Exact Domain Integration in the Boundary Element Method for 2D Poisson Equation

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***Abstract*—Boundary value problems for Poisson equation often appear in electrical engineering applications, such as magnetic and electric field modeling and so on. In such context, effective techniques of solving such equations are subject of continuous development. This article reports an exact formula for domain integral in boundary-integral form of 2D Poisson Equation. This formula is derived for rectangle domain element.**

I. Introduction

Boundary element approach is known as an effective way to solve linear partial differential equations, particularly Poisson equation, which often used in electrical engineering problems, such as magnetic and electric field computation [1], some simplified hydrodynamic models [2] and many others. In such problems the source term of boundary integral equation is computed numerically, especially in the cases, when it cannot be reduced to boundary integral. This is common practice, when source term is not differentiable [3]. But in some cases, such as inverse problems, control problems and so on it is preferably to have exact formulas for source-term domain integral. In this work, integration formula is given for rectangle domain element with constant value.

II. Problem statement

Consider Poisson equation

$$\Delta\vartheta = b \,, \tag{1}$$

where ϑ and f are functions of two variables, defined in region $\Omega \subset \mathrm{R}^2$ with boundary S. Let boundary be divided in two part: $S = S_1 \cup S_2$. On first one S_1 Dirichlet boundary conditions are prescribed: $\vartheta = \vartheta^*$. On boundary part S_2 Neumann boundary condition is precribed: $q = q^*$, $q_i = \vartheta_{,i} n_i$, where n_i is outer normal. Then integral representation of (1) is:

$$\begin{aligned}\vartheta(\xi) = &\int_{S_1} \big[q(x)\cdot u^*(\xi,x) - \vartheta^*(x)\cdot f^*(\xi,x)\big]dS(x) + \\ &+\int_{S_1} \big[q(x)^*\cdot u^*(\xi,x) - \vartheta(x)\cdot f^*(\xi,x)\big]dS(x) - \\ &-\int_{\Omega} b(x) * G(\xi,x)d\Omega\,,\end{aligned} \tag{2}$$

where $x \in S$ is boundary point and xi is inner point of Ω; $u^*(\xi,x)$ and $f^*(\xi,x)$ are influence functions; and $G(\xi,x)$ is Green's function. With fundamental solution

$$G = (-1/2\pi)ln(\vec{r}) \tag{3}$$

influence functions are:

$$\begin{aligned} u^*(\xi,x) &= -\frac{1}{2\pi}ln(\vec{r}), \\ f^*(\xi,x) &= -\frac{1}{2\pi r^2}r_i n_i\,. \end{aligned} \tag{4}$$

It is assumed in this work, that collocation approach is used, with constant value interpolation along boundary element. The main subject of this work is evaluation of domain integral

$$I_s = \int_{\Omega} f(x)\cdot G(\xi,x)d\Omega\,, \tag{5}$$

which defines influence of source terms on potential ϑ at inner point ξ.

III. Results

An approach, proposed in [4] for analytical evaluation of boundary integrals in 2D potential problem is used here. Let source b is defined as the set of rectangle subdomains (which are later called domain elements) with constant value source term approximation and zero value on other inner points of Ω. This situation is typical for electrical engineering applications, where rectangle elements can represent conducting parts.

Consider domain element D of rectangle form sides L_1 and L_2 with constant source value b on D. Let $\tilde{x} = Kx$ be coordinate transformation with following properties:

1) all the points of rectangle are in positive quadrant;
2) one of corners of rectangle is $\tilde{x} = (0,0)$, i.e. rectangle sides coincide with coordinate axes and lower left corner lies in $(0,0)$
3) inner point $\tilde{\xi} = (\tilde{\xi_1}, \tilde{\xi_2})$ lies in positive quadrant.

With this coordinate transformantion defined, integral (4) is reduced to summ of integrals over domain elements of type:

$$I = -\frac{b}{4\pi}\int_0^{L_1}\int_0^{L_2} ln\big|\big|\tilde{x} - \tilde{\xi}\big|\big|^2 d\tilde{x}_1 d\tilde{x}_2 \\ = -\frac{b}{4\pi}\Big[I_1 - I_2 + I_3 + I_4 + I_5\Big]\,, \tag{6}$$

where

$$I_1 = J_1^2 atan\Big(\frac{J2}{J1}\Big) + J_2 J_1 - J_2^2 atan\Big(\frac{J1}{J2}\Big) + \\ + \tilde{\xi}_1^2 atan\Big(\frac{J2}{\tilde{\xi}_1}\Big) + J_2\tilde{\xi}_1 - J_2^2 atan\Big(\frac{\tilde{\xi}_1}{J2}\Big)\,; \tag{7}$$

$$I_2 = -J_1^2 atan\Big(\frac{\tilde{\xi}_2}{J1}\Big) - \tilde{\xi}_2 L_1 - \tilde{\xi}_2^2 * atan\Big(\frac{J_1}{\tilde{\xi}_2}\Big) + \\ + \tilde{\xi}_1^2 * atan\Big(\frac{\tilde{\xi}_2}{\tilde{\xi}_1}\Big) + \tilde{\xi}_2^2 atan\Big(\frac{\tilde{\xi}_1}{\tilde{\xi}_2}\Big)\,; \tag{8}$$

$$I_3 = -J_2\Big[J_1 * ln\big(J_1^2 + J_2^2\big) + \tilde{\xi}_1 ln\big(\tilde{\xi}_1^2 + J_2^2\big) + \\ + 2J_2 atan\Big(\frac{J_1}{J_2}\Big) + 2J_2 atan\Big(\frac{\tilde{\xi}_1}{J_2}\Big) - 2L_1\Big]\,; \tag{9}$$

$$I_4 = \tilde{\xi}_2 \cdot \Big[J_1 ln\big(J_1^2 + \tilde{\xi}_2^2\big) - 2J_1 + 2\tilde{\xi}_2 atan\Big(\frac{J_1}{\tilde{\xi}_2}\Big) + \\ + \tilde{\xi}_1 ln\big(\tilde{\xi}_1^2 + \tilde{\xi