sufficient to prove that $G - V(C)$ has no edges. Indeed, the vertices of any edge e in $G - V(C)$ must have degree at most 3 in G, which implies that e is incident to at most 4 other edges of G, a contradiction to the minimum degree assumption. Thus, $L(G)$ is hamiltonian. $\square$

Using the claw-free closure concept developed by Ryjáček [21], Theorem 4 can be extended to claw-free graphs. Let us recall the main result of [21]:

Theorem 24. *Let G be a claw-free graph. Then there is a well-defined graph $cl(G)$ (called the* closure *of G) such that the following holds:*

(i) *G is a spanning subgraph of $cl(G)$,*
(ii) *$cl(G)$ is the line graph of a triangle-free graph,*
(iii) *the length of a longest cycle in G is the same as in $cl(G)$.*

Corollary 25. *Every 5-connected claw-free graph G of minimum degree at least 6 is hamiltonian.*

Proof. Apply Theorem 24 to obtain the closure $cl(G)$ of G. Since $G \subseteq cl(G)$, the closure is 5-connected and has minimum degree at least 6. Being a line graph, $cl(G)$ is hamiltonian by Theorem 4. Since G is a spanning subgraph of $cl(G)$, property (iii) in Theorem 24 implies that G is hamiltonian. $\square$

11. Hamilton-connectedness

Recall from Section 1 that a graph is Hamilton-connected if for every pair of distinct vertices u, v, there is a Hamilton path from u to v. The method used to prove Theorem 4 and Corollary 25 can be adapted to yield the following stronger result:

Theorem 26. *Every 5-connected claw-free graph of minimum degree at least 6 is Hamilton-connected.*

In this section, we sketch the necessary modifications to the argument. For a start, let $H = L(G)$ be a 5-connected line graph of minimum degree at least 6. By considerations similar to those in the proof of Proposition 23, it may be assumed that the minimum degree of G is at least 3 and that no vertex of G is incident with a pair of parallel edges, so we may associate with G a 3-hypergraph H just as in that proof. Moreover, H may again be assumed to satisfy the assumptions of Theorem 5.

Let $V_{\geq 4} \subseteq V(G)$ be the set of vertices of degree at least 4 in G.

First, we will need a replacement of Theorem 20(ii) that translates the Hamilton-connectedness of H to a property of G. A *trail* F is a sequence of edges of G such that each pair of consecutive edges is adjacent in G, and F contains each edge of G at most once. We will say that F *spans* a set Y of vertices if each vertex in Y is incident with an edge of F. A trail is an (e_1, e_2)-*trail* if it starts with e_1 and ends with e_2. Furthermore, an (e_1, e_2)-trail F is *internally dominating* if every edge of G has a common endvertex with some edge in F other than e_1 and e_2. The following fact is well-known (see, e.g., [17]):

Theorem 27. *Let G be a graph with at least 3 edges. Then $L(G)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has an internally dominating (e_1, e_2)-trail.*

One way to find an internally dominating (e_1, e_2)-trail (where e_1, e_2 are edges) is by using a connection to X-joins as defined in Section 10. For each edge e of G, fix an endvertex u_e of degree at least 4 in G (which exists since G is essentially 5-edge-connected). If e_1 and e_2 are edges, set

$$X(e_1, e_2) = \begin{cases} \{u_{e_1}, u_{e_2}\} & \text{if } u_{e_1} \neq u_{e_2}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Suppose now that the graph $G - e_1 - e_2$ happens to contain a connected $X(e_1, e_2)$-join J spanning all of $V_{\geq 4}$. By the classical observation of Euler, all the edges of J can be arranged in a trail T_J whose first edge is incident with u_{e_1} and whose last edge is incident with u_{e_2}. Adding e_1 and e_2, we obtain an (e_1, e_2)-trail T in G. (If $u_1 = u_2$, we use the fact that u_1 is incident with an edge of T_J.) Since G contains no adjacent vertices of degree 3, T is an internally dominating (e_1, e_2)-trail.

Summing up, the Hamilton-connectedness of $L(G)$ will be established if we can show that for every $e_1, e_2 \in E(G)$, the graph $G - e_1 - e_2$ contains a connected $X(e_1, e_2)$-join spanning $V_{\geq 4}$.

How to find such $X(e_1, e_2)$-joins? Recall that in Section 10, the existence of a connected dominating eulerian subgraph of G (a connected dominating $\emptyset$-join) was guaranteed by Lemma 22 based on the assumption that H contains an even quasitree. As shown by Lemma 18, an even quasitree in H exists whenever H contains a quasitree with tight complement. A rather straightforward modification of the proofs of these two lemmas (which we omit) leads to the following generalization:

Lemma 28. *Let H' be a 3-hypergraph containing a quasitree π with tight complement, and let $X \subseteq V(H')$. Then there is a quasigraph τ such that $E(\pi)$ and $E(\tau)$ are disjoint, and $\pi^* + \tau^*$ is a connected X-join in $Gr(H')$ spanning all vertices in $V(H')$.*

Roughly speaking, Lemma 28 will reduce our task to showing that for each pair of edges e_1, e_2 of G, a suitably defined 3-hypergraph H' admits a quasitree with tight complement.

Let us define the 3-hypergraph H' to which Lemma 28 is to be applied. Suppose that e_1 and e_2 are given edges of G, and let w_i $(i = 1, 2)$ be the endvertex of e_i distinct from u_i. We distinguish two cases:

(1) if e_1 and e_2 have a common vertex of degree 3 (namely, the vertex $w_1 = w_2$), then H' is obtained from H by removing the 3-hyperedge corresponding to w_1;
(2) otherwise, H' is the hypergraph obtained by performing the following for $i = 1, 2$:
  (2a) if w_i has degree 3, then the 3-hyperedge e_{w_i} of H corresponding to w_i is replaced by the 2-hyperedge $e_{w_i} - \{u_i\}$,
  (2b) otherwise, the 2-hyperedge e_i of H is deleted.

By Lemma 28 and the preceding remarks, it suffices to show that H' admits a quasitree with tight complement. To do so, we apply to H' the proof of Theorem 5, which works well as far as equation (14). However, the inequality (15) may fail since H' is not necessarily 4-edge-connected. It has to be replaced as follows.

For an arbitrary hypergraph H^*, let $s(H^*)$ be the sum of all vertex degrees in H^*. Let $\mathscr{P}$ be the partition of $V(H')$ obtained in the proof of Theorem 5. Furthermore, let n_4^* be the number of vertices of degree 4 in $H/\mathscr{P}$, and let $n_{5+}^* = n - n_4^*$. (All the symbols such as n, m, m_3 etc., used in the proof of Theorem 5, are now related to the hypergraph H' rather than H.)

It is not hard to relate $s(H')$ to $s(H)$. Indeed, the operations in cases (1), (2a) and (2b) above decrease the degree sum by 3, 1 and 2, respectively. It follows that $s(H') \geq s(H) - 4$ and, in fact,

$$s(H'/\mathscr{P}) \geq s(H/\mathscr{P}) - 4.$$

Since H is 4-edge-connected, we know that

$$s(H/\mathscr{P}) \geq 4n_4^* + 5n_{5+}^*$$

and thus we can replace (15) by

$$4n_4^* + 5n_{5+}^* - 4 \leq s(H'/\mathscr{P}) = 2m + m_3 + \overline{m_3}.$$

This eventually leads to

$$m_3 \geq \overline{m_3} + n_{5+}^* + 2$$

as a replacement for (16). Thus, the contradiction is much the same as before, since we have (by the same argument as in the old proof) that $m_3 \leq n_{5+}^*$. This proves Theorem 26 in the case of line graphs.

If G is a claw-free graph, we will use a closure operation again. However, the claw-free closure described in Section 10 is not applicable, since the closure of G may be Hamilton-connected even if G is not. Instead, we use the *M-closure* which was defined in [22] and applied there to prove that 7-connected claw-free graphs are Hamilton-connected. Let us list its relevant properties [22, Theorem 9]:

Theorem 29. *If G is a connected claw-free graph, then there is a well-defined graph $cl^M(G)$ with the following properties:*

(i) *G is a spanning subgraph of $cl^M(G)$,*
(ii) *$cl^M(G)$ is the line graph of a multigraph H,*
(iii) *$cl^M(G)$ is Hamilton-connected if and only if G is Hamilton-connected.*

Using this result (and the fact that parallel edges are allowed throughout our argument), it is easy to prove Theorem 26 just like Corollary 25 is proved using the claw-free closure.

12. Conclusion

We have developed a method for finding dominating eulerian subgraphs in graphs, based on the concept of a quasitree with tight complement. Using this method, we have made some progress on Conjecture 2, although the conjecture itself is still wide open. It is conceivable that a refinement in some part of the analysis may improve the result a bit — perhaps to all 5-connected line graphs. On the other hand, the 4-connected case would certainly require major new ideas. For instance, the preimage G of a 4-connected line graph may be cubic, in which case we do not even know how to associate a 3-hypergraph with G in the first place.

As mentioned in Section 1, a simpler variant of our method yields a short proof of the tree-packing theorem of Tutte and Nash-Williams. It is well known that spanning trees in a graph G are the bases of a matroid, the cycle matroid of G, and thus matroid theory provides a very natural setting for the tree-packing theorem. Interestingly, quasitrees with tight complement do not quite belong to the realm of matroid theory, although quasitrees themselves do. Is there an underlying abstract structure, more general than the matroidal one, which forms the 'reason' for the existence of both disjoint spanning trees in graphs, and quasitrees with tight complement in hypergraphs?

It remains a question for further research whether our approach may be useful for other problems on the packing of structures similar to spanning trees, but also lacking their matroidal properties. These include the packing of Steiner trees [13,14] or T-joins [3,20].

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Equivalence of Jackson's and Thomassen's conjectures

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Abstract

A graph G is said to be claw-free if G has no induced subgraph isomorphic to $K_{1,3}$. For a cycle C in a graph G, C is called a Tutte cycle of G if C is a Hamilton cycle of G, or the order of C is at least 4 and every component of $G - C$ has at most three neighbors on C. In [On a closure concept in claw-free graphs, J. Combin. Theory Ser. B **70** (1997), 217–224], Ryjáček proved that the conjectures by Matthews and Sumner (every 4-connected claw-free graph is Hamiltonian) and by Thomassen (every 4-connected line graph is Hamiltonian) are equivalent. In this paper, we show the above conjectures are equivalent with the conjecture by Jackson in 1992 (every 2-connected claw-free graph has a Tutte cycle).

Keywords: Hamiltonian, Claw-free graph, Line graph, Tutte cycle
AMS Subject Classification: 05C38, 05C45

1 Introduction

In this paper, we consider finite graphs without loops. For terminology and notation not defined in this paper, we refer the readers to [5]. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. The *degree* of a vertex v of G is the number of edges incident with v in G, and we denote by $\delta(G)$ the minimum degree of G. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph induced by X in G, and let $G - X = G[V(G) - X]$. For

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a subgraph H of G, let $G - H = G - V(H)$. A graph G is said to be *Hamiltonian* if G has a *Hamilton cycle*, i.e., a cycle containing all vertices of G, and *Hamilton-connected* if G has a *Hamilton path* between any pair of vertices, i.e., a path containing all vertices of G. A graph G is said to be *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$. For a cycle C of G, C is said to be *maximal* if there exists no cycle C' such that $V(C) \subsetneq V(C')$.

In this paper, we will deal with many statements which are unknown to be true or not. We call two statements *equivalent* if the correctness of one statement implies that of the other and vice versa. Most of the results in this paper are motivated by the following two conjectures due to Matthews and Sumner [16] and Thomassen [22], respectively.

Conjecture A (Matthews and Sumner [16], Thomassen [22]) *The following statements are true.*

(A1) *Every* 4*-connected claw-free graph is Hamiltonian.*

(A2) *Every* 4*-connected line graph is Hamiltonian.*

Since every line graph is claw-free, statement (A2) is a special case of statement (A1). However it is known that a result on closures due to Ryjáček [17] implies that statements (A1) and (A2) are even equivalent.

Theorem B (Ryjáček [17]) *Statements* (A1) *and* (A2) *are equivalent.*

Like Theorem B, many statements that are seemingly stronger or weaker than statements (A1) and (A2) have been proven to be equivalent to it as follows (see a survey [4] for more details). Note that statements (A5) and (A6) were conjectured by Ash and Jackson [1] and Fleischner [7], respectively.

Theorem C *All of the following statements are equivalent to statements* (A1) *and* (A2).

(A3) *Every* 4*-connected claw-free graph is Hamilton-connected* [18].

(A4) *Every* 4*-connected line graph is* 1*-Hamilton-connected* (2*-edge-Hamilton-connected*) [14].

(A5) *Every essentially* 4*-edge-connected graph has a dominating closed trail* [8].

(A6) *Every cyclically* 4*-edge-connected cubic graph has a dominating cycle* [8].

(A7) *Every cyclically* 4*-edge-connected cubic graph that is not* 3*-edge-colorable has a dominating cycle* [11].

(A8) *Every snark has a dominating cycle* [2].

Recently, as a positive result related to Conjecture A, Kaiser and the fourth author [15] proved that every 5-connected claw-free graph with minimum degree at least 6 is Hamilton-connected.

On the other hand, it is known that considering "Tutte cycles" is an effective approach to some problems on Hamiltonicity, where a cycle C of a graph G is called a *Tutte cycle* of G if (i) C is a Hamilton cycle of G, or (ii) $|V(C)| \geq 4$ and every component of $G - C$ has at most three neighbors on C. Note that every Tutte cycle C of a 4-connected graph G is a Hamilton cycle, since otherwise the neighbors of a component of $G - C$ form a cut set of order at most three, contradicting 4-connectedness of G. One can show that every 4-connected planar graphs are Hamiltonian by proving assertions on the existence of certain Tutte cycles in 2-connected planar graphs (see [21, 23]). Starting with this result, many researchers have studied about the existence of certain Tutte cycles not only in planar graphs but also in projective planar graphs or graphs on other surfaces in order to show Hamiltonicity of such graphs, (for example, see [19, 20, 24]). Thus, it has succeeded to show Hamiltonicity of 4-connected planar graphs or graphs on surfaces, considering stronger concept "Tutte cycles".

Motivated by the above situation for planar graphs, in this paper, we concentrate on Tutte cycles in claw-free graphs. As a possible approach to solve Conjecture A, Jackson [10] proposed the following conjecture (also see a survey [6, Conjecture 2a.5]).

Conjecture D (Jackson [10]) *The following statement is true.*
(A9) *Every 2-connected claw-free graph has a Tutte cycle.*

As mentioned above, Tutte cycles in 4-connected graphs are Hamilton cycles, and hence statement (A9) implies statement (A1). The main result of this paper is to show that the converse also holds. In fact, we prove the following theorem.

Theorem 1 *Statements* (A1) *and* (A9) *are equivalent.*

On the other hand, if a graph has a Tutte cycle, then we can expect that it is long since it can avoid only vertices in a component of the graph after deleting a cut set of order at most three. Actually, Tutte cycles in 4-connected graphs are Hamilton cycles, i.e., Tutte cycles in 4-connected graphs are longest cycles of the graphs. How about 2-connected (or 3-connected) claw-free graphs? In view of Theorem 1, it would be natural to ask that every 2-connected (or 3-connected) claw-free graph has a Tutte cycle which is longest. As an answer of this problem, in Section 6, we will give a 3-connected claw-free graph in which any Tutte cycle is not longest. Thus it is not always true that a 2-connected (or 3-connected) claw-free graph has a longest one. However, the following theorem, which is also our main theorem, implies that if every 2-connected claw-free graph has a Tutte cycle, then we can always take it so that it is maximal.

Theorem 2 *Statement* (A9) *is equivalent to the following statement.*
(A10) *Every 2-connected claw-free graph has a Tutte cycle which is a maximal cycle of the graph.*

In Sections 3 and 4, we prove Theorems 1 and 2 by using closure concepts and other related results, some of which are also new.

2 Notation and terminology

In this section, we prepare terminology and notation which we use subsequent sections. Let G be a graph. For a vertex v of G, we denote by $d_G(v)$ and $N_G(v)$ the degree and the neighborhood of v in G, respectively, and let $N_G[v] = N_G(v) \cup \{v\}$. For an integer l, let $V_l(G) = \{v \in V(G) \mid d_G(v) = l\}$, and let $V_{\geq l}(G) = \bigcup_{m \geq l} V_m(G)$ and $V_{\leq l}(G) = \bigcup_{m \leq l} V_m(G)$. For a subgraph H of G and a vertex v in $G - H$, let $N_H(v) = N_G(v) \cap V(H)$. For subgraphs H and F of G with $V(F) \cap V(H) = \emptyset$, we define $N_H(F) = \bigcup_{v \in V(F)} N_H(v)$. We use $L(G)$ for the line graph of G. Let $e \in E(G)$. We denote by v_e a vertex in $L(G)$ corresponding to e. Let $V(e)$ be the set of end vertices of e, and we define $E_G(e) = \{f \in E(G) \mid V(f) \cap V(e) \neq \emptyset\}$. The *edge degree* of e in G is defined by the number of elements of $E_G(e) - \{e\}$, i.e., the number of edges incident with e. Note that for a graph G, the minimum edge degree of G is d if and only if the minimum degree of $L(G)$ is d. For subsets X and Y of $V(G)$ with $X \cap Y = \emptyset$, let $E_G(X, Y)$ denote the set of edges between X and Y, and let $e_G(X, Y) = |E_G(X, Y)|$. We often identify a subgraph H of G with its vertex set $V(H)$. For example, we write $E_G(H, F)$ instead of $E_G(V(H), V(F))$ for two disjoint subgraphs H and F of G. For a graph H and an edge set X, $H + X$ means the graph with vertex set $V(H) \cup \left(\bigcup_{e \in X} V(e)\right)$ and the edge set $E(H) \cup X$. For a subgraph H of G, let $E_G(H) = E(G[V(H)]) \cup E_G(H, G - H)$. A *star* is a graph consisting of a vertex and edges incident with the vertex (note that a star is not necessary a tree in this paper).

3 Closure

In this and the next sections, we will prove Theorems 1 and 2. In order to prove them, here we consider a new statement and divide the proof into two theorems. Before mentioning those, we need some definitions.

A connected graph T is called a *closed trail* (abbreviated as CT) if all vertices of T have even degree in T. Let H be a multigraph, and let T be a CT of H. We call T a *dominating closed trail* of H if $H - T$ is edgeless (in case that T is a cycle, we call T a *dominating cycle*), and T is said to be *edge-maximal* if there exists no closed trail T' of H such that $E_H(T) \subsetneq E_H(T')$. Note that a dominating CT of H is an edge-maximal CT of H. In [9], it is shown that for a connected multigraph H with $|E(H)| \geq 3$, H has a dominating CT if and only if $L(H)$ is Hamiltonian. Hence by the definition of an edge-maximal CT, we can easily obtain the following lemma.

Lemma 1 *Let H be a graph, and let T be an edge-maximal CT of H and $H^* = H[V(T)] + E_H(T, H - T)$. Then $L(H^*)$ has a Hamiltonian cycle which is a maximal cycle of $L(H)$.*

Let H be a graph with $|E(H)| \geq 3$. For a closed trail T of H, T is called a *Tutte closed trail* of H if (i) $E_H(T) = E(H)$, or (ii) $|E_H(T)| \geq 4$ and $e_H(F, T) \leq 3$ for every component F of $H - T$, and T is called a *weakly Tutte closed trail* of H if (i) $E_H(T) = E(H)$, or (ii)

$|E_H(T)| \geq 4$ and $e_H(F,T) \leq 3$ for all $F \in \mathcal{F}_H(T)$, where let $\mathcal{F}_H(T) = \{F \mid F$ is a component of $H - T$ with $|V(F)| \geq 2\}$. If T is a Tutte closed trail (resp. a weakly Tutte closed trail) and an edge-maximal closed trail of H, then we call T a *Tutte* (resp. a *weakly Tutte*) *edge-maximal closed trail* of H. Furthermore, we need the following terminology and notation. Now let H be a connected multigraph. For an edge-cut set X of H, X is called an *essential k-edge-cut set* of H if $|X| = k$ and $G - X$ has exactly two components of orders at least 2. We define $\mathcal{E}_k(H) = \{X \subseteq E(H) \mid X$ is an essential k-edge-cut set of $H\}$. For an integer $k \geq 2$, H is called *essentially k-edge-connected* if $|E(H)| \geq k + 1$ and $\mathcal{E}_l(H) = \emptyset$ for all $l < k$. It is known that for a multigraph H such that $L(H)$ is not complete, H is essentially k-edge-connected if and only if $L(H)$ is k-connected and that if H is essentially 2-edge-connected and H is not a star, then $H - V_1(H)$ is 2-edge-connected.

We are ready to state a new statement that plays a crucial role in the proofs of Theorems 1 and 2. We also give two theorems.

(A11) *Every essentially* 2*-edge-connected multigraph has a weakly Tutte edge-maximal CT.*

Theorem 3 *If statement* (A1) *is true, then statement* (A11) *is also true.*

Theorem 4 *If statement* (A11) *is true, then statement* (A10) *is also true.*

Here we prove Theorems 1 and 2 assuming Theorems 3 and 4.

Proof of Theorem 1. It is clear that statement (A10) implies statement (A9) and statement (A9) implies statement (A1). On the other hand, if statement (A1) is true, then by Theorem 3, statement (A11) is true, and by Theorem 4, statement (A10) is also true. This completes the proofs of Theorems 1 and 2. $\square$

Thus, to prove Theorems 1 and 2, it suffices only to show Theorems 3 and 4. We will prove Theorems 3 and 4 in the next section and in the rest of this section, respectively. Notice that by Theorems 3 and 4, we have that statement (A11) is also equivalent to statement (A1).

Before preparing some results to prove Theorem 4, we also state other statements and a theorem as follows.

(A12) *Every essentially* 2*-edge-connected multigraph has a weakly Tutte CT.*

(A13) *Every essentially* 2*-edge-connected multigraph has a Tutte CT.*

Theorem 5 *If statement* (A12) *is true, then statement* (A13) *is also true.*

We can easily see that statement (A11) implies statement (A12). Moreover, by the definition of a Tutte CT, it is easy to check that statement (A13) implies statement (A5) "every essentially 4-edge-connected graph has a dominating CT". Therefore, combining this with Theorems C, 3 and 5, we have that statement (A1) is also equivalent to statements (A12) and (A13). Note that it is not necessary to prove Theorem 5 for the proofs of Theorems 3 and 4, but we prove it since it may itself be interesting (we will prove Theorem 5 in Section 5).

Now we introduce some concepts to prove Theorem 4. We use Ryjáček closure [17] and certain cycles with a particular property. In [17], Ryjáček introduced the concept of a closure for claw-free graphs as follows. For a vertex v of a graph G, we call v a *locally connected vertex* of G if $G[N_G(v)]$ is connected. For a locally connected vertex v of a graph G, we call v an *eligible vertex* of G if $G[N_G(v)]$ is not compete. Let G be a claw-free graph. For an eligible vertex v of G, the operation of adding all possible edges between vertices in $N_G(v)$ is called *local completion* at v. In [17], it is shown that this operation preserves the claw-freeness of the original graph. Iterating local completions as long as possible, we obtain the graph G^* in which $G^*[N_{G^*}(v)]$ is a complete graph for every locally connected vertex v, i.e., there is no eligible vertex in G^*. We call this graph the *closure* of G, and denote it $\mathrm{cl}(G)$. In [17], it is shown that the closure of a graph has the following property.

Theorem E (Ryjáček [17]) *Let G be a claw-free graph. Then the following hold.*

(i) $\mathrm{cl}(G)$ *is well-defined, (i.e., uniquely defined).*

(ii) *There exists a triangle-free simple graph H such that $L(H) = \mathrm{cl}(G)$.*

(iii) *The length of a longest cycle in G and in $\mathrm{cl}(G)$ is the same.*

To obtain Theorem E (iii), Ryjáček actually proved the following, where for an eligible vertex v of a claw-free graph G, let G_v be the graph obtained from G by local completion at v.

Proposition F (Ryjáček [17]) *Let G be a claw-free graph and v be an eligible vertex of G. If C' is a longest cycle of G_v, then G has a cycle C such that $V(C) = V(C')$.*

We further consider about certain cycles with a particular property as follows. For a vertex v of G, we call v a *simplicial vertex* of G if $G[N_G(v)]$ is complete. A cycle C of a claw-free graph G is called a *cycle with* $(*)$*-property* if C satisfies the following property:

$$(*)\qquad \begin{array}{l}\text{for every vertex } x \text{ in } C \text{ such that } x \text{ is a simplicial vertex of } \mathrm{cl}(G),\\ \qquad\qquad \text{if } E(G[N_G[x]]) \cap E(C) \neq \emptyset, \text{then } N_G[x] \subseteq V(C).\end{array}$$

Proposition F might not hold for a cycle C' which is not a longest cycle of G_v. However, in the proof of Proposition F, the maximality of $|V(C')|$ is only used for the fact that $N_{G_v}[v] \subseteq V(C')$ if $E(G_v[N_{G_v}[v]]) \cap E(C') \neq \emptyset$. Therefore by the definition of the $(*)$-property, the same argument can work in the proof of the following proposition. Note that every eligible vertex of G is a simplicial vertex of $\mathrm{cl}(G)$. Note also that $N_G(x) \subseteq N_{G_v}(x)$ for all $x \in V(G)$, and hence the $(*)$-property is a heredity property from G_v to G.

Proposition 6 *Let G be a claw-free graph and v be an eligible vertex of G. If C' is a cycle with $(*)$-property of G_v, then G has a cycle C with $(*)$-property such that $V(C) = V(C')$.*

As a corollary of Proposition 6, we can obtain the following, where for convenience, we call a cycle C of a graph G a *Tutte maximal cycle* of G if C is a Tutte cycle and a maximal cycle of G. Note that if C' is a Tutte cycle (resp. a maximal cycle) of G_v, then C is a Tutte cycle (resp. a maximal cycle) of G for any cycle C in G such that $V(C) = V(C')$.

Corollary 7 *Let G be a claw-free graph. If $\mathrm{cl}(G)$ has a Tutte maximal cycle with $(*)$-property, then G has a Tutte maximal cycle with $(*)$-property.*

By the definition of a weakly Tutte edge-maximal CT, the following holds.

Proposition 8 *Let G be a claw-free graph, and let H be a graph with $L(H) = \mathrm{cl}(G)$. If H has a weakly Tutte edge-maximal CT, then $L(H)$ has a Tutte maximal cycle with $(*)$-property.*

Proof of Proposition 8. Let T be a weakly Tutte edge-maximal CT of H and $H^* = H[V(T)] + E_H(T, H - T)$. Then by Lemma 1, $L(H^*)$ has a Hamilton cycle C which is a maximal cycle of $L(H)$. Let $e \in E(H^*)$. Then by the definition of H^*, if $e \in E_H(F, T)$ for some $F \in \mathcal{F}_H(T)$, then v_e is not a simplicial vertex of $L(H)$ ($= \mathrm{cl}(G)$). Also, if $e \notin E_H(F, T)$ for all $F \in \mathcal{F}_H(T)$, then $E_H(e) \subseteq E(H^*)$, and hence $N_{L(H)}(v_e) \subseteq V(C)$. This implies that C is a cycle with $(*)$-property of $L(H)$. On the other hand, by the definition of a weakly Tutte CT, $e_H(F, T) \leq 3$ for all $F \in \mathcal{F}_H(T)$. Since $E_H(F) \cap E(H^*) = E_H(F, T)$ for each $F \in \mathcal{F}_H(T)$, we have that $|N_C(L(F))| = |E_H(F) \cap E(H^*)| = e_H(F, T) \leq 3$ for each $F \in \mathcal{F}_H(T)$. Moreover, by again the definition of a weakly Tutte CT, $V(C) = E(H^*) = E_H(T) = E(H)$ or $|V(C)| = |E(H^*)| = |E_H(T)| \geq 4$ holds. These imply that C is a Tutte cycle of $L(H)$. Thus C is a Tutte maximal cycle with $(*)$-property of $L(H)$. $\square$

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Suppose that statement (A11) is true. Let G be a 2-connected claw-free graph. By Theorem E (ii), there exists a triangle-free simple graph H such that $L(H) = \mathrm{cl}(G)$. If $L(H)$ is complete, then $L(H)$ clearly has a Hamilton cycle, and hence by Theorem E (iii), G has a Hamilton cycle, that is, G has a Tutte maximal cycle. Thus we may assume that $L(H)$ is not complete, and hence H is essentially 2-edge-connected. Since we assumed that statement (A11) is true, H has a weakly Tutte edge-maximal CT. Then, by Proposition 8, $L(H)$ has a Tutte maximal cycle with $(*)$-property. Hence by Corollary 7, G has a Tutte maximal cycle. Thus statement (A10) is also true and this completes the proof of Theorem 4. $\square$

4 Proof of Theorem 3

4.1 Set up for the proof of Theorem 3

In the end of this section, we will prove Theorem 3, that is, prove statement (A11) assuming (A1), by induction on the number of elements of $\mathcal{E}_2(H) \cup \mathcal{E}_3(H)$, where H is a given essentially 2-edge-connected multigraph. In order to do that, we need the following for the first step of the induction. Here for a graph H and a subset S of $E(H) \cup V(H)$, a closed trail T of H is a called an *S-closed trail* (abbreviated as S-CT) if $S \subseteq E(T) \cup V(T)$. Furthermore, if T is a dominating closed trail (resp. a weakly Tutte closed trail) and an S-closed trail of H, we call T a *dominating* (resp. a weakly Tutte) *S-closed trail* of H.

Lemma 2 *Statement* (A1) *is equivalent to the following statement.*

(A14) *Every essentially* 4*-edge-connected multigraph H has a dominating $V_{\geq 4}(H)$-CT, i.e., H has a Tutte edge-maximal CT.*

Proof of Lemma 2. By Theorem C, it is easy to see that statement (A14) implies statement (A1). So it suffices to show the converse. Assume that statement (A1) is true. Then by Theorem C, every essentially 4-edge-connected graph has a dominating CT. Let H be an essentially 4-edge-connected multigraph. Let H^* be the graph obtained from H by adding a pendant edge to each vertex in $V_{\geq 4}(H)$. Then H^* is also essentially 4-edge-connected and $V_{\geq 4}(H^*) = V_{\geq 4}(H)$. By the assumption, H^* has a dominating closed trail T. Since each vertex in $V_{\geq 4}(H^*)$ is incident with a pendant edge, $V_{\geq 4}(H^*) \subseteq V(T)$. Therefore by the definition of H^*, since $V_{\geq 4}(H^*) = V_{\geq 4}(H)$, we have that T is a dominating $V_{\geq 4}(H)$-CT of H. $\square$

We next prepare some results to prove the case of $\mathcal{E}_2(H) = \emptyset$ and $\mathcal{E}_3(H) \neq \emptyset$. To show this case, we actually consider about weakly Tutte closed trails passing through specified vertices and edges. Before mentioning the statement, we prepare the following terminology. Let H be a multigraph. For three distinct edges e_1, e_2 and e_3 in H, (e_1, e_2, e_3) is called a 3-*star* of H if there exists a vertex u of H such that $d_H(u) = 3$, $u \in V(e_1) \cap V(e_2) \cap V(e_3)$ and $V(e_3) - \{u\} \subseteq V_{\geq 3}(H)$, and u is called the *center* of (e_1, e_2, e_3).

(A15) *Let H be an essentially* 4*-edge-connected multigraph, and let (e_1, e_2, e_3) be a* 3*-star of H. Then H has a dominating $\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)$-CT.*

In order to consider statement (A15), we need the concept called "$V_2(H)$-*dominated*". A graph H is said to be $V_2(H)$-*dominated* if for any distinct four vertices u_1, u_2, v_1 and v_2 in H with $\{u_1, u_2, v_1, v_2\} = V_2(H)$, the graph $H + \{u_1u_2, v_1v_2\}$ has a dominating $\{u_1u_2, v_1v_2\}$-CT. The following was proven by Kužel [13].

Theorem G (Kužel [13]) *Statement* (A1) *is equivalent to the following statement.*

(A16) *Any subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated.*

Actually, we show the following theorem in this section.

Theorem 9 *If statement* (A16) *is true, then statement* (A15) *is also true.*

We prove Theorem 9 in the next subsection and prove Theorem 3 in Subsections 4.3 and 4.4. At the end of this subsection, we give another theorem as follows.

Theorem 10 *If statement* (A15) *is true, then statement* (A1) *is also true.*

Combining Theorem 10 with Theorems G and 9, statement (A1) is also equivalent to statement (A15). Note that it is not necessary to prove Theorem 10 for the proof of Theorem 9, but we prove it since it may itself be interesting (we will prove Theorem 10 in Section 5).

4.2 Proof of Theorem 9

We first prove Theorem 9. We need some concepts and results.

Let $k \geq 3$ be an integer, and let H be an essentially 3-edge-connected graph such that $L(H)$ is not complete. Note that $V_{\leq 2}(H)$ is an independent set of H. The *core* of a graph H denoted by $\mathrm{core}(H)$, is the graph obtained by recursively deleting all vertices of degree 1, recursively deleting a vertex z with degree 2 in H and adding the edge xy with $N_H(z) = \{x, y\}$, and recursively deleting the created loops. It is easy to see that if H is an essentially k-edge-connected graph such that $L(H)$ is not complete, then $\mathrm{core}(H)$ is a 3-edge-connected essentially k-edge-connected multigraph (in particular, $\delta(\mathrm{core}(H)) \geq 3$). Moreover we can see that the following holds.

Lemma 3 *Let H be an essentially 4-edge-connected graph such that $L(H)$ is not complete, and let $H^* = \mathrm{core}(H)$. Suppose that H^* has a dominating $V_{\geq 4}(H^*)$-closed trail T^*. Then H has a dominating $V_{\geq 4}(H)$-closed trail T which satisfies the following*:

- *If $xy \in E(T^*)$, then $xy \in E(T)$ or $xz, yz \in E(T)$ for some $z \in V_2(H)$.*

Proof of Lemma 3. By the definition of a core, for each $xy \in E(H^*)$, $xy \in E(H)$ or there exists a vertex z in $V_2(H)$ such that $xz, yz \in E(H)$. Let $X = \{e \in E(H^*) \mid e \notin E(H)\}$. For each $e = xy \in X$, let z_e be a vertex in $V_2(H)$ such that $N_H(z_e) = \{x, y\}$. Then by replacing e with a path xz_ey for each $e = xy \in E(T^*) \cap X$, we can obtain a closed trail T of H such that $V(T) = V(T^*) \cup \{z_e \mid e \in E(T^*) \cap X\}$ and $E(T) = \{xz_e, yz_e \mid e = xy \in E(T^*) \cap X\} \cup (E(T^*) - X)$. Moreover, since $V_{\geq 4}(H^*) = V_{\geq 4}(H)$ by the definition of a core and the assumption, $V_{\geq 4}(H) = V_{\geq 4}(H^*) \subseteq V(T^*) \subseteq V(T)$. Therefore, to complete the proof, we have only to prove that T is a dominating CT of H. Note that $|E(H)| \geq 5$ because H is essentially 4-edge-connected. Let $x \in V(H-T)$. Since $V(T^*) \subseteq V(T)$, $x \notin V(T^*)$. Suppose that

$N_H(x) \not\subseteq V(T)$, and let $z \in N_H(x) - V(T)$. If $\{x, z\} \subseteq V_{\geq 3}(H)$, then by the definition of a core, $\{x, z\} \subseteq V(H^*)$ and $xz \in E(H^*)$. Since $x, z \notin V(T^*)$, this contradicts that T^* is a dominating CT of H^*. Thus $\{x, z\} \cap V_{\leq 2}(H) \neq \emptyset$. Since H is essentially 4-edge-connected and $L(H)$ is not complete, we also have that $\{x, z\} \cap V_{\geq 3}(H) \neq \emptyset$. Since $x, z \in V(H - T)$ and $xz \in E(H)$, we may assume that $x \in V_{\geq 3}(H)$ and $z \in V_{\leq 2}(H)$. Since $V_{\geq 4}(H) \subseteq V(T)$, $x \in V_3(H)$. Then $E_H(xz) - \{xz\} \in \mathcal{E}_2(H) \cup \mathcal{E}_3(H)$, a contradiction. Thus $N_H(x) \subseteq V(T)$. Since x is an arbitrary vertex in $H - T$, this implies that T is a dominating CT of H. $\Box$

We also need the following operation (see [8] for more details). Let H be a graph and $z \in V_{\geq 4}(H)$, and let $u_1, u_2, \ldots, u_d$ ($d = d_H(z)$) be an ordering of neighbors of z (we allow repetition in case of parallel edges). Then the graph H_z obtained from the disjoint union of $G - z$ and the cycle $C_z = z_1 z_2 \ldots z_d z_1$ by adding the edges $u_i z_i$ for each $1 \leq i \leq d$ is called an *inflation* of H at z. If $\delta(H) \geq 3$, then, by successively taking an inflation at each vertex of degree greater than 3, we can obtain a cubic graph H^I, called a *cubic inflation* of H. An inflation of a graph at a vertex is not unique (since it depends on the ordering of neighbors of z) and the operation may decrease the edge-connectivity. However, the following was proven in [8].

Theorem H (Fleischner and Jackson [8]) *Let H be an essentially 4-edge-connected graph with $\delta(H) \geq 3$. Then some cubic inflation of H is also essentially 4-edge-connected.*

Let H^I be a cubic inflation of a graph H and for each $z \in V(H)$, set $I(z) = V(C_z)$ if $z \in V_{\geq 4}(H)$; otherwise, set $I(z) = \{z\}$. Observing that a dominating cycle in H^I must contain at least one vertex in $I(z)$ for each $z \in V_{\geq 4}(H)$, we immediately have the following fact (which is implicit in [8]).

Lemma I (Fleischner and Jackson [8]) *Let H be a graph with $\delta(H) \geq 3$, and let H^I be a cubic inflation of H. Suppose that H^I has a dominating cycle C. Then H has a dominating $V_{\geq 4}(H)$-closed trail T which satisfies the following*:

- *If $uv \in E(C)$ with $u \in I(x)$ and $v \in I(y)$ for some $x, y \in V(H)$ ($x \neq y$), then $xy \in E(T)$.*

Proof of Theorem 9. Suppose that statement (A16) is true. Let H be an essentially 4-edge-connected multigraph, and let (e_1, e_2, e_3) be a 3-star of H (note that $V(e_3) \subseteq V_{\geq 3}(H)$ and that $V(e_1) \cup V(e_2) \subseteq V_{\geq 2}(H)$ because H is essentially 4-edge-connected). We will find a dominating $\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)$-CT of H.

If $L(H)$ is complete, then we can easily see that (i) H is a star such that $V(e_1) = V(e_2) = V(e_3)$, or (ii) H is a triangle such that e_3 is an unique simple edge in H or $V(e_i) = V(e_3)$ and $V(e_{3-i}) \neq V(e_3)$ for some $i = 1$ or 2. In either case, clearly H has a spanning closed trail T such that $\{e_1, e_2\} \subseteq E(T)$, that is, H has a desired closed trail.

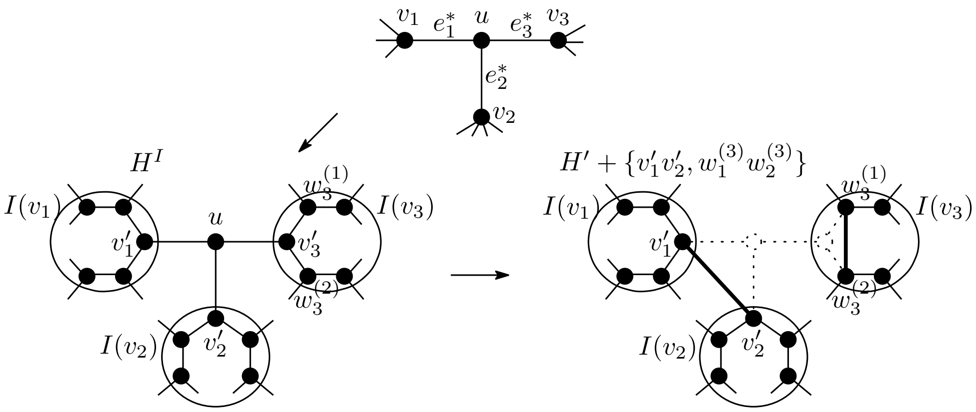

Figure 1: The subgraph H' of H^I

Thus we may assume that $L(H)$ is not complete. Let u be the center of (e_1, e_2, e_3). Let $H^* = \mathrm{core}(H)$. Then H^* is an essentially 4-edge-connected graph with $\delta(H^*) \geq 3$. Note that $e_3 \in E(H^*)$ since $V(e_3) \subseteq V_{\geq 3}(H)$. Let e_1^* and e_2^* be two distinct edges incident with u in H^* such that $e_i^* \neq e_3$ for each $i = 1, 2$, and let $e_3^* = e_3$. Note that (e_1^*, e_2^*, e_3^*) is a 3-star with center u of H^*.

By Theorem H, there exists a cubic inflation H^I of H^* such that H^I is essentially 4-edge-connected. Note that H^I is a simple graph. Note also that by the definition of a 3-star, $I(u) = \{u\}$. For each i with $1 \leq i \leq 3$, let $v_i \in V(e_i^*) - \{u\}$, and let $v_i' \in I(v_i)$ such that $uv_i' \in E(H^I)$. We claim that H^I has a dominating cycle containing uv_1', uv_2' and v_3'. Since H^I is essentially 4-edge-connected, if $v_k'v_l' \in E(H^I)$ for some k and l with $1 \leq k < l \leq 3$, then it is easy to check that $H^I \cong K_4$, and hence H^I has a desired dominating cycle. Thus we may assume that $v_k'v_l' \notin E(H^I)$ for each k and l with $1 \leq k < l \leq 3$.

Let $\{w_1^{(3)}, w_2^{(3)}\} = N_{H^I}(v_3') - \{u\}$. Then since $H' := H^I - \{u, v_3'\}$ is a subgraph of H^I such that $\delta(H') = 2$ and $V_2(H') = \{v_1', v_2', w_1^{(3)}, w_2^{(3)}\}$ and we assumed that statement (A16) is true, $H' + \{v_1'v_2', w_1^{(3)}w_2^{(3)}\}$ has a dominating cycle C' containing $v_1'v_2'$ and $w_1^{(3)}w_2^{(3)}$ (see Figure 1). Hence $(C' - \{v_1'v_2', w_1^{(3)}w_2^{(3)}\}) + \{uv_1', uv_2', v_3'w_1^{(3)}, v_3'w_2^{(3)}\}$ is a desired dominating cycle of H^I. Thus the assertion holds. Then by Lemma I, H^* has a dominating $\{e_1^*, e_2^*\} \cup V(e_3^*) \cup V_{\geq 4}(H^*)$-CT. Hence by Lemma 3 and the definition of e_1^*, e_2^* and e_3^*, H has a dominating $\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)$-CT. Therefore, statement (A15) is true, and this completes the proof of Theorem 9. $\square$

4.3 Preparation for the proof of Theorem 3

In this subsection, we prepare some technical lemmas to prove Theorem 3.

In the proof of Theorem 3, we will restrict maximal cycles on H to some component. To show that the resulting graph is a weakly Tutte CT, we use the following lemma.

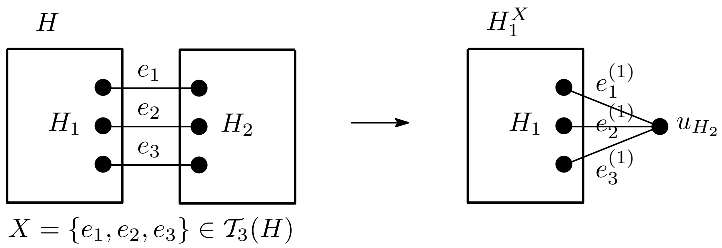

Figure 2: The graph H_1^X

Lemma 4 *Let H be a graph, and let T be a weakly Tutte CT of H. If T' is a CT of H such that $E_H(T') = E_H(T)$, then T' is also a weakly Tutte CT of H.*

Proof of Lemma 4. Let T' be a CT of H such that $E_H(T') = E_H(T)$, and suppose that T' is not a weakly Tutte CT of H. Then there exists $F' \in \mathcal{F}_H(T')$ with $e_H(F', T') \geq 4$. Write $E_H(F', T') = \{e_1, \ldots, e_l\}$ $(l \geq 4)$. Since $E_H(F', T') \subseteq E_H(T') = E_H(T)$, $V(T) \cap \left(\bigcup_{i=1}^{l} V(e_i)\right) \neq \emptyset$. Let $S = V(T) \cap \left(\bigcup_{i=1}^{l} V(e_i)\right)$, and suppose that $S \subseteq V(T') \cap \left(\bigcup_{i=1}^{l} V(e_i)\right)$. Then $\{e_1, \ldots, e_l\} = E_H(F', T') \subseteq E_H(T, H - T)$ and there exists a component F of $H - T$ such that $V(F') \subseteq V(F)$, which contradicts the assumption that T is a weakly Tutte CT of H. Thus $S \cap V(F') \cap \left(\bigcup_{i=1}^{l} V(e_i)\right) \neq \emptyset$, and hence $E(F') \cap E_H(T) \neq \emptyset$. Since $E(F') \cap E_H(T') = \emptyset$, this contradicts the assumption that $E_H(T') = E_H(T)$. $\square$

In the rest of this subsection, we fix the following notation. Let k be an integer with $2 \leq k \leq 3$, and let H be an essentially k-edge-connected graph.

To prove Theorem 3, we prepare the following terminology and notation. Let $\mathcal{T}_k(H) = \{(X, H_1, H_2) \mid X \in \mathcal{E}_k(H)$ and, H_1 and H_2 are distinct components of $G - X\}$. Let $(X, H_1, H_2) \in \mathcal{T}_k(H)$. We define two graphs H_1^X and H_2^X as follows. For each $i = 1, 2$, let H_i^X be the graph obtained from H by contracting H_{3-i} to a vertex $u_{H_{3-i}}$. Note that H_i^X is also an essentially k-edge-connected multigraph. If $X = \{e_1, \ldots, e_k\}$, then for each i, j with $1 \leq i \leq 2$ and $1 \leq j \leq k$, let $e_j^{(i)}$ be the edge in H_i^X corresponding to e_j (see Figure 2).

Now we fix the following notation. Let $(X, H_1, H_2) \in \mathcal{T}_k(H)$, and write $X = \{e_1, \ldots, e_k\}$.

Lemma 5 *Let $1 \leq i \leq 2$. If H_i^X has a weakly Tutte edge-maximal closed trail T_i such that $E(T_i) \cap \{e_1^{(i)}, \ldots, e_k^{(i)}\} = \emptyset$, then T_i is a weakly Tutte edge-maximal closed trail of H, or H_i^X has a weakly Tutte edge-maximal closed trail R_i such that $E(R_i) \cap \{e_1^{(i)}, \ldots, e_k^{(i)}\} \neq \emptyset$.*

Proof of Lemma 5. We may assume that $i = 1$. Note that T_1 is a weakly Tutte CT of H because $E(T_1) \cap \{e_1^{(1)}, \ldots, e_k^{(1)}\} = \emptyset$. Suppose that T_1 is not a weakly Tutte edge-maximal CT of H. Then there exists an edge-maximal closed trail T' of H such that $E_H(T_1) \subsetneq E_H(T')$. Note that $E(T') \cap X \neq \emptyset$ because T_1 is an edge-maximal CT of H_1^X such that $E(T_1) \cap \{e_1^{(1)}, \ldots, e_k^{(1)}\} = \emptyset$. Note also that $|E(T') \cap X| = 2$ because $2 \leq k \leq 3$. We may assume that $E(T') \cap X = \{e_1, e_2\}$, and

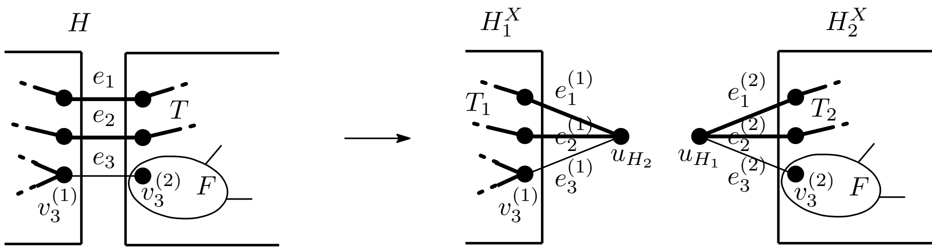

Figure 3: The component F of $H-T$

let $R_1=(T'-V(H_2))+\{e_1^{(1)},e_2^{(1)}\}$. Then R_1 is a CT of H_1^X. Since $E_H(T_1)\subseteq E_H(T')$, $E_{H_1^X}(T_1)-\{e_1^{(1)},\dots,e_k^{(1)}\}=E_H(T_1)\cap E(H_1)\subseteq E_H(T')\cap E(H_1)$. Moreover, by the definition of R_1 and since $\{e_1^{(1)},\dots,e_k^{(1)}\}\subseteq E_{H_1^X}(R_1)$ because $u_{H_2}\in V(R_1)$, $(E_H(T')\cap E(H_1))\cup\{e_1^{(1)},\dots,e_k^{(1)}\}=E_{H_1^X}(R_1)$. This implies that $E_{H_1^X}(T_1)\subseteq E_{H_1^X}(R_1)$. Since T_1 is an edge-maximal CT of H_1^X, we have that $E_{H_1^X}(T_1)=E_{H_1^X}(R_1)$, and hence R_1 is also an edge-maximal CT of H_1^X. Furthermore, since T_1 is a weakly Tutte CT of H_1^X and $E_{H_1^X}(T_1)=E_{H_1^X}(R_1)$, it follows from Lemma 4 that R_1 is also a weakly Tutte CT of H_1^X. Thus R_1 is a weakly Tutte edge-maximal CT of H_1^X such that $E(R_1)\cap\{e_1^{(1)},\dots,e_k^{(1)}\}\neq\emptyset$. $\square$

We further fix the following notation in the following three lemmas (Lemmas 6 through 8). Let $e_i=v_i^{(1)}v_i^{(2)}$ with $v_i^{(1)}\in V(H_1)$ and $v_i^{(2)}\in V(H_2)$ for each $1\le i\le k$. Let l_1 and l_2 be integers with $1\le l_1<l_2\le k$, and for each $i=1,2$, let T_i be a $\{e_{l_1}^{(i)},e_{l_2}^{(i)}\}$-CT of H_i^X and $T=\big((T_1-u_{H_2})\cup(T_2-u_{H_1})\big)+\{e_{l_1},e_{l_2}\}$.

Lemma 6 *If T_i is a weakly Tutte CT of H_i^X for each $i=1,2$ and $\{v_1^{(i)},\dots,v_k^{(i)}\}\subseteq V(T_i)$ for some $i=1$ or 2, then T is a weakly Tutte CT of H.*

Proof of Lemma 6. We may assume that $l_1=1$ and $l_2=2$, and hence $\{v_1^{(i)},v_2^{(i)}\}\subseteq V(T_i)$ for each $i=1,2$. By the symmetry of T_1 and T_2, we also may assume that $\{v_1^{(1)},\dots,v_k^{(1)}\}\subseteq V(T_1)$. Let F be a component of $H-T$. Since $\{v_1^{(1)},\dots,v_k^{(1)}\}\subseteq V(T_1)-\{u_{H_2}\}\subseteq V(T)$ and $\{e_1,e_2\}\subseteq E(T)$, we have that if $v_k^{(2)}\notin V(F)$, then F is a component of $H_i^X-T_i$ for some $i=1$ or 2, and hence $E_H(F,T)=E_{H_i^X}(F,T_i)$ for some $i=1$ or 2; if $v_k^{(2)}\in V(F)$ (note that in this case, $k=3$), then F is a component of $H_2^X-T_2$ and $e_k^{(2)}\in E_{H_2^X}(F,T_2)$, and hence $E_H(F,T)=\big(E_{H_2^X}(F,T_2)-\{e_k^{(2)}\}\big)\cup\{e_k\}$ (see Figure 3). Since T_i is a weakly Tutte CT of H_i^X for each $i=1,2$, this implies that T is a weakly Tutte CT of H. $\square$

Lemma 7 *If T_i is an edge-maximal CT of H_i^X for each $i=1,2$ and $\{v_1^{(i)},\dots,v_k^{(i)}\}\subseteq V(T_i)$ for some $i=1$ or 2, then T is an edge-maximal CT of H.*

Proof of Lemma 7. If $\{v_1^{(1)},\dots,v_k^{(1)}\}\subseteq V(T_1)$, then let $A=\{v_1^{(1)},\dots,v_k^{(1)}\}$; otherwise, let

$A = \{v_1^{(2)}, \dots, v_k^{(2)}\}$. Suppose that T is not an edge-maximal CT of H. Then there exists an edge-maximal closed trail T' of H such that $E_H(T) \subsetneq E_H(T')$. Note that $E(T') \cap X \neq \emptyset$. Let m_1 and m_2 be integers with $1 \leq m_1 < m_2 \leq k$ such that $E(T') \cap X = \{e_{m_1}, e_{m_2}\}$. For each $i = 1, 2$, let $R_i = (T' - V(H_{3-i})) + \{e_{m_1}^{(i)}, e_{m_2}^{(i)}\}$. Then R_i is a CT of H_i^X for each $i = 1, 2$. Let $1 \leq i \leq 2$. Since $E_H(T) \subseteq E_H(T')$, we have that $E_{H_i^X}(T_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\} = E_H(T) \cap E(H_i) \subseteq E_H(T') \cap E(H_i) = E_{H_i^X}(R_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\}$. Since $\{e_1^{(i)}, \dots, e_k^{(i)}\} \subseteq E_{H_i^X}(T_i) \cap E_{H_i^X}(R_i)$ because $u_{H_{3-i}} \in V(T_i) \cap V(R_i)$, this implies that $E_{H_i^X}(T_i) \subseteq E_{H_i^X}(R_i)$. Since T_i is an edge-maximal CT of H_i^X, we obtain $E_{H_i^X}(T_i) = E_{H_i^X}(R_i)$, i.e., $E_{H_i^X}(T_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\} = E_{H_i^X}(R_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\}$. Since i is an arbitrary integer with $1 \leq i \leq 2$, $E_{H_i^X}(T_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\} = E_{H_i^X}(R_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\}$ holds for each $i = 1, 2$. On the other hand, since $A \subseteq (V(T_1) - \{u_{H_2}\}) \cup (V(T_2) - \{u_{H_1}\}) = V(T)$, $X \subseteq E_H(T)$, and hence $X \subseteq E_H(T')$. Thus we obtain $E_H(T) = (E_{H_1^X}(T_1) - \{e_1^{(1)}, \dots, e_k^{(1)}\}) \cup (E_{H_2^X}(T_2) - \{e_1^{(2)}, \dots, e_k^{(2)}\}) \cup X = (E_{H_1^X}(R_1) - \{e_1^{(1)}, \dots, e_k^{(1)}\}) \cup (E_{H_2^X}(R_2) - \{e_1^{(2)}, \dots, e_k^{(2)}\}) \cup X = E_H(T')$, a contradiction. $\square$

We call $(X, H_1, H_2) \in \mathcal{T}_k(H)$ a *minimal* 3*-tuple* of H if there exists no $X' \in \mathcal{E}_k(H)$ such that $H - X'$ has a component H_2' such that $V(H_2') \subsetneq V(H_2)$. Then by the definition of a minimal 3-tuple, we can obtain the following.

Lemma 8 *Suppose that $k = 3$ and (X, H_1, H_2) is a minimum 3-tuple of H. If $d_H(v_j^{(2)}) = 2$ for some j with $1 \leq j \leq 3$, then H_2 is isomorphic to K_2.*

Proof of Lemma 8. We may assume that $j = 3$. Since H is essentially 3-edge-connected, $X \in \mathcal{E}_3(H)$ and $d_H(v_3^{(2)}) = 2$, it follows that there exists an unique vertex v' in $N_H(v_3^{(2)}) \cap V(H_2)$. Note that $v' \in V_{\geq 3}(H)$ and $H_2 - v_3^{(2)}$ is connected. Then $X' := \{e_1, e_2, v_3^{(2)} v'\}$ is an edge-cut set of H, and $H_1 + \{e_3\}$ and $H_2 - v_3^{(2)}$ are components of $H - X'$. Therefore, since (X, H_1, H_2) is a minimal 3-tuple of H, we have $|V(H_2 - v_3^{(2)})| = 1$. $\square$

4.4 Proof of Theorem 3

We finally prove Theorem 3.

Proof of Theorem 3. Assume that statement (A1) is true. Let H be an essntially 2-edge-connected multigraph. We will prove that H has a weakly Tutte edge-maximal CT by induction on $g(H) := |\mathcal{E}_2(H) \cup \mathcal{E}_3(H)|$. If $g(H) = 0$, then H is essentially 4-edge-connected. By the assumption that statement (A1) is true and Lemma 2, H has a desired CT, and we are done. Hence we may assume that $g(H) \geq 1$.

By way of a contradiction, suppose that

$$H \text{ has no weakly Tutte edge-maximal CT.} \tag{4.1}$$

Suppose first that $\mathcal{E}_2(H) \neq \emptyset$, let $(X, H_1, H_2) \in \mathcal{T}_2(H)$ and write $X = \{e_1, e_2\}$. Then H_i^X is also essentially 2-edge-connected and $g(H_i^X) < g(H)$ for each $i = 1, 2$. Hence by the induction hypothesis, H_i^X has a weakly Tutte edge-maximal closed trail T_i for each $i = 1, 2$. By Lemma 5 and (4.1), we may assume that $E(T_i) \cap \{e_1^{(i)}, e_2^{(i)}\} \neq \emptyset$ for each $i = 1, 2$, and hence $\{e_1^{(i)}, e_2^{(i)}\} \subseteq E(T_i)$ for each $i = 1, 2$. Then by Lemmas 6 and 7, $T := \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_2\}$ is a weakly Tutte edge-maximal CT of H, which contradicts (4.1) again. Thus $\mathcal{E}_2(H) = \emptyset$.

Then H is essentially 3-edge-connected. Let (X, H_1, H_2) be a minimal 3-tuple of H in $\mathcal{T}_3(H)$. Write $X = \{e_1, e_2, e_3\}$ and $e_i = v_i^{(1)} v_i^{(2)}$ with $v_i^{(1)} \in V(H_1)$ and $v_i^{(2)} \in V(H_2)$ for each $1 \leq i \leq 3$. Note that H_i^X is also essentially 3-edge-connected, and $g(H_i^X) < g(H)$ for each $i = 1, 2$, and hence by the induction hypothesis, H_1^X has a weakly Tutte edge-maximal CT. We define $\mathcal{T} = \{T_1 \mid T_1$ is a weakly Tutte edge-maximal CT of H_1^X such that $E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} \neq \emptyset\}$. By Lemma 5 and (4.1), $\mathcal{T} \neq \emptyset$ (note that $|E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}| = 2$ for all $T_1 \in \mathcal{T}$).

We divide the proof of Theorem 3 into two cases.

Case 1. $d_{H_2^X}(v_j^{(2)}) \geq 3$ for each j with $1 \leq j \leq 3$.

Let $T_1 \in \mathcal{T}$, and we may assume that $E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = \{e_1^{(1)}, e_2^{(1)}\}$. Then by the assumption of Case 1, $(e_1^{(2)}, e_2^{(2)}, e_3^{(2)})$ is a 3-star with center u_{H_1} in H_2^X. Moreover, by the definition of a minimal 3-tuple and since $\mathcal{E}_2(H) = \emptyset$, H_2^X is essentially 4-edge-connected. Since we assumed that statement (A1) is true, it follows from Theorems G and 9 that statement (A15) is also true. Thus H_2^X has a dominating $\{e_1^{(2)}, e_2^{(2)}\} \cup V(e_3^{(2)}) \cup V_{\geq 4}(H_2^X)$-closed trail T_2, i.e., T_2 is a weakly Tutte edge-maximal CT of H_2^X, $\{e_1^{(2)}, e_2^{(2)}\} \subseteq E(T_2)$ and $\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\} \subseteq V(T_2)$. Hence by Lemmas 6 and 7, $T := \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_2\}$ is a weakly Tutte edge-maximal CT of H, which contradicts (4.1).

Case 2. $d_{H_2^X}(v_j^{(2)}) \leq 2$ for some j with $1 \leq j \leq 3$.

We may assume that $d_{H_2^X}(v_3^{(2)}) \leq 2$. Then by the denition of H_2^X and since $X \in \mathcal{E}_3(H)$, $d_H(v_3^{(2)}) = d_{H_2^X}(v_3^{(2)}) = 2$. Hence by Lemma 8, $H_2 \cong K_2$, i.e., $v_1^{(2)} = v_2^{(2)}$ and $v_1^{(2)} \neq v_3^{(2)}$. Let $T_1 \in \mathcal{T}$. We choose T_1 so that $e_3^{(1)} \in E(T_1)$ or $\{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\} \subseteq V(T_1)$ if possible.

Suppose that $e_3^{(1)} \in E(T_1)$. By the symmetry of $e_1^{(1)}$ and $e_2^{(1)}$, we may assume that $E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}\} = \{e_1^{(1)}\}$. Let $T_2 = H_2^X - \{e_2^{(2)}\}$. Then T_2 is clearly a weakly Tutte $\{e_1^{(2)}, e_3^{(2)}\} \cup V(e_2^{(2)})$-CT of H_2^X such that $E_{H_2^X}(T_2) = E(H_2^X)$, i.e., T_2 is a weakly Tutte edge-maximal CT of H_2^X, $\{e_1^{(2)}, e_3^{(2)}\} \subseteq E(T_2)$ and $\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\} \subseteq V(T_2)$. Hence by Lemmas 6 and 7, $T := \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_3\}$ is a weakly Tutte edge-maximal CT of H, which contradicts (4.1). Thus $e_3^{(1)} \notin E(T_1)$, that is, $E(T_1) \cap \{e_1^{(1)}, e_2^{(2)}, e_3^{(2)}\} = \{e_1^{(1)}, e_2^{(2)}\}$.

Let $T_2 = H_2^X - v_3^{(2)}$ and $T = \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_2\}$ ($= (T_1 - u_{H_2}) + \{e_1, e_2\}$). Then T_2 is clearly a weakly Tutte $\{e_1^{(2)}, e_2^{(2)}\}$-CT of H_2^X such that $E_{H_2^X}(T_2) = E(H_2^X)$, i.e., T_2 is a weakly Tutte edge-maximal CT of H_2^X and $\{e_1^{(2)}, e_2^{(2)}\} \subseteq E(T_2)$. Then by Lemma 6, we also have that T is a weakly Tutte CT of H. Hence by Lemma 7 and (4.1), $v_3^{(1)} \notin V(T_1)$ and there exists an edge-maximal closed trail T' of H such that $E_H(T) \subsetneq E_H(T')$. In particular, since

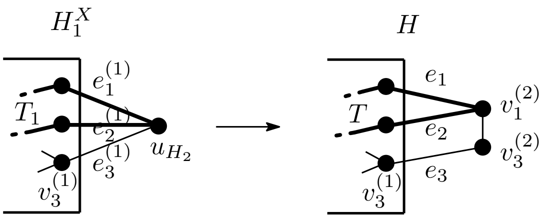

Figure 4: The closed trail T of H

$v_3^{(1)} \notin V(T_1)$,

$$E_H(T) = (E_{H_1^X}(T_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}) \cup \{e_1, e_2, v_1^{(2)}v_3^{(2)}\} \text{ (see Figure 4).} \tag{4.2}$$

Note that since T_1 is an edge-maximal closed trail of H_1^X and $E_H(T) \subsetneq E_H(T')$, $E_H(T') \cap X \neq \emptyset$. Let l_1 and l_2 be integers with $1 \leq l_1 < l_2 \leq 3$ such that $E_H(T') \cap X = \{e_{l_1}, e_{l_2}\}$. Let $R_1 = (T' - V(H_2)) + \{e_{l_1}^{(1)}, e_{l_2}^{(1)}\}$. Then R_1 is a CT of H_1^X. Since $E_H(T) \subseteq E_H(T')$, $E_{H_1^X}(T_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = E_H(T) \cap E(H_1) \subseteq E_H(T') \cap E(H_1) = E_{H_1^X}(R_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}$. Since $\{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} \subseteq E_{H_1^X}(T_1) \cap E_{H_1^X}(R_1)$ because $u_{H_2} \in V(T_1) \cap V(R_1)$, this implies that $E_{H_1^X}(T_1) \subseteq E_{H_1^X}(R_1)$. Since T_1 is an edge-maximal CT of H_1^X, we have $E_{H_1^X}(T_1) = E_{H_1^X}(R_1)$. Since T_1 is a weakly Tutte CT of H_1^X, this together with Lemma 4 implies that R_1 is a weakly Tutte CT of H_1^X. Therefore R_1 is a weakly Tutte edge-maximal CT of H_1^X such that $E(R_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(2)}\} \neq \emptyset$, i.e., $R_1 \in \mathcal{T}$. Then by the choice of T_1, we have that $\{l_1, l_2\} = \{1, 2\}$ and $v_3^{(1)} \notin V(R_1)$. Then by the definition of R_1, $E(T') \cap X = \{e_1, e_2\}$ and $v_3^{(1)} \notin V(T')$. Therefore we obtain

$$E_H(T') = (E_{H_1^X}(R_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}) \cup \{e_1, e_2, v_1^{(2)}v_3^{(2)}\}. \tag{4.3}$$

Since $E_{H_1^X}(T_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = E_{H_1^X}(R_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}$, it follows from (4.2) and (4.3) that $E_H(T) = E_H(T')$, which contradicts the fact that $E_H(T) \subsetneq E_H(T')$.

This completes the proof of Theorem 3. $\square$

5 Proofs of Theorems 5 and 10

As mentioned in the paragraph following Theorem 5 and the paragraph following Theorem 10 in Sections 3 and 4, respectively, we prove Theorems 5 and 10 in this section.

Proof of Theorem 5. Assume that statement (A12) is true. Let H be an essentially 2-edge-connected multigraph. Let H^* be a graph obtained from H by adding a pendant edge to each vertex in $V_{\geq 4}(H)$. Then H^* is also essentially 2-edge-connected and $V_{\geq 4}(H^*) = V_{\geq 4}(H)$. Since we assumed that statement (A12) is true, H^* has a weakly Tutte closed trail T. Then by the definition of H^*, T is also a weakly Tutte CT of H. We show that T is a Tutte CT of H. Suppose

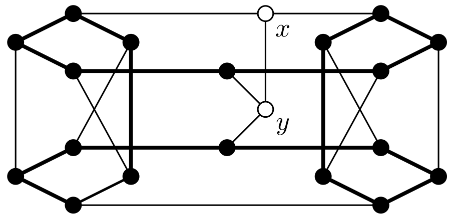

Figure 5: The cubic graph to construct the example

that T is not a Tutte CT of H. Since T is a weakly Tutte CT of H, there exists a component F of $H-T$ such that $|V(F)| = 1$, say $V(F) = \{x\}$, and $x \in V_{\geq 4}(H)$. Then by the definition of H^*, there exists a vertex y in $N_{H^*}(x) \cap V_1(H^*)$. Since $x \notin V(T)$ and $V(H-T) \subseteq V(H^*-T)$, we have that xy is a graph in $\mathcal{F}_{H^*}(T)$ such that $e_{H^*}(\{x,y\},T) = d_H(x) \geq 4$, which contracts that T is a weakly Tutte CT of H^*. Thus T is a Tutte CT of H^*. Hence statement (A13) is also true, and this completes the proof of Theorem 5. $\square$

Proof of Theorem 10. By Lemma 2, it is enough to show that statement (A15) implies statement (A14). Assume that statement (A15) is true. Let H be an essentially 4-edge-connected multigraph. We will find a dominating $V_{\geq 4}(H)$-CT. If $L(H)$ is complete, then H is a star or a triangle, and hence we can easily see that H has a desired dominating CT. Thus, we may assume that $L(H)$ is not complete.

Then $H^* := \mathrm{core}(H)$ is an essentially 4-edge-connected graph with $\delta(H^*) \geq 3$. By Theorem H, there exists a cubic inflation H^I of H^* such that H^I is essentially 4-edge-connected. Since we assumed that statement (A15) is true, taking any vertex in H^I as the center of a 3-star, we can find a dominating cycle of H^I. By Lemma I, H^* has a dominating $V_{\geq 4}(H^*)$-CT. By Lemma 3, H also has a dominating $V_{\geq 4}(H)$-CT. Hence statement (A14) is also true, and this completes the proof of Theorem 10. $\square$

6 Concluding remarks

In 1992, Jackson posed the possible approach to the well-known conjecture on the existence of a Hamilton cycle in 4-connected claw-free graphs (Conjecture A), using a Tutte cycle. Indeed, he conjectured that statement (A9) "every 2-connected claw-free graph has a Tutte cycle" is true (Conjecture D), which directly implies Conjecture A. In this paper, we have concentrated on a Tutte cycle on claw-free graphs and seen that many statements (A1)–(A16) are equivalent (see Theorems B, C, G, 1–5, 9, 10 and Lemma 2).

By the above fact, we have that statement (A10) "every 2-connected claw-free graph has a

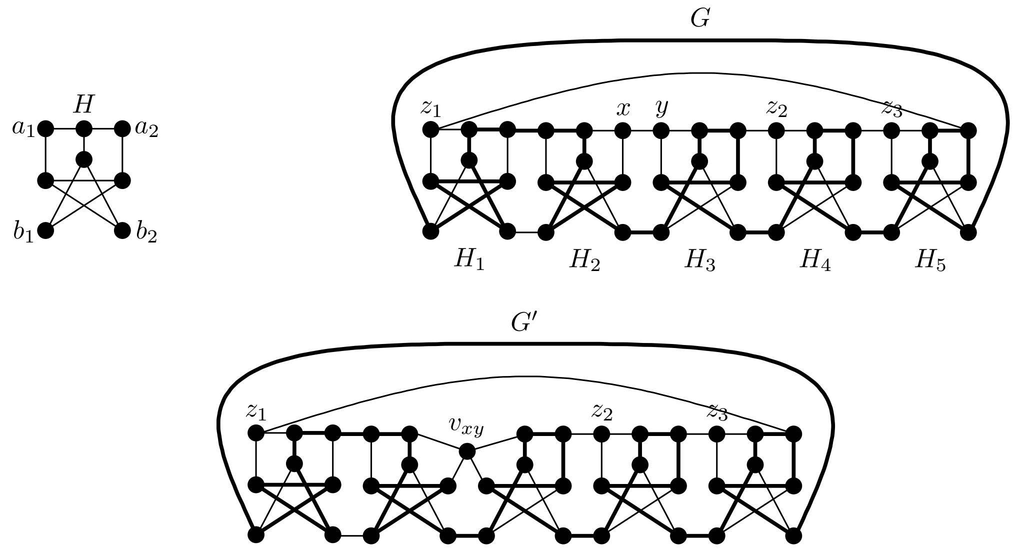

Figure 6: The cubic graph introduced by Kochol [12]

Tutte maximal cycle" is seemingly stronger than statement (A9), that is, if (A9) is true, then we can always take a Tutte cycle so that it is maximal. However, as mentioned in Section 1, it is not always true that a 3-connected claw-free graph has a Tutte cycle which is longest even if statement (A9) is true. The following is the 3-connected claw-free graph showing this. Let G be the graph illustrated in Figure 5. Then it is easy to check that G is an essentially 3-edge-connected (3-connected) cubic graph which is not Hamiltonian. Moreover, the edges depicted in Figure 5 by bold lines induce a cycle C such that $V(C) = V(G) - \{x, y\}$ and C is a maximal cycle of G. Let $d \geq 3$ be an integer. Let G^* be the graph obtained from G by adding $d - 2$ pendant edges to each vertex in $\{x, y\}$ and at least $2d - 2$ pendant edges to each vertex in $V(G) - \{x, y\}$, and let X be the set of pendant edges which are incident with $\{x, y\}$ in G^*. Note that $|X \cup \{xy\}| = 2d - 3$. Then by the definition of G^* and since G is essentially 3-edge-connected, we have that G^* is also essentially 3-edge-connected and the minimum edge degree of G^* is just d. Furthermore, since G is not Hamiltonian and C is a maximal cycle of G satisfying $V(C) = V(G) - \{x, y\}$, for every closed trail (cycle) T of G^* with $T \neq C$, $|E_{G^*}(T)| < |E_{G^*}(C)|$ holds. These imply that $L(G^*)$ is a 3-connected claw-free graph with $\delta(L(G^*)) = d$, and for any longest cycle D of $L(G^*)$, $V(D) = E_{G^*}(C) = E(G^*) - (X \cup \{xy\})$ holds. Since $|E_{G^*}(C) \cap E_{G^*}(xy)| = e_{G^*}(\{x, y\}, V(G^*) - \{x, y\}) = 4$, every cycle D of $L(G^*)$ with $V(D) = E_{G^*}(C)$ is not a Tutte cycle of $L(G^*)$. Thus any Tutte cycle of $L(G^*)$ is not longest.

In addition, if statement (A9) is true, then we can also take Tutte closed trails (weakly Tutte closed trails, weakly Tutte edge-maximal closed trails) in essentially 2-edge-connected graphs (see statements (A11)–(A13)). Moreover, it is also true that every essentially 4-edge-connected graph has a Tutte edge-maximal CT if statement (A9) is true (see statement (A14)).

However, it is not always true that an essentially 3-edge-connected graph has a Tutte edge-maximal CT. We finally give the graph showing this. We use the methods of Kochol [12] for constructions of snarks with a maximal cycle that is not a dominating cycle. (Note that by using this method, we can also construct a 3-connected claw-free graph in which any Tutte cycle is not longest other than the above graph.) Let G be the graph in the right side of Figure 6. It arises from five copies of the graph H $(H_1, H_2, H_3, H_4, H_5)$ illustrated in the left side of Figure 6 after joining the vertices a_i and b_i of degree 2 as in depicted in the figure. Then G is an essentially 3-edge-connected (3-connected) cubic graph and the cycle C depicted by bold lines is a maximal cycle of G such that $V(C) = V(G) - \{x, y, z_1, z_2, z_3\}$. Let G' be the graph obtained from G by contracting xy to a vertex v_{xy} (see Figure 6), and let G^* be the graph obtained from G' by adding a pendant edge to each vertex in $V(G') - \{v_{xy}, z_1, z_2, z_3\}$. Then G^* is also essentially 3-edge-connected and C is a dominating CT of G^*, i.e., C is an edge-maximal CT of G^*. Since each vertex in $V(G') - \{v_{xy}, z_1, z_2, z_3\}$ is incident with a pendant edge in G^* and $E_{G^*}(C) = E(G^*)$, every edge-maximal closed trail of G^* contains $V(G') - \{v_{xy}, z_1, z_2, z_3\}$. On the other hand, since C is a maximal cycle of G satisfying $V(G) - \{x, y, z_1, z_2, z_3\}$ and by the definition of H, G, G' and G^*, we can see that for every closed trail T of G^* with $v_{xy} \in V(T)$, $V(G) - \{x, y, z_1, z_2, z_3\} = V(G') - \{v_{xy}, z_1, z_2, z_3\} \not\subseteq V(T)$ holds (note that there exists no Hamilton path in H from a_1 to $\{a_2, b_2\}$ and H has no two disjoint paths covering $V(H)$ from a_1 to $\{a_2, b_2\}$ and from b_1 to $\{a_2, b_2\}$, respectively, see [12] for more details). Thus C is an unique edge-maximal CT of G^*. But since C is not a Tutte CT of G^*, any Tutte CT of G^* is not an edge-maximal CT of G^*.

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On 1-Hamilton-connected claw-free graphs

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Abstract

A graph G is 1-Hamilton-connected if $G - x$ is Hamilton-connected for every vertex $x \in V(G)$. In the paper we introduce a closure concept for 1-Hamilton-connectedness in claw-free graphs. The closure of a graph G is the line graph of a multigraph H such that, for some $e \in E(H)$, $H - e$ has at most two triagles or one double edge, and is 1-Hamilton-connected if and only if G is 1-Hamilton-connected. As an application, we prove that

(i) the Thomassen's conjecture (every 4-connected line graph is hamiltonian) is equivalent to the statement that every 4-connected claw-free graph is 1-Hamilton-connected,

(ii) every 4-connected claw-free and hourglass-free graph is 1-Hamilton-connected.

1 Introduction

A well-known concept in Hamiltonian graph theory is the closure operation $\mathrm{cl}(G)$ for claw-free graphs, introduced in [17]. The closure operation turns a claw-free graph into the line graph of a triangle-free graph while preserving the hamiltonicity of the graph. While $\mathrm{cl}(G)$ also preserves many weaker graph properties (such as traceability or the existence of a 2-factor), stronger properties, such as Hamilton-connectedness, turn out not to be preserved [4], [18]. The first attempt to develop a closure for Hamilton-connectedness was by Brandt [3], the technique was further developed in [19] and [10]. In the present paper, we further strengthen these techniques to the property of 1-Hamilton-connectedness (where a graph G is k-Hamilton-connected if $G - M$ is Hamilton-connected for any set of vertices $M \subset V(G)$ with $|M| = k$).

The concept of k-Hamilton-connectedness was introduced already in 1970 by Lick [15] and since then, studied in many papers (see e.g. [12], [7]). The property of 1-Hamilton-connectedness is closely related to a well-known conjecture by Thomassen [20] which states

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that every 4-connected line graph is hamiltonian, as it was recently shown [9] that the Thomassen's conjecture is equivalent with the statement that every 4-connected line graph is 1-Hamilton-connected. Having in mind that 4-connectedness is a necessary condition for a graph to be 1-Hamilton-connected, we observe that the Thomassen's conjecture, if true, would imply that a line graph is 1-Hamilton-connected if and only if it is 4-connected, which means that 1-Hamilton-connectedness would be polynomial in line graphs. Note that there are many further known equivalent versions of the conjecture (see [5] for a survey on this topic).

In the present paper, we

- in Section 3, develop a closure concept for 1-Hamilton-connectedness in claw-free graphs,
- in Section 4, as applications of the closure, prove that
  - the Thomassen's conjecture is equivalent with the statement that every 4-connected claw-free graph is 1-hamilton-connected,
  - every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected (which gives a partial solution to the conjecture).

We follow the most common graph-theoretical terminology and for concepts and notations not defined here we refer e.g. to [2]. Specifically, by a *graph* we mean a finite undirected graph $G = (V(G), E(G))$; in general, we allow a graph to have multiple edges. The precise way of using (simple) graphs and multigraphs will be specified later in Section 2. We use $d_G(x)$ to denote the *degree* of a vertex x, and we set $V_i(G) = \{x \in V(G)|\ d_G(x) = i\}$. The *neighborhood* of a vertex x, denoted $N_G(x)$, is the set of all neighbors of x, and we define the *closed neighborhood* of x as $N_G[x] = N_G(x) \cup \{x\}$. For a set $M \subset V(G)$, $\langle M\rangle_G$ denotes the *induced subgraph* on M, and for a graph F, G is said to be F*-free* if G does not contain an induced subgraph isomorphic to F. Specifically, for $F = K_{1,3}$ we say that G is *claw-free*.

If $\{x, y\} \subset V(G)$ is a vertex-cut of G and K_1, K_2 are components of $G - \{x, y\}$; then the subgraphs $\langle V(K_1) \cup \{x, y\}\rangle_G$ and $\langle V(K_2) \cup \{x, y\}\rangle_G$ are called the *bicomponents* (of G at $\{x, y\}$).

For $x \in V(G)$, $G - x$ is the graph obtained from G by removing x and all edges adjacent to it. If $x, y \in V(G)$ are such that $e = xy \notin E(G)$, then $G + e$ is the graph with $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$, and, conversely, for $e = xy \in E(G)$ we denote $G - e$ the graph with $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$.

We use $\alpha(G)$ to denote the *independence number* of G, $\nu(G)$ to denote the *matching number* of G (i.e., the size of a largest matching in G), and $\omega(G)$ stands for the *number of components* of G. A *clique* is a set $K \subset V(G)$ such that $\langle K\rangle_G$ is a complete graph.

A graph G is *hamiltonian* if G contains a *hamiltonian cycle*, i.e. a cycle of length $|V(G)|$, and G is Hamilton-connected if, for any $a, b \in V(G)$, G contains a *hamiltonian* (a, b)*-path*, i.e., an (a, b)-path P with $V(P) = V(G)$. For $k \geq 1$, G is k*-Hamilton-connected* if $G - X$ is Hamilton-connected for every set of vertices $X \subset V(G)$ with $|X| = k$. Note that a hamiltonian graph is necessarily 2-connected, a Hamilton-connected graph must be 3-connected and if G is k-Hamilton-connected, then G must be $(k + 3)$-connected.

2 Preliminary results

In this section we summarize some background knowledge that will be needed for our results.

The *line graph* of a graph (multigraph) H, denoted $L(H)$, is the graph with $E(H)$ as vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. Recall that every line graph is claw-free.

It is well-known that if G is a line graph of a simple graph, then the graph H such that $G = L(H)$ (called the *preimage* of G) is uniquely determined, with one exception of $G = K_3$. However, in line graphs of multigraphs this is, in general, not true, as can be seen from the graphs in Fig. 1, where $L(H_1) = L(H_2) = G$. This difficulty can be

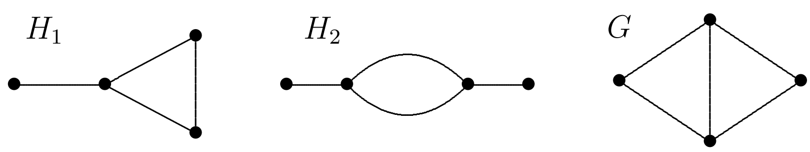

Figure 1

overcome by imposing an additional requirement that simplicial vertices in the line graph correspond to pendant edges.

Proposition A [19]. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

For a line graph G, we will always consider its preimage to be the unique multigraph with the properties given in Proposition A; this preimage will be denoted $L^{-1}(G)$. This means that, throughout the paper, when working with a claw-free graph or with a line graph G, we always consider G to be a simple graph, while if G is a line graph, for its preimage $H = L^{-1}(G)$ we always admit H to be a multigraph, i.e. we always allow H to have multiple edges.

We will also use the notation $e = L^{-1}(a)$ and $a = L(e)$ in situations when $H = L^{-1}(G)$, $a \in V(G)$ and $e \in E(H)$ is the edge of H corresponding to the vertex a. Note that our special choice of the line graph preimage already implies some restrictions on its structure: for example, it is not difficult to observe that $H = L^{-1}(G)$ can never contain a triangle with two vertices of degree 2, for if $\langle\{x_1, x_2, x_3\}\rangle_H$ is such a triangle with $d_H(x_1) = d_H(x_2) = 2$, then $L(x_1x_2)$ is a simplicial vertex in G, but x_1x_2 is not a pendant edge in H (see the graphs H_1 and G in Fig. 1). More generally, if $\langle\{x_1, x_2\}\rangle_H$ is a multiedge in $H = L^{-1}(G)$, then both x_1 and x_2 must have a neighbor outside the set $\{x_1, x_2\}$, and if $\langle\{x_1, x_2, x_3\}\rangle_H$ is a triangle or a multitriangle (a triangle with some multiple edges) in H, then at most one of the vertices x_1, x_2, x_3 can have no neighbor outside the set $\{x_1, x_2, x_3\}$ (for otherwise G contains a simplicial vertex corresponding to a nonpendant edge of H).

We will need the following characterization of line graphs of multigraphs by Krausz [8].

Theorem B [8]. *A nonempty graph G is a line graph of a multigraph if and only if $V(G)$ can be covered by a system of cliques $\mathcal{K}$ such that every vertex of G is in exactly two cliques of $\mathcal{K}$ and every edge of G is in at least one clique of $\mathcal{K}$.*

If G is a line graph and $\mathcal{K} = \{K_1, ..., K_m\}$ is a partition with the properties given in Theorem B, then a graph H such that $G = L(H)$ can be obtained from $\mathcal{K}$ as the intersection graph (multigraph) of the set system $\{V(K_1), ..., V(K_m)\}$, in which the number of vertices shared by two cliques equals the multiplicity of the (multi)edge joining the corresponding vertices of H. A system of cliques $\mathcal{K} = \{K_1, ..., K_m\}$ with the properties given in Theorem B is called a *Krausz partition* of G, and its elements are called *Krausz cliques*. Note that not every clique (and even not every maximal clique) in a line graph G has to be a Krausz clique. If $G = L(H)$, then such non-Krausz cliques in G can correspond to (some of the) triangles, multiple edges or multitriangles (i.e., triangles with some multiple edges) in H.

In general, for a given line graph G, a Krausz partition is not uniquely determined, but every such partition uniquely determines a graph H with the property $G = L(H)$ as its intersection graph. However, by Proposition A, every line graph G has a unique Krausz partition $\mathcal{K}$ such that a vertex $x \in V(G)$ is simplicial if and only if one of the two cliques containing x is of order 1. Thus, whenever we will be working with Krausz cliques and Krausz partitions, we will be always using this particular uniquely determined partition (which gives the unique preimage $L^{-1}(G)$).

Harary and Nash-Williams [6] showed that a line graph G of order at least 3 is hamiltonian if and only if $H = L^{-1}(G)$ contains a *dominating closed trail*, i.e. a closed trail (eulerian subgraph) T such that every edge of H has at least one vertex on T. A similar argument gives the following analogue for Hamilton-connectedness (see e.g. [13]). Here an *internally dominating trail* (abbreviated IDT) is a trail T such that every edge of H has one vertex on T as its internal vertex, and, for $e_1, e_2 \in E(H)$, an (e_1, e_2)-IDT is an IDT having e_1v and e_2 as terminal edges.

Theorem C [13]. *A line graph G of order at least 3 is Hamilton-connected if and only if $H = L^{-1}(G)$ has an (e_1, e_2)-IDT for any pair of edges $e_1, e_2 \in E(H)$.*

An edge cut R of a graph H is *essential* if $H - R$ has at least two nontrivial components. For an integer $k > 0$, H is *essentially k-edge-connected* if every essential edge cut R of G contains at least k edges. Obviously, a line graph $G = L(H)$ is k-connected if and only if the graph H is essentially k-edge-connected.

A vertex $x \in V(G)$ is *locally connected* (*eligible*), if $\langle N(x) \rangle$ is a connected (connected noncomplete) subgraph of G, respectively. The set of all eligible vertices in G will be denoted $V_{EL}(G)$. It is an easy observation that in the special case when G is a line graph and $H = L^{-1}(G)$, a vertex $x \in V(G)$ is locally connected if and only if the edge $e = L_G^{-1}(x)$ is in a triangle or in a multiedge in H, and $G_x^* = L(H|_e)$, where the graph $H|_e$ is obtained from H by contraction of e into a vertex and replacing the created loop(s) by pendant edge(s).

For $x \in V(G)$, the *local completion of G at x* is the graph $G^*_x = (V(G), E(G) \cup \{y_1y_2|\ y_1, y_2 \in N_G(x)\})$, i.e. the graph obtained from G by adding all the missing edges with both vertices in $N_G(x)$).

As shown in [17], if G is claw-free and $x \in V_{EL}(G)$, then G^*_x is hamiltonian if and only if G is hamiltonian. The *closure* $\mathrm{cl}(G)$ of a claw-free graph G is then defined [17] as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely: $\mathrm{cl}(G) = G_k$, where $G_1, \ldots, G_k$ is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G)$, $i = 1, \ldots, k-1$, and $V_{EL}(G_k) = \emptyset$). We say that G is *closed* if $G = \mathrm{cl}(G)$.

The following result from [17] summarizes basic properties of the closure operation.

Theorem D [17]. *For every claw-free graph G:*
(i) $\mathrm{cl}(G)$ *is uniquely determined,*
(ii) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph,*
(iii) $\mathrm{cl}(G)$ *is hamiltonian if and only if G is hamiltonian.*

Recall that the closure operation $\mathrm{cl}(G)$ does not preserve the Hamilton-connectedness of G [18], [4]. Thus, more generally, for $k \geq 1$, we say that a vertex x is *k-eligible* if $\langle N(x)\rangle$ is k-connected noncomplete. The following fact was conjectured in [1] and proved in [18].

Proposition E [18]. *If G is claw-free and $x \in V(G)$ is 2-eligible, then G is Hamilton-connected if and only if G^*_x is Hamilton-connected.*

We will often use the following observation. Let T_1, T_2 be the graphs shown in Fig. 2 (the graph T_1 will be referred to as the *diamond* and T_2 as the *multitriangle*). Let $G =$

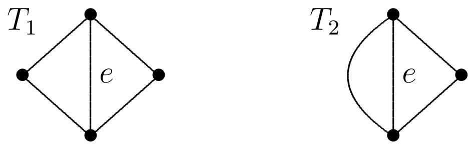

Figure 2

$L(H)$, suppose that H contains a subgraph F isomorphic to T_1 or T_2 (in case of T_2 such that at least vertex of e has a neighbor outside F), and set $x = L(e)$. Then it is easy to see that x is 2-eligible in G and, consequently, by Proposition E, $G = L(H)$ is Hamilton-connected if and only if $G^*_x = L(H|_e)$ is Hamilton-connected (or, equivalently, H has an (f_1, f_2)-IDT for any $f_1, f_2 \in E(H)$ if and only if $H|_e$ has an (f_1, f_2)-IDT for any $f_1, f_2 \in E(H|_e)$).

By recursively performing the local completion operation at k-eligible vertices, we can define [1] the k-closure $\mathrm{cl}_k(G)$ of G, which is uniquely determined [1] and, if G is claw-free, $\mathrm{cl}_2(G)$ is Hamilton-connected if and only if so is G [18].

It can be easily seen that, in general, $\mathrm{cl}_2(G)$ is not a line graph, and even not a line graph of a multigraph. To overcome this drawback, the authors developed in [19] the concept of the *multigraph closure* (or briefly *M-closure*) $\mathrm{cl}^M(G)$ of a graph G: the graph $\mathrm{cl}^M(G)$ is obtained from $\mathrm{cl}_2(G)$ by performing local completions at some (but not all) eligible vertices, where these vertices are chosen in a special way such that the resulting graph is a line graph of a multigraph while still preserving the (non-)Hamilton-connectedness of G. We do not give technical details of the construction since these will not be needed for our proofs; we refer the interested reader to [18], [19].

The concept of M-closure was further strengthened in [10] in such a way that the closure of a claw-free graph is the line graph of a multigraph with either at most two triangles and no multiedge, or with at most one double edge and no triangle.

For a given claw-free graph G, we construct a graph G^M by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$,
- G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
- for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

A graph G^M obtained by the above construction will be called a *strong M-closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure will be said to be *SM-closed*.

The following theorem summarizes basic properties of the SM-closure operation.

Theorem F [10]. *Let G be a claw-free graph and let G^M be its SM-closure. Then G^M has the following properties:*

(i) $V(G) = V(G^M)$ *and* $E(G) \subset E(G^M)$,

(ii) G^M *is obtained from* G *by a sequence of local completions at eligible vertices,*

(iii) G *is Hamilton-connected if and only if* G^M *is Hamilton-connected,*

(iv) *if* G *is Hamilton-connected, then* $G^M = \mathrm{cl}(G)$,

(v) *if* G *is not Hamilton-connected, then either*

 (α) $V_{EL}(G^M) = \emptyset$ *and* $G^M = \mathrm{cl}(G)$, *or*

 (β) $V_{EL}(G^M) \neq \emptyset$ *and* $(G^M)^*_x$ *is Hamilton-connected for any* $x \in V_{EL}(G^M)$,

(vi) $G^M = L(H)$, *where* H *contains either*

 (α) *at most 2 triangles and no multiedge, or*

 (β) *no triangle, at most one double edge and no other multiedge,*

(vii) *if* G *contains no hamiltonian* (a, b)*-path for some* $a, b \in V(G)$ *and*

 (α) X *is a triangle in* H, *then* $E(X) \cap \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\} \neq \emptyset$,

 (β) X *is a multiedge in* H, *then* $E(X) = \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\}$.

Note that, by (vi), the structure of $L^{-1}(G^M)$ is very close to that of $L^{-1}(\mathrm{cl}(G))$ (only at most two triangles or at most one double edge). In some cases (specifically, in cases (iv) and $(v)(\alpha)$ of Theorem F), we have $V_{EL}(G^M) = \emptyset$ and $G^M = \mathrm{cl}(G)$, implying that G^M is uniquely determined. However, if $V_{EL}(G^M) \neq \emptyset$, then, for a given graph G, its SM-closure G^M is in general not uniquely determined and, as will be seen from the proof, the construction of G^M requires knowledge of a pair of vertices a, b for which there is no hamiltonian (a, b)-path in G. Consequently, there is not much hope to construct G^M in polynomial time (unless P=NP). Nevertheless, the special structure of G^M will be very useful for our considerations in the next sections.

3 Closure for 1-Hamilton-connectedness

Let G be a claw-free graph and let $x \in V(G)$ be such that $G-x$ is not Hamilton-connected. Let $\widetilde{G}_x$ be a graph obtained by the following construction.

(1) Set $G_0 := G$, $i := 0$.

(2) If there is a $u_i \in V(G_i)$ such that u_i is eligible in $G_i - x$ and $(G_i)^*_{u_i} - x$ is not Hamilton-connected, then set $G_{i+1} = (G_i)^*_{u_i}$ and go to (3),

otherwise set $\widetilde{G}_x := G_i$ and stop.

(3) Set $i := i + 1$ and go to (2).

Then we say that $\widetilde{G}_x$ is a *partial x-closure* of the graph G.

The following proposition summarizes main properties of a partial x-closure of a claw-free graph. Here the *5-wheel*, denoted W_5, is the graph consisting of a 5-cycle C_5 and a vertex (the *center* of the W_5) adjacent to all vertices of the C_5.

Proposition 1. *Let G be a claw-free graph, let $x \in V(G)$ be such that $G - x$ is not Hamilton-connected, and let $\widetilde{G}_x$ be a partial x-closure of G. Then $\widetilde{G}_x - x$ is an SM-closed line graph and $\widetilde{G}_x$ satisfies one of the following:*

(i) $\widetilde{G}_x$ is a line graph;

(ii) x is a center of an induced W_5, and there are $u_1, u_2 \in N_{\widetilde{G}_x}(x)$ such that
- *(α) $\{u_1, u_2\}$ is a cut set of $\widetilde{G}_x - x$,*
- *(β) one of the bicomponents of $\widetilde{G}_x - x$ at $\{u_1, u_2\}$ is isomorphic to $K_3 - e$,*
- *(γ) the graph $(\widetilde{G}_x + \{u_1, u_2\}) - x$ contains no induced W_5 with center at x,*
- *(δ) the graph $(\widetilde{G}_x + \{u_1, u_2\}) - x$ is SM-closed;*

(iii) there are Krausz cliques K_1, K_2 in $\widetilde{G}_x - x$ such that
- *(α) $N_{\widetilde{G}_x}(x) \subset K_1 \cup K_2$,*
- *(β) the graph $(V(\widetilde{G}_x), E(\widetilde{G}_x) \cup \{xv \mid v \in K_1 \cup K_2\})$ is a line graph.*

Proof of Proposition 1 is postponed to Section 5. ■

Note that if G is such that $\widetilde{G}_x$ satisfies (ii) of Proposition 1, then the graph $\widetilde{G}_x + uv$ contains no induced W_5 with center at x, hence $\widetilde{G}_x + uv$ satisfies (i) or (iii) of Proposition 1.

It is also easy to see that, in case (ii), $\{L^{-1}(u_1), L^{-1}(u_2)\}$ is a 2-element edge cut of $H = L^{-1}(\widetilde{G}_x - x)$ separating a single edge from the rest of H.

Let now G be a claw-free graph, and let $\overline{G}$ be a graph obtained by the following construction:

(1) If G is 1-Hamilton-connected, set $\overline{G} = \mathrm{cl}(G)$.

(2) If G is not 1-Hamilton-connected, choose a vertex $x \in V(G)$ such that $G - x$ is not Hamilton-connected and a partial x-closure $\widetilde{G}_x$ of G.

(3) If $\widetilde{G}_x$ satisfies (ii) of Proposition 1 (i.e., x is a center of an induced W_5 in $\widetilde{G}_x$), choose a cut set $\{u_1, u_2\}$ of $\widetilde{G}_x - x$, add the edge u_1u_2 to $\widetilde{G}_x$ (i.e., set $\widetilde{G}_x := \widetilde{G}_x + u_1u_2$), and proceed to (4).

(4) If $\widetilde{G}_x$ is a line graph, set $\overline{G} = \widetilde{G}_x$.

Otherwise, $\widetilde{G}_x$ satisfies (iii) of Proposition 1, i.e. some two Krausz cliques K_1, K_2 in $\widetilde{G}_x - x$ cover all vertices in $N_{\overline{G}}(x)$, and then set $\overline{G} = (V(\widetilde{G}_x), E(\widetilde{G}_x) \cup \{xv \mid v \in (K_1 \cup K_2)\})$.

Then we say that the resulting graph $\overline{G}$ is a *1HC-closure* of the graph G.

The following result summarizes basic properties of a 1HC-closure of a graph G.

Theorem 2. *Let G be a claw-free graph and let $\overline{G}$ be its 1HC-closure. Then*
(i) *$\overline{G}$ is a line graph,*
(ii) *for some $x \in V(\overline{G})$, the graph $\overline{G} - x$ is SM-closed,*
(iii) *$\overline{G}$ is 1-Hamilton-connected if and only if G is 1-Hamilton-connected.*

Proof. Properties (i) and (ii) follow immediately by the definition of $\overline{G}$. Also clearly $\overline{G}$ is 1-Hamilton-connected if so is G, and if G is not 1-Hamilton-connected, then neither is $\widetilde{G}_x$ (for some $x \in V(G)$ which is used in the construction). It remains to show that $\overline{G}$ is not 1-Hamilton-connected if $\widetilde{G}_x$ is not. This is clear if $\widetilde{G}_x$ satisfies (i) or (iii) of Proposition 1. Finally, if $\widetilde{G}_x$ satisfies (ii), then $\overline{G}$ is not 1-Hamilton-connected since neither $\widetilde{G}_x$ nor $\overline{G}$ is 4-connected. ■

Note that (ii) is equivalent to the statement that $H = L^{-1}(\overline{G})$ contains an edge $e \in E(H)$ such that $L(H - e)$ is SM-closed.

Also note that, for a given claw-free graph G, its 1-Hamilton-connected closure is not uniquely determined.

We finish this section with a result which shows that steps (3) and (4) in the definition of a 1HC-closure of a graph can be also accomplished by adding (some) edges in neighborhoods of eligible vertices.

Proposition 3. *Let G be a claw-free graph. Then there is a sequence of graphs $G_0, \dots, G_k$ such that*

(i) $G_0 = G$,

(ii) $V(G_i) = V(G_{i+1})$ *and* $E(G_i) \subset E(G_{i+1}) \subset E((G_i)^*_{x_i})$ *for some* $x_i \in V(G_i)$ *eligible in* G_i,

(iii) G_k *is a 1HC-closure of* G.

Proof. Steps (1) and (2) of the definition of a 1HC-closure clearly satisfy the conditions of the proposition, and so does step (3), since the added edge has both vertices in $N_G(x)$ and x is eligible. It remains to verify the statement in step (4). Suppose, to the contrary, that, in step (4), for some Krausz clique K_i in $\widetilde{G}_x - x$, adding the edges joining K_i to x does not satisfy the conditions.

If $|K_i \cap N_G(x)| \geq 2$, then K_i and $\langle N_G(x)\rangle_{\widetilde{G}_x}$ share an edge, say, v_1v_2, but then v_1 is eligible, a contradiction. Hence $|K_i \cap N_G(x)| = 1$. Let $K_i \cap N_G(x) = \{u\}$. By the properties of the Krausz partition, u is, besides K_i, in some other Krausz clique K_j. If $\langle N_G(x)\rangle_{\widetilde{G}_x}$ is disconnected, then u is a simplicial vertex in $G-x$ (otherwise u centers a claw in G) and, since simplicial vertices in $G-x$ correspond to pendant edges in $H = L^{-1}(G)$, one of K_i, K_j (say, K_j) is of size 1. But then, extending K_j to x adds no new edge to $\widetilde{G}_x$.

Finally, if $\langle N_G(x)\rangle_{\widetilde{G}_x}$ is connected, then there is an edge e in $\langle N_G(x)\rangle_{\widetilde{G}_x}$ containing u, and necessarily e is in K_j. But then, for the clique K_j, we have $|K_j \cap N_G(x)| \geq 2$ and we are in the previous case. ■

4 Applications of the closure

In this section we show two applications of the 1HC-closure. The first of them, Theorem 4, is related to a famous conjecture by Thomassen [20] stating that every 4-connected line graph is hamiltonian. There are many known equivalent versions of the conjecture (see [5] for a survey on this topic). We show the following equivalence.

Theorem 4. *The following statements are equivalent:*

(i) *Every 4-connected line graph is hamiltonian.*

(ii) *Every 4-connected claw-free graph is 1-Hamilton-connected.*

Proof. Obviously, (ii) implies (i). Conversely, first recall that, by a recent result [9], (i) is equivalent to the statement that every 4-connected line graph is 1-Hamilton-connected. Thus, if G be a counterexample to (ii), then its 1HC-closure provides a counterexample to (i). ■

As another application, we prove a theorem on hourglass-free graphs. Our result, Theorem 5, is a strengthening of the main result of [14] and can be considered as a partial solution to the statement (ii) of Theorem 4, i.e., equivalently, to the Thomassen's conjecture.

Here the *hourglass* is the unique graph Γ with degree sequence $4, 2, 2, 2, 2$. The vertex $x \in V(\Gamma)$ of degree 4 is called the *center* of Γ and we also say that Γ is *centered at x*. Note that Γ is a line graph and, in multigraphs, it has three nonisomorphic preimages (see Fig. 3).

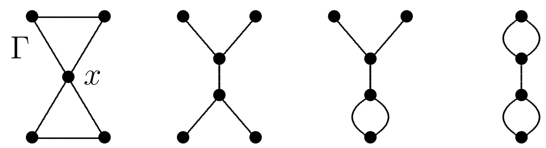

Figure 3

The following theorem is our second application.

Theorem 5. *Every 4-connected claw-free and hourglass-free graph is 1-Hamilton-connected.*

For the proof of Theorem 5, we will need several auxiliary results.

Lemma 6. *Let G be a claw-free graph such that every induced hourglass in G is centered at an eligible vertex and let $\overline{G}$ be a 1HC-closure of G satisfying the statement of Proposition 3. Then every induced hourglass in $\overline{G}$ is centered at an eligible vertex.*

Proof. Let $G_0, \ldots, G_k$ be a sequence of graphs with the properties given in Proposition 3, let $\overline{G} = G_k$, and let i, $0 \leq i \leq k-1$, be the smallest integer such that G_{i+1} contains an induced hourglass Γ centered at a locally disconnected vertex. Denote $V(\Gamma) = \{u_0, u_1, u_2, u_3, u_4\}$ such that $E(\Gamma) = \{u_0u_1, u_0u_2, u_0u_3, u_0u_4, u_1u_2, u_3u_4\}$ (i.e., u_0 is the center of Γ). By the choice of i, $E(\Gamma) \not\subset E(G_i)$. If G_i contains all the edges of Γ containing u_0, then u_0 centers a claw in G_i; hence we can choose the notation such that $u_0u_1 \notin E(G_i)$. By Proposition 3, there is a vertex v eligible in G_i such that $u_0u_1 \in N_{G_i}(v)$. Let u_5 be the first vertex of a (u_0, u_1)-path in $\langle N_{G_i}(v)\rangle_{G_i}$. Then $\langle\{u_0, v, u_5, u_3, u_4\}\rangle_{G_i}$ is an induced hourglass in G_i, centered at u_0. This contradicts the choice of i since u_0 is locally disconnected in G_i. ■

Lemma 7. *Let G be a 4-connected claw-free hourglass-free graph. Then there is a 1HC-closure $\overline{G}$ of G such that $L^{-1}(\overline{G})$ has at most three vertices of degree three.*

Proof. Let $\overline{G}$ be a 1HC-closure of G with the properties given in Proposition 3 and let $H = L^{-1}(\overline{G})$. Recall that H is essentially 4-edge-connected and that a vertex of $\overline{G}$ is eligible if and only if the corresponding edge of H is in a triangle or in a multiedge.

Claim 1. *Let $x \in V(H)$ be of degree 3 in H. Then there is a subgraph $T \subset H$ such that T is isomorphic to the graph T_1 or T_2 of Fig. 2 and $d_T(x) = 3$.*

Proof. Let $N_H(x) = \{u, v, w\}$. We distinguish two possibilities.

First suppose that u, v, w are distinct. Since H is essentially 4-edge-connected, we have $d_H(w) \geq 3$, and since the vertex $L(xw)$ does not center in $\overline{G}$ a hourglass with a locally disconnected center, xw is in a triangle. Since $d_H(x) = 3$, we have, up to a symmetry, $uw \in E(H)$. The same idea, applied to the edge xv, implies $vw \in E(H)$. But then x, w, u, v are vertices of a T_1 in H.

Secondly, let $u = v$. Similarly as before, the edge xw is in a triangle, implying $uw \in E(H)$ and then x, u, v are vertices of a T_2 in H. $\square$

Let now $x \in V(H)$ be of degree 3 in H. We distinguish two cases.

Case 1: *All vertices of degree 3 in H are in $N_H[x]$.*

If x is in a T_2, then $|N_H[x]| = 3$ and we are done. Thus, suppose x is in a $T_1 \subset H$. If all vertices of T_1 are of degree 3, then either T_1 is connected to $H - T_1$ with exactly two edges, in which case H is not essentially 4-edge-connected, or H is in a K_4, but then removal of any edge from H yields a diamond, contradicting the fact that $\overline{G}$ contains a vertex the removal of which yields an SM-closed graph. Hence H contains at most three vertices of degree 3.

Case 2: *There is $y \in V(H)$ such that $d_H(y) = 3$ and $xy \notin E(H)$.*

By Claim 1, there are subgraphs T_x and T_y of H (not necessarily induced) such that x or y is of degree 3 in T_x or T_y, respectively, and each of T_x, T_y is isomorphic to T_1 or to T_2. By the properties of the 1HC-closure, there is an edge $e \in E(H)$ such that $L(H-e)$ is SM-closed, i.e., $H - e$ contains at most two triangles or at most one double edge. Thus, e is an edge of both T_x and T_y and, since x, y are nonadjacent, e contains neither x nor y. Now, if one of T_x, T_y is a T_2, then removal of any edge leaves in $H - e$ two double edges or a double edge or a triangle, which is not possible. Hence both T_x and T_y is the diamond T_1.

Denote $e = wz$, and let u and v be the fourth vertex in T_x and T_y, respectively. Then we have, up to a symmetry, the following two possibilities (see Fig. 4):

(a) $d_{T_x}(w) = d_{T_y}(w) = 3$ (implying $d_{T_x}(z) = d_{T_y}(z) = 2$),

(b) $d_{T_x}(w) = d_{T_y}(z) = 3$ (implying $d_{T_x}(z) = d_{T_y}(w) = 2$).

We consider these possibilites separately.

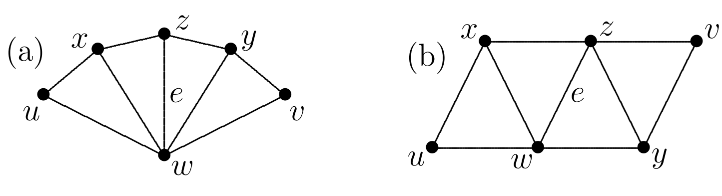

Figure 4

(a) Let first $d_{T_x}(w) = d_{T_y}(w) = 3$. If $u = v$, then $H - e$ contains a diamond, hence $u \neq v$. If $d_H(u) = 3$, then, by the previous observations, u is a vertex of degree 3 of a diamond T_u. This implies either $uu' \in E(H)$ and $u'w \in E(H)$ for some other vertex u', or $uz \in E(T_u)$, but then, in both cases, T_u is a diamond also in $H - e$, a contradiction. If $d_H(u) = 2$, then $\{uw, xw, xz\}$ is an edge-cut separating the edge ux, a contradiction. Hence $d_H(u) > 3$ and, symmetrically, $d_H(v) > 3$. Thus, among the vertices in $V(T_x) \cup V(T_y)$, only x, y and possibly z are of degree 3. If H contains another vertex t of degree 3, then t is adjacent to neither x nor y and, by Claim 1, t is in a diamond T_1. But then, for any edge $f \in E(H)$, $H - f$ contains at least three triangles, a contradiction.

(b) Secondly, let $d_{T_x}(w) = d_{T_y}(z) = 3$. For $u = v$ immediately $d_H(u) = d_H(v) > 3$; for $u \neq v$, similarly as before, $d_H(u) = 3$ implies that $uz \in E(H)$ and $H - e$ contains a diamond, and $d_H(u) = 2$ contradicts the connectivity assumption. Thus, in both cases, we have $d_H(u) > 3$ and, symmetrically, $d_H(v) > 3$. Hence x and y are the only vertices of degree 3 in $V(T_x) \cup V(T_y)$. Similarly, if $d_H(t) = 3$ for some other $t \in V(H)$, then t is adjacent to neither x nor y, t is in a diamond and, for any $f \in E(H)$, $H - f$ contains at least three triangles, a contradiction. ∎

The *core* of a graph H, denoted $\mathrm{co}(H)$, is the graph obtained from H by deleting all vertices of degree 1 and suppressing all vertices of degree 2 (i.e., contracting exactly one of the edges xy, yz for each path xyz with $d_H(y) = 2$). Note that, by the definition of the core, all vertices of degree one or two are deleted or suppressed, hence $\delta(\mathrm{co}(H)) \geq 3$.

For the proof of Theorem 5, we will need two more results.

Theorem G [11]. *Let H be a graph such that $co(H)$ has two edge-disjoint spanning trees and $G = L(H)$ is 3-connected. Then, for any any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

Theorem H [16], [21]. *A graph G has k edge-disjoint spanning trees if and only if*

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

Proof of Theorem 5. Let G be a 4-connected claw-free hourglass-free graph and, by Lemma 7, let $\overline{G}$ be a 1HC-closure of G such that $H = L^{-1}(\overline{G})$ has at most three vertices of degree 3. Recall that H is essentially 4-edge-connected.

By Theorem C, we need to show that for any $f, e_1, e_2 \in E(H)$, the graph $H - f$ has an (e_1, e_2)-IDT. Since the graph $L(H - f) = \overline{G} - x$ (where $x = L(F)$) is clearly 3-connected, by Theorem G, it is sufficient to show that the graph $\mathrm{co}(H - f)$ has two edge-disjoint spanning trees. Thus, let $f \in E(H)$.

<u>Claim 1.</u> *The graph $\mathrm{co}(H) - f$ has two edge-disjoint spanning trees.*

<u>Proof.</u> First note that possibly $f \notin E(\mathrm{co}(H))$ if f is a pendant edge of H; in this case $\mathrm{co}(H) - f = \mathrm{co}(H)$. Obviously, $\mathrm{co}(H)$ is essentially 4-edge-connected (since so is H) and has at most three vertices of degree 3 (since, by the connectivity assumption, pendant edges in H can be incident only to vertices of degree at least 4 in $\mathrm{co}(H)$). Hence, for any

set $E \subset E(\mathrm{co}(H))$, every component C of $\mathrm{co}(H) - E$ is connected to $(\mathrm{co}(H) - E) - C$ by at least 4 edges, except for the case when C is a trivial component consisting of one of the at most three vertices of degree 3. This implies $2|E| \geq 4(\omega(\mathrm{co}(H) - E) - 3) + 3 \cdot 3 = 4\omega(\mathrm{co}(H) - E) - 3$, from which, by parity, $2|E| \geq 4\omega(\mathrm{co}(H) - E) - 2$, i.e., $|E| \geq 2\omega(\mathrm{co}(H) - E) - 1$.

Now, set $H' = \mathrm{co}(H) - f$ and let $E_0 \subset E(H')$. Set $E = E_0 \cup \{f\}$ if $f \in E(\mathrm{co}(H))$ and $E = E_0$ otherwise . Then clearly $|E_0| \leq |E| \leq |E_0| + 1$, $E \subset E(\mathrm{co}(H))$ and $\omega(\mathrm{co}(H) - E) = \omega(H' - E_0)$. Hence $|E_0| \geq |E| - 1 \geq 2\omega(\mathrm{co}(H) - E) - 2 = 2\omega(H' - E_0) - 2$. By Theorem H, H' has two edge-disjoint spanning trees. $\square$

Claim 2. *The graph* $\mathrm{co}(\mathrm{co}(H) - f)$ *has two edge-disjoint spanning trees.*

Proof. Suppose that $f \in E(\mathrm{co}(H))$ (otherwise there is nothing to do by Claim 1) and note that, since $V_1(\mathrm{co}(H)) = V_2(\mathrm{co}(H)) = \emptyset$, we have $V_1(\mathrm{co}(H) - f) = \emptyset$ and $V_2(\mathrm{co}(H) - f) = V_3(\mathrm{co}(H)) \cap V(f)$. By Claim 1, let T_1, T_2 be two edge-disjoint spanning trees in $\mathrm{co}(H) - f$, let $u \in V_2(\mathrm{co}(H) - f)$ and let u_1, u_2 be the neighbors of u. Then each of the edges u_1u, u_2u is in one of T_1, T_2, say, $u_1u \in E(T_1)$ and $u_2u \in E(T_2)$ and, removing for every $u \in V_2(\mathrm{co}(H) - f)$ the edge u_iu from T_i, $i = 1, 2$, we obtain two edge-disjoint spanning trees in $\mathrm{co}(\mathrm{co}(H) - f)$. $\square$

Claim 3. $\mathrm{co}(H - f) = \mathrm{co}(\mathrm{co}(H) - f)$.

Proof. The claim is trivially true if f is a pendant edge of H, so suppose f is nonpendant. As already noted, we have $V_1(\mathrm{co}(H) - f) = \emptyset$ and $V_2(\mathrm{co}(H) - f) = V_3(\mathrm{co}(H)) \cap V(f)$, from which $V(\mathrm{co}(\mathrm{co}(H) - f)) = V(H) \setminus [V_1(H) \cup V_2(H) \cup (V_3(H) \cap V(f))]$. On the other hand, $V_1(H - f) = V_1(H) \cup (V_2(H) \cap V(f))$ (note that $V_2(H)$ is an independent set by the connectivity assumption) and $V_2(H - f) = (V_2(H) \setminus V(f)) \cup (V_3(H) \cap V(f))$, from which $V_1(H - f) \cup V_2(H - f) = V_1(H) \cup V_2(H) \cup (V_3(H) \cap V(f))$, implying $V(\mathrm{co}(H - f)) = V(H) \setminus [V_1(H) \cup V_2(H) \cup (V_3(H) \cap V(f))]$. Thus, $\mathrm{co}(H - f)$ and $\mathrm{co}(\mathrm{co}(H) - f)$ are graphs on the same vertex set.

In the construction of $\mathrm{co}(H - f)$, each of the vertices in $V_1(H - f) = V_1(H) \cup (V_2(H) \cap V(f))$ was removed together with a pendant edge; in $\mathrm{co}(\mathrm{co}(H) - f)$, in the construction of $\mathrm{co}(H)$, the set $V_1(H)$ was removed, and in the step from $\mathrm{co}(H) - f$ to $\mathrm{co}(\mathrm{co}(H) - f)$, $V_2(H) \cap V(f)$ was removed. Thus, in the construction of both graphs, the sets of removed vertices are the same. Consequently, the sets of suppressed vertices are also the same and the claim follows. $\square$

Now, $\mathrm{co}(H - f)$ has two edge-disjoint spanning trees by Claims 3 and 2. $\blacksquare$

5 Proof of Proposition 1

For our proof we will need four lemmas describing subgraphs that cannot occur in the preimage of an SM-closed graph.

Lemma 8. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a triangle with a vertex of degree 2 in H.*

For the proof of Lemma 8, we will need the following proposition from [4].

Proposition I [4]. *Let x be an eligible vertex of a claw-free graph G, G^*_x the local completion of G at x, and a, b two distinct vertices of G. Then for every longest (a,b)-path $P'(a,b)$ in G^*_x there is a path P in G such that $V(P) = V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an (a,b)-path $P(a,b)$ in G such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between a and b):*

- (i) *There is an induced subgraph $F \subset G$ isomorphic to the graph S in Fig. 5 such that both a and x are vertices of degree 4 in F. In this case G contains a path P_b such that b is an endvertex of P and $V(P_b) = V(P')$. If, moreover, $b \in V(F)$, then G contains also a path P_a with endvertex a and with $V(P_a) = V(P')$.*
- (ii) *$x = a$ and $ab \in E(G)$. In this case there is always both a path P_a in G with endvertex a and with $V(P_a) = V(P')$ and a path P_b in G with endvertex b and with $V(P_b) = V(P')$.*

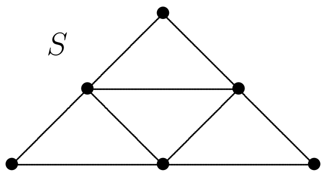

Figure 5

Proof of Lemma 8. Let G be an SM-closed graph. If G is Hamilton-connected, the lemma is obvious since $H = L^{-1}(G)$ is triangle-free by the definition of the SM-closure. Thus, suppose that G is not Hamilton-connected. Let, to the contrary, $T = \langle\{v_1, v_2, v_2\}\rangle_H$ be a triangle in H with $d_H(v_1) = 2$, and set $x_i = L(v_i v_{i+1})$, $i = 1, 2, 3$ (indices mod 3). Observe that $L^{-1}(S)$ (where S is the graph in Fig. 5) is isomorphic to the net N, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle. Since $d_H(v_1) = 2$, T is not contained in a copy of N, hence the triangle $L(T) = \langle\{x_1, x_2, x_3\}\rangle_G$ is not contained in an induced subgraph of $G = L(H)$ isomorphic to $S = L(N)$.

Since the edge $L^{-1}(x_2) = v_2 v_3$ is in the triangle T, and T cannot have two vertices of degree 2 by the definition of the preimage L^{-1}, x_2 is eligible in G and, by the definition of the SM-closure, $G^*_{x_2}$ is not Hamilton-connected, i.e., there is no hamiltonian (a,b)-path in $G^*_{x_2}$ for some $a, b \in V(G)$ for which there is no hamiltonian (a,b)-path in G. By Proposition I(ii), for every such hamiltonian (a,b)-path in $G^*_{x_2}$, one of a, b is x_2 (say, $a = x_2$), and $b \in N(x_2)$.

Now, x_1 is also eligible in G, and since $N_G(x_1) \subset N_G(x_2)$ (this follows easily from $d_h(v_1) = 2$), also $G^*_{x_1} \subset G^*_{x_2}$, hence every hamiltonian path in $G^*_{x_1}$ is also a hamiltonian path in $G^*_{x_2}$. We already know that every such (a,b)-path satisfies $a = x_1$, and, applying Proposition I(ii) to x_1, we have $b = x_1$.

Thus, we conclude that the only possible vertices for which there is a hamiltonian path in $G^*_{x_2}$ but not in G are the vertices x_1 and x_2. However, x_3 is also eligible in G and

$N_G(x_3) \subset N_G(x_2)$, thus, by a symmetric argument, we obtain the same conclusion for x_3 and x_2, a contradiction. ■

In the proof of the next three lemmas we will need the following slight extension of a technical lemma from [10].

For a graph H, $u \in V(H)$ with $d_H(u) = 2$ and $N_H(u) = \{v_1, v_2\}$, $H|_{(u)}$ denotes the graph obtained from H by suppressing the vertex u (i.e., by replacing the path v_1uv_2 by the edge v_1v_2) and by adding one pendant edge to each of v_1 and v_2.

Lemma J [10]. *Let H be a graph and $u \in V(H)$ of degree 2 with $N_H(u) = \{v_1, v_2\}$ and $h_i = uv_i$, $i = 1, 2$. Set $H' = H|_{(u)}$, $h = v_1v_2 \in E(H')$, and let $f_1, f_2 \in E(H') \setminus E(H)$ be the two pendant edges attached to v_1 and v_2, respectively.*

- *(i) If $L(H)$ is Hamilton-connected, then H' has an (e_1, e_2)-IDT for every $e_1, e_2 \in E(H')$ such that either*
  - *(α) $h \notin \{e_1, e_2\}$, or*
  - *(β) $h \in \{e_1, e_2\}$ and $\{e_1, e_2\} \cap \{f_1, f_2\} \neq \emptyset$.*
- *(ii) If $L(H')$ is Hamilton-connected, then H has an (e_1, e_2)-IDT for every $e_1, e_2 \in E(H)$ such that $\{e_1, e_2\} \neq \{h_1, h_2\}$.*
- *(iii) If moreover H contains a pendant edge attached to v_1 and H has an (h_1, e)-IDT for every $e \in E(H)$, then H' has an (h, e')-IDT for every $e' \in E(H')$*

Proof. Parts (i) and (ii) are a reformulation of Lemma 3 from [10]. We prove (iii). Thus, for any $e' \in E(H')$, we construct an (h, e')-IDT in H'. Let f denote the pendant edge at v_1 in H. If $e' \in \{f, f_1, f_2\}$, then, for any (h_1, h_2)-IDT in H, an appropriate replacement of h_1 and h_2 with h and e' gives the desired (h, e')-IDT in H'. Thus, let $e' \notin \{f, f_1, f_2\}$. Let $e \in E(H)$ be the edge corresponding to e', and let T be an (h_1, e)-IDT in H. If $h_2 \in E(T)$, then necessarily $v_1 \in V(T)$ (otherwise f is not dominated), and then T' obtained from T by replacing h_1, h_2 with h is an (h, e')-IDT in H'. Similarly, if $h_2 \notin E(T)$, then necessarily $v_2 \in V(T)$ (otherwise h_2 is not dominated), and then T' obtained from T by replacing h_1 with h is a desired (h, e')-IDT in H'. ■

Lemma 9. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a subgraph $\overline{H}$ isomorphic to a cycle C_5 with a vertex of degree 2 in H and with a chord.*

Proof. If G is Hamilton-connected, the lemma is obvious. Thus, suppose that G is not Hamilton-connected and let, to the contrary, $\overline{H} \subset H$ be a graph consisting of a cycle $C = v_1v_2v_3v_4v_5v_1$ with a chord e, and choose the notation such that $d_H(v_4) = 2$. If $e = v_3v_5$, we have a contradiction with Lemma 8, hence without loss of generality suppose that $e = v_2v_5$. First observe that e is the only chord of C in H, for otherwise H contains a diamond, a contradiction. Denote $v_iv_{i+1} = h_{i+1}$, $i = 1, \ldots, 5$ (indices mod 5) and set $H_1 = H|_{(v_4)}$. Then $L(H_1)$ is not Hamilton-connected by Lemma J(ii). It is straightforward to see that in $L(H_1)$, the neighborhood of the vertex $L(e)$ is 2-connected. By Proposition E, the graph $(L(H_1))^*_{L(e)} = L(H_1|_e)$ is not Hamilton-connected. Set

$H_2 = H_1|_e$ (denoting v_2 the vertex obtained by merging $v_2, v_5 \in V(H_1)$). Now, the subgraph of H_2 corresponding to $\overline{H} \subset H$ consists of three vertices v_1, v_2, v_3, a double edge h_1, h_2 joining v_1 and v_2, a double edge h_3, h_4 joining v_1 and v_2, two pendant edges at v_2 and one pendant edge at v_3.

Now we return back the suppressed vertex v_4: let H_3 be the graph obtained from H_2 by subdividing the edge h_4 with a vertex v_4 (denoting $h_5 = v_4v_2$) and removing a pendant edge from each of v_2, v_3. If $L(H_3)$ is Hamilton-connected, then H_2 has, for $e_1, e_2 \in E(H_2)$, an (e_1, e_2)-IDT for $e_1, e_2 \neq h_4$ by Lemma J(i), and for $h_4 \in \{e_1, e_2\}$ by Lemma J(iii), hence $L(H_2)$ is Hamilton-connected, a contradiction. Thus, $L(H_3)$ is not Hamilton-connected. But H_3 can be alternatively obtained from H by contracting the chord e, i.e., $H_3 = H|_e$, or, equivalently, $L(H_3) = G^*_{L(e)}$. As $L(H_3)$ is not Hamilton-connected and $L(e)$ is eligible in G (since e is in a triangle in H), we have a contradiction with the fact that G is SM-closed. ■

Lemma 10. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a cycle C of length 5 such that some two vertices of C are of degree 2 in H and some edge of C is in a double edge or in a triangle in H.*

Proof. If G is Hamilton-connected, the lemma is obvious. Thus, suppose that G is not Hamilton-connected, let $C = v_1v_2v_3v_3v_5v_1 \subset H$ and let v_j, v_k, $j < k$, be of degree 2 in H. Set $v_iv_{i+1} = h_{i+1}$, $i = 1, \ldots, 5$ (indices mod 5).

Suppose first that v_j, v_k are consecutive on C, say, $j = 1$, $k = 2$. Then $R = \{h_1, h_2\}$ is an essential edge-cut separating h_2 from the rest of H. By the assumptions, some of h_4, h_5 (say, h_4), is in a triangle or in a double edge, implying $L(h_4)$ is eligible in G. But R is an essential edge-cut also in $H|_{h_4} = L^{-1}(G^*_{L(h_4)})$, hence $G^*_{L(h_4)}$ is not Hamilton-connected, contradicting the definition of SM-closure. Thus, v_j, v_k are not consecutive on C.

Choose the notation such that $j = 3$ and $k = 5$, i.e., $d_H(v_3) = d_H(v_5) = 2$. Then the only possible chords of C are the edges v_1v_4 and v_2v_4, but if some of them is present, we have a contradiction with Lemma 8. Thus, C is chordless. This implies that either

(i) h_2 is in a double edge, or

(ii) h_2 is in a triangle $T = v_1v_2z$ with $z \in V(H) \setminus V(C)$.

In case (i), we use h_2' to denote the edge parallel with h_2 and $\overline{H}$ to denote the graph with $V(\overline{H}) = V(C)$ and $E(\overline{H}) = E(C) \cup \{h_2'\}$; in case ($ii$) we set $h_2' = zv_1$, $h_2'' = zv_2$, $V(\overline{H}) = V(C) \cup \{z\}$ and $E(\overline{H}) = E(C) \cup \{h_2', h_2''\}$. Recall that in both cases $d_H(v_3) = d_H(v_5) = 2$.

By the properties of the SM-closure, for each pair $e, fE(H)$, for which there is no (e, f)-IDT in H, we have $\{e, f\} = \{h_2, h_2'\}$ in case (i), or $\{e, f\} \cap \{h_2, h_2', h_2''\}$ in case (ii), respectively. Thus, by Lemma J(ii), for the graph $H_1 = H|_{(v_5)}$ (in which we denote $v_1v_4 = h_1$), $L(H_1)$ is not Hamilton-connected. Similarly, the graph $L(H_2)$, where $H_2 = H_1|_{(v_3)}$ (in which we set $v_2v_4 = h_3$) is also not Hamilton-connected. But now $\langle \{v_1, v_2, v_4\} \rangle_{H_2}$ is a triangle with a double edge h_2, h_2' in case (i), or $\langle \{v_1, v_2, v_4, z\} \rangle_{H_2}$ is a diamond in case (ii). In both cases, it is straightforward to verify that, in $L(H_2)$, the neighborhood of the vertex $x_2 = L(h_2)$ is 2-connected. Thus, setting $H_3 = H_2|_{h_2}$, we obviously have $L(H_3) = (L(H_2))^*_{x_2}$ and, by Proposition E, $L(H_3)$ is also not Hamilton-connected. Note that in H_3 the subgraph corresponding to $\overline{H}$ consists of: in case (i) two vertices v_1, v_4 joined by h_1 and h_3, 4 pendant edges at v_1 and 2 pendant edges at v_4, or in case (ii) three vertices z, v_1, v_4, where z, v_1 are joined by h_2', h_2'' and v_1, v_4 are joined by h_1, h_3, and there are 3 pendant edges at v_1 and 2 pendant edges at v_4.

Now we return back the suppressed vertices of degree 2: H_4 is obtained from H_3 by subdividing h_3 with v_3 (denoting $v_3v_4 = h_4$) and removing a pendant edge from each of v_1, v_4, and, similarly, H_5 is obtained from H_4 by subdividing h_1 with v_5 (denoting $v_4v_5 = h_5$), and removing a pendant edge from each of v_1, v_4. If $L(H_4)$ is Hamilton-connected, then H_3 has, for $e, f \in E(H_3)$, an (e, f)-IDT for $e, f \neq h_3$4 by Lemma J(i), and for $h_3 \in \{e, f\}$ by Lemma J(iii), hence $L(H_3)$ is Hamilton-connected, a contradiction. Thus, $L(H_4)$ is not Hamilton-connected. By a similar argument, $L(H_5)$ is also not Hamilton-connected. But now we observe that $H_5 = H|_{h_2}$, or, equivalently, $L(H_5) = G^*_{x_2}$. As h_2 is in a double edge or in a triangle, x_2 is eligible in G and we have a contradiction with the fact that G is SM-closed. ■

Lemma 11. *Let G be an SM-closed graph, let $H = L^{-1}(G)$ and let F be the graph with $V(F) = \{v_1, v_2, v_3, v_4, v_5, z\}$ and $E(F) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_3v_5, zv_1, zv_2\}$ (see Fig. 6). Then H does not contain a subgraph $\overline{H}$ isomorphic to the graph F such that $N_H(\{v_1, v_2, v_3, v_5\}) \subset V(\overline{H})$.*

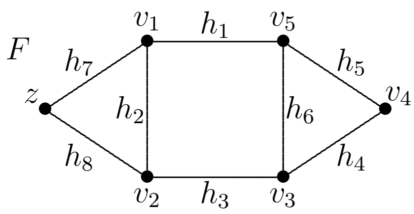

Figure 6

Proof. If G is Hamilton-connected, the lemma is obvious. Thus, suppose that G is not Hamilton-connected and let $\overline{H}$ be a subgraph of H with the properties given in the lemma. Let $h_1, \ldots, h_8$ denote the edges of $\overline{H}$ as shown in Fig. 6 and denote $T_1 = v_1v_2zv_1$ and $T_2 = v_3v_4v_5v_3$ the two triangles in $\overline{H}$. Observe that H contains no multiple edge since H already contains two triangles, and that neither of the vertices v_1, v_2, v_3, v_5 can have another neighbor in $\overline{H}$ for otherwise H contains a diamond, a contradiction. Thus, $\overline{H}$ is either induced, or $\langle V(\overline{H})\rangle_H = \overline{H} + zv_4$. Moreover, if $zv_4 \notin E(H)$, then, by the connectivity assumption, the graph $H - \{v_1, v_2, v_3, v_5\}$ contains a (z, v_4)-path (since otherwise $\{h_1, h_3\}$ is an essential edge-cut of size 2 in H). Specifically, we have $d_H(z) \geq 3$ and $d_H(v_4) \geq 3$.

Since h_2 is in a triangle, $x_2 = L(h_2)$ is eligible in G, implying $G^*_{x_2} = L(H|_{h_2})$ is Hamilton-connected since G is SM-closed. Thus, the graph $H_1 = H|_{h_2}$ has an (e_1, e_2)-IDT for any $e_1, e_2 \in E(H_1)$. We will show that H has an (f_1, f_2)-IDT for any $f_1 \in E(T_1)$ and $f_2 \in E(T_2)$, contradicting the fact that $G = L(H)$ is SM-closed.

Thus, choose any $e_1, e_2 \in E(H_1)$, let T' be an (e_1, e_2)-IDT in H_1 and let T be the part of T' that is outside $\overline{H}|_{h_2}$ (in the special case when $V(\overline{H})$ dominates all edges of H and

$T' \subset \overline{H}|_{h_2}$, necessarily $zv_4 \in E(H)$ or $zw, wv_4 \in E(H)$ for some $w \in V(H) \setminus V(\overline{H})$, and we choose $T = zv_4$ or $T = zwv_4$, respectively).

Then T is also a trail in $H - \overline{H}$, with initial and terminal edges incident to z and/or v_4 and dominating all edges in $H - \overline{H}$. We distinguish two possibilities:

(α) both $d_T(z)$ and $d_T(v_4)$ is odd,

(β) both $d_T(z)$ and $d_T(v_4)$ is even (possibly zero).

In the case (β), only one of $d_T(z)$, $d_T(v_4)$ can be zero and, by symmetry, we choose the notation such that $d_T(z) \neq 0$. Up to a symmetry, we have the following possibilities for $f_1 \in E(T_1)$ and $f_2 \in E(T_2)$. In each of them, we find an (f_1, f_2)-IDT in H for both possibilities (α) and (β).

Case	f_1	f_2	An (f_1, f_2)-IDT for the possibility (α)	(β)
1	h_7	h_5	$zv_1v_2zTv_4v_3v_5v_4$	$v_1zTzv_2v_1v_5v_3v_4v_5$
2	h_7	h_4	$zv_1v_2zTv_4v_5v_3v_4$	$v_1zTzv_2v_1v_5v_4v_3$
3	h_7	h_6	$zv_1v_2zTv_4v_3v_5$	$v_1zTzv_2v_1v_5v_4v_3v_5$
4	h_2	h_4	Symmetric to 3(α)	$v_2v_1zTzv_2v_3v_5v_4v_3$
5	h_2	h_6	$v_2v_1zTv_4v_3v_5$	$v_2v_1zTzv_2v_3v_4v_5v_3$

■

Proof of Proposition 1. Let G_0 be a claw-free graph and $x \in V(G_0)$ such that $G_0 - x$ is not Hamilton-connected, and let $(\widetilde{G_0})_x$ be a partial x-closure of G_0. In the rest of the proof, we will simply denote $G := (\widetilde{G_0})_x$.

Immediately by the construction of G, G is claw-free and $G - x$ is SM-closed. Thus, it remains to show that G satisfies (i), (ii) or (iii).

We introduce the following notation:

$N_G(x) = \{x_1, \ldots, x_d\}$ (i.e., $d_G(x) = d$),

$\mathcal{K}$ – Krausz partition of $G - x$,

$K'_1, \ldots, K'_k$ – all cliques in $\mathcal{K}$ with $K'_i \cap N_G(x) \neq \emptyset$, $i = 1, \ldots, k$,

$H' = L^{-1}(G - x)$,

$K_i = K'_i \cap N_G(x)$, $i = 1, \ldots, k$.

The cliques $K_1, \ldots, K_k \subset \langle N_G(x) \rangle_G$ satisfy the conditions of Theorem B (applied on $\langle N_G(x) \rangle_G$), and we use H to denote the intersection graph of the system $\{K_1, \ldots, K_k\}$. Then we have $H \subset H'$ and $L(H) = \langle N_G(x) \rangle_G$. However, note that not necessarily $H = L^{-1}(\langle N_G(x) \rangle_G)$ (since the graph H can be another "preimage" of $\langle N_G(x) \rangle_G$, see e.g. the example in Fig. 1).

Using the correspondence between a line graph and its preimage, we will identify Krausz cliques in $G - x$ with the vertices of H' (the centers of the stars in H' that correspond to the cliques in $\mathcal{K}$). Thus, $\{K'_1, \ldots, K'_k\} \subset V(H')$ and $\{K_1, \ldots, K_k\} = V(H)$.

Note that if $N_G(x)$ can be covered by two Krausz cliques, then at most two cliques from $\mathcal{K}$ have at least two vertices in $N_G(x)$ (hence at least one edge in $\langle N_G(x) \rangle_G$), and extending these cliques to x we get a Krausz partition of G. Thus, to show that G satisfies (iii), it is sufficient to show that $N_G(x)$ can be covered by two Krausz cliques.

Suppose first that $\langle N_G(x)\rangle_G$ is disconnected, and let F_1, F_2 be its components. Then both F_1 and F_2 are cliques since G is claw-free. If F_1, F_2 are subcliques of Krausz cliques in $G-x$, we are done; so, suppose that, say, F_1 is not. Then, as noted in Section 2, $L^{-1}(F_1)$ is a (multi)triangle or a multiedge in $H' = L^{-1}(G-x)$; since $G-x$ is SM-closed, $L^{-1}(F_1)$ is a triangle or a double edge.

If $L^{-1}(F_1)$ is a double edge, then $L^{-1}(F_1) = \langle\{K'_a, K'_b\}\rangle_H$ for some $a, b \in \{1, \ldots, k\}$, and since F_1 is a clique, one of K'_a, K'_b, say, K'_b, has no neighbor w with $L(K'_b w) \in N_G(x)$, but then F_1 is a subclique of $K'_a \in \mathcal{K}$, a contradiction. So, suppose that $L^{-1}(F_1)$ is a triangle, set $L^{-1}(F_1) = \langle\{K'_a, K'_b, K'_c\}\rangle_H$ (where $a, b, c \in \{1, \ldots, k\}$), and let $z \in V(F_2)$ be arbitrary. By the properties of the preimage L^{-1} (see Section 2), at least two of the vertices K'_a, K'_b, K'_c have a neighbor outside $\{K'_a, K'_b, K'_c\}$. Let, say, $K'_a w_1, K'_b w_2 \in E(H')$, where $w_1, w_2 \in V(H') \setminus \{K'_a, K'_b, K'_c\}$. Then $w_1 \neq w_2$ (otherwise H' contains a diamond), both $L(K'_a w_1) \notin N_G(x)$ and $L(K'_b w_2) \notin N_G(x)$ (for if e.g. $L(K'_a w_1) \in N_G(x)$, then $\langle\{x, L(K'_a w_1), L(K'_b K'_c), z\}\rangle_G$ is a claw), but then $\langle\{L(K'_a K'_b), L(K'_a w_1), L(K'_b w_2), z\}\rangle_G$ is a claw, a contradiction again.

Thus, we can suppose that $\langle N_G(x)\rangle_G$ (and therefore also H) is connected.

Claim 1. *If H contains a triangle and does not contain a C_5, then $L(H) = \langle N_G(x)\rangle_G$ can be covered by two Krausz cliques.*

Proof. Let, say, $T = \langle\{K'_1, K'_2, K'_3\}\rangle_H$ be a triangle in H and denote $h_1 = K'_1 K'_3$, $h_2 = K'_1 K'_2$, $h_3 = K'_2 K'_3$. By Lemma 8, $d_{H'}(K'_i) \geq 3$, $i = 1, 2, 3$. Let $e_i \in E(H') \setminus E(T)$ be an edge incident to K'_i, and set $y_i = L(e_i)$ and $x_i = L(h_i)$, $i = 1, 2, 3$. Since H' does not contain a diamond, the edges e_1, e_2, e_3 have no vertex in common, i.e., $\{e_1, e_2, e_3\}$ is a matching in H'. Hence the vertices y_1, y_2, y_3 are independent in $G - x$.

Now, if all y_i, $i = 1.2.3$, are in $N_G(x)$, then $\langle\{x, y_1, y_2, y_3\}\rangle_G$ is a claw in G, and if, say, $y_1, y_2 \in V(G) \setminus N_G(x)$, then $\langle\{x_2, y_1, y_2, x\}\rangle_G$ is a claw in G, a contradiction. Hence exactly two x_i's are in $N_G(x)$. Choose the notation such that $x_1, x_2 \in N_G(x)$ and $x_3 \in V(G) \setminus N_G(x)$. Then, since the edge e_3 was chosen arbitrarily, we have $d_H(K'_3) = 2$.

If all other edges of H are incident to K'_1 or K'_2, then $E(H)$ can be covered by two stars centered at K'_1, K'_2, hence $\langle N_G(x)\rangle_G$ can be covered by two cliques and we are done. Hence suppose that there is an $f \in E(H)$ that is incident to none of K'_1, K'_2, K'_3. since H is connected, we can choose f such that f has a common vertex with, say, e_1. Set $L(f) = z$.

But now, if f has a common vertex with e_2, then e_1, h_1, h_3, e_2, f determine a C_5 in H, contradicting the assumption, and if f does not share a vertex with e_2, then $\{f, h_1, e_2\}$ is a matching in H, implying $\langle\{x, z, x_1, y_2\}\rangle_G$ is a claw in G, a contradiction again. □

We now distinguish two cases.

Case 1: *$\langle N_G(x)\rangle_G$ does not contain an induced cycle of length 5.*

Then, equivalently, H does not contain a cycle C_5 (not necessarily induced).

First observe that $\alpha(\langle N_G(x)\rangle_G) = \nu(H) \leq 2$, for otherwise x is a center of an induced claw in G, This immediately implies that H does not contain a cycle C_ℓ of length $\ell \geq 6$, since such a cycle contains a matching of size 3. If H contains a triangle, then $\langle N_G(x)\rangle_G$ can be covered by two cliques by Claim 1 and we are done. Thus, the only possible cycles in H are of length 4.

Let $C = x_1x_2x_3x_4x_1$ be a cycle of length 4 in H. Since H is triangle-free, C is chordless. If $V(H) = V(C)$, then H can be covered by two stars (hence $\langle N_G(x)\rangle_G$ can be covered by two cliques) and we are done; if H contains an edge $e = uv$ with $\{u, v\} \cap V(C) = \emptyset$, then e together with two edges from $E(C)$ form a matching of size 3 in H, a contradiction. Hence every edge in $E(H) \setminus E(C)$ has exactly one vertex in $V(C)$.

Now, if some two consecutive vertices of C have a neighbor outside C, say, $x_1y_1 \in E(H)$ and $x_2y_2 \in E(H)$ for some $y_1, y_2 \in V(H) \setminus V(C)$, then $y_1 \neq y_2$ (since H is triangle-free) and $\{x_1y_1, x_2y_2, x_3x_4\}$ is a matching in H, a contradiction. Hence all edges in $E(H)\setminus E(C)$ are incident to some pair of nonconsecutive vertices of C, implying H can be covered by two stars.

Thus, it remains to consider the case when H is a tree. Let $D = \{d_1, \ldots, d_\gamma\}$ be a minimum dominating set in H. By the minimality of D, for every i, $1 \leq i \leq \gamma$, there is a vertex $w_i \in V(H) \setminus D$ such that d_i is the only neighbor of w_i in D. If $\gamma \geq 3$, then $\{d_1w_1, d_2w_2, d_3w_3\}$ is a matching in H, hence $\gamma \leq 2$, implying $\{d_1\}$ (if $\gamma = 1$) or $\{d_1, d_2\}$ (if $\gamma = 2$) are centers of stars covering all edges of H.

Case 2: *$\langle N_G(x)\rangle_G$ contains an induced cycle of length 5.*

Let C be an induced cycle of length 5 in $\langle N_G(x)\rangle_G$. Then $L^{-1}(C)$ is a C_5 (not necessarily induced) in H. If $k \geq 6$, then there is an edge $e \in E(H) \setminus E(C)$ with at least one vertex outside C, but then e together with two edges of C form a matching of size 3 in H, a contradiction. Hence $k = 5$ and $N_G(x) = V(C)$.

We choose the notation such that $C = x_1x_2x_3x_4x_5x_1$ and $x_ix_{i+1} \in E(K'_i)$ (i.e., $x_{i+1} \in K'_i \cap K'_{i+1}$), $i = 1, \ldots, 5$ (indices mod 5). Then $C_H = K'_1K'_2K'_3K'_4K'_5K'_1$ is the corresponding 5-cycle in $H' = L^{-1}(G - x)$, and we denote its edges $h_i = L^{-1}(x_i)$ (i.e., $h_{i+1} = K'_iK'_{i+1}$), $i = 1, \ldots, 5$ (indices mod 5).

Claim 2. *For any $y \in N_G(x)$, $y \in K_i \cap K_j$ for some $i, j = 1, \ldots, 5$, $i \neq j$.*

Proof. If e.g. $y \in K_1 \setminus (\cup_{i=2}^5 K_i)$ for some $y \in N_G(x)$, then $y \in K'$ for some other $K' \in \mathcal{K}$ (since every vertex is in 2 Krausz cliques), implying $k \geq 6$, a contradiction. $\square$

We introduce the following notation:
$\mathcal{K}_x = \{K'_1, \ldots, K'_5\}$,
$K_x = \cup_{i=1}^5 K'_i$,
$R = V(G) \setminus (\{x\} \cup K_x)$,
$\mathcal{K}_R = \mathcal{K} \setminus \mathcal{K}_x$,
$I(K'_i) = K'_i \setminus (\cup_{j\in(\{1,\ldots,5\}\setminus\{i\})} K'_j)$, $i = 1, \ldots, 5$.
The vertices in $I(K'_i)$ will be referred to as the *internal vertices* of the clique K'_i. Note that, by Claim 2, $I(K'_i) \cap N_G(x) = \emptyset$, $i = 1, \ldots, 5$.

Claim 3. *If $y \in K_x$ has a neighbor in R, then $y \in I(K'_i)$ for some $i = 1, \ldots, 5$.*

Proof. By the properties of the Krausz cliques and by Claim 2, only vertices in $I(K'_i)$ can have a neighbor in R, since if a vertex $y \in K'_i \cap K'_j$ (for some $i, j \in i, \ldots, 5$) has a neighbor in R, then y is in three Krausz cliques, a contradiction. $\square$

Claim 4. *If $y_1 \in I(K'_i)$ and $y_2 \in I(K'_{i+1})$ for some $i = 1, \ldots, 5$, then*

(*i*) $y_1y_2 \in E(G)$,

(*ii*) $y_1y_2 \in E(\langle K \rangle_G)$ *for some* $K \in \mathcal{K}_R$,

(*iii*) $\langle \{K'_i, K'_{i+1}, K\} \rangle_G$ *is a traingle in* $H' = L^{-1}(G - x)$,

(*iv*) $|I(K'_i)| = |I(K'_{i+1})| = 1$.

Proof. Let e.g. $y_1 \in I(K'_1)$ and $y_2 \in I(K'_2)$.

(*i*) If $y_1y_2 \notin E(G)$, then $\langle \{x_2, x, y_1, y_2\} \rangle_G$ is a claw in G.

(*ii*) If $y_1y_2 \in E(\langle K'_i \rangle_G)$ for some $i = 2, \ldots, 5$, then $y_1 \in K'_1 \cap K'_i$, contradicting the assumption $y_1 \in I(K'_i)$. Hence $y_1y_2 \in E(\langle K \rangle_G)$ for some $K \in \mathcal{K}_R$,

(*iii*) Follows immediately by the structure of K'_1, K'_2 and K.

(*iv*) If e.g. $y_1, y'_1 \in I(K'_1)$, $y_1 \neq y'_1$, then $y_1, y'_1 \in K_i \cap K$, implying H' contains a triangle and a double edge, a contradiction. $\square$

Claim 5. *There is no j, $1 \leq j \leq 5$, such that $I(K'_i) \neq \emptyset$ for $i = j, j+1, j+2$.*

Proof. Let e.g. $I(K'_i) \neq \emptyset$ for $i = 1, 2, 3$. By Claim 4, the edge y_1y_2 is in some clique $K^1 \in \mathcal{K}_R$, and y_2y_3 is in some $K^2 \in \mathcal{K}_R$. Since y_2 cannot be in three Krausz cliques, we have $K^1 = K^2$, implying that $y_1y_3 \in E(G)$ and y_1y_3 is also in K^1. Then we have $y_1 \in K^1 \cap K'_1$, $y_2 \in K^1 \cap K'_2$, $y_3 \in K^1 \cap K'_3$, $x_2 \in K'_1 \cap K'_2$ and $x_3 \in K'_2 \cap K'_3$, implying that K^1, K'_1, K'_2, K'_3 are vertices of a diamond in H', a contradiction. $\square$

Claim 6. $|\{i|\ 1 \leq i \leq 5, I(K'_i) \neq \emptyset\}| \leq 3$.

Proof. Otherwise we have $I(K'_i) \neq \emptyset$ for some three consecutive cliques K'_i, contradicting Claim 5. $\square$

Claim 7. $|K'_i \cap K'_{i+1}| = 1$, $i = 1, \ldots, 5$.

Proof. Let, to the contrary, e.g. $|K'_1 \cap K'_2| \geq 2$. Then $\langle \{K'_1, K'_2\} \rangle_{H'}$ is a multiedge, implying $|K'_1 \cap K'_2| = 2$ and $|K'_i \cap K'_{i+1}| = 1$ for $i = 2, 3, 4, 5$. Moreover, there is no i, $1 \leq i \leq 5$, such that both $I(K'_i) \neq \emptyset$ and $I(K'_{i+1}) \neq \emptyset$, for otherwise, by Claim 4, H' contains a triangle, contradicting the fact that H' already contains a double edge. Hence $|\{i|\ 1 \leq i \leq 5, I(K'_i) \neq \emptyset\}| \leq 2$, and the vertices K'_i with $I(K'_i) \neq \emptyset$ are nonconsecutive on the 5-cycle $C_H = K'_1K'_2K'_3K'_4K'_5K'_1$ in H'. Moreover, if $I(K'_i) \neq \emptyset$ and $I(K'_j) \neq \emptyset$ for some i, j, then $K'_i \cap K'_j = \emptyset$, for otherwise the edge $K'_iK'_j \in E(H')$ is a chord in C_H, contradicting again the properties of SM-closed graphs.

This means that the 5-cycle C_H is chordless, $\langle \{K'_1, K'_2\} \rangle_{H'}$ is the only double edge, at most two vertices of C_H can have a neighbor outside C_H (namely, those for which the corresponding clique in $G - x$ has some internal vertices), and these verties are nonconsecutive.

Now, if $I(K_1') = I(K_2') = \emptyset$, then $\{K_1'K_5', K_2'K_3'\}$ is an essential edge-cut in both H' and $H|_{K_1'K_2'}$, implying that neither $G-x = L(H')$ nor $(G-x)^*_{x_2} = L(H'|_{K_1'K_2'})$ is Hamilton-connected, contradicting the fact that $G-x$ is SM-closed (note that x_2 is eligible since $x_2 = L^{-1}(K_1'K_2')$ and $K_1'K_2'$ is in a double edge). Thus, we can suppose $I(K_1') \neq \emptyset$. But then at least two vertices of C_H are of degree 2 in H' and we have a contradiction with Lemma 10. $\square$

Now we can finish the proof of Proposition 1. Clearly, $I(K_i') \neq \emptyset$ for at least one i, $1 \leq i \leq 5$, for otherwise $V(G) = N_G(x)$ and there is nothing to do. Thus, by Claim 6, one, two or three cliques K_i' have $I(K_i') \neq \emptyset$. We consider these possibilities separately.

Subcase 2.1: $|\{i|\ 1 \leq i \leq 5, I(K_i') \neq \emptyset\}| = 3$.

By Claim 5, we have $I(K_i') \neq \emptyset$ for at most two consecutive cliques K_i'. Thus, without loss of generality let $I(K_i') \neq \emptyset$ for $i = 1, 2, 4$ (i.e., $I(K_3') = I(K_5') = \emptyset$). By Claim 4, there is a vertex $y \in V(H') \setminus V(H)$ such that $\langle\{K_1', K_2', y\}\rangle_{H'}$ is a triangle. If $d_{H'}(K_3') = d_{H'}(K_5') = 2$, we have a contradiction by Lemma 10. Thus we have, say, $d_{H'}(K_3') \geq 3$, i.e., besides K_2' and K_4', K_3' has at least one more neighbor, say, z. Then $z \in \{K_1', K_2', K_4', K_5'\}$ since $I(K_3') = \emptyset$, and the only possibility that does not create a double edge or a diamond (recall that H' already contains a triangle) is $z = K_5'$ and $d_{H'}(K_3') = d_{H'}(K_5') = 3$. Set $\overline{H} = \langle\{K_1', K_2', K_3', K_4', K_5', y\}\rangle_{H'}$ and note that $T_1 = \langle\{K_1', K_2', y\}\rangle_{H'}$ and $T_2 = \langle\{K_3', K_4', K_5'\}\rangle_{H'}$ are two triangles in $\overline{H}$ (hence also in H') and, by Claim 4(iv), y and K_4' are the only vertices of $\overline{H}$ that can have adjacencies outside $\overline{H}$. But then $\overline{H}$ (or possibly $\overline{H} - yK_4'$, if $yK_4' \in E(H')$), has the structure shown in Fig. 6 and we have a contradiction by Lemma 11.

Subcase 2.2: $|\{i|\ 1 \leq i \leq 5, I(K_i') \neq \emptyset\}| = 2$.

By symmetry, we can choose the notation such that $I(K_1') \neq \emptyset$ and either $I(K_2') \neq \emptyset$ or $I(K_3') \neq \emptyset$.

Let first $I(K_1') \neq \emptyset$, $I(K_2') \neq \emptyset$. By Claim 4, there is a vertex $y \in V(H') \setminus V(H)$ such that $\langle\{y, K_1', K_2'\}\rangle_{H'}$ is a triangle and y is the only neighbor of K_1' and K_2' outside H. If the cycle C_H is cordless, we have a contradiction by Lemma 10, and if C_H has a chord, we have a contradiction by Lemma 8.

Thus, suppose that $I(K_1') \neq \emptyset$, $I(K_3') \neq \emptyset$. By Claim 7 and by the properties of SM-closed graphs, C_H has no multiedge and at most one chord, but if C_H has a chord, we have a contradiction with Lemma 9. Hence C_H is chordless. Then $\{h_1, h_4\}$ is an essential edge-cut in H', separating h_5 from the rest of H', hence $\{x_1, x_4\}$ is a vertex-cut in $G - x$, separating x_5 from the rest of $G - x$. The graph $(G - x) + x_1x_4$ is SM-closed, since it is the line graph of a graph obtained from H' by contracting the edge h_5 and adding a pendant edge to the contracted vertex, and this operation creates neither a triangle nor a multiedge. Thus, the graph $G - x$ satisfies all conditions of part (ii) of Proposition 1.

Subcase 2.3: $|\{i|\ 1 \leq i \leq 5, I(K_i') \neq \emptyset\}| = 1$.

If C_H has a chord, we have a contradiction with Lemma 9, hence C_H is chordless. But then again, e.g. $\{h_1, h_4\}$ is an edge-cut in H' and we can add the edge x_1x_4 to $G - x$ to satisfy all conditions of part (ii) of Proposition 1. ■

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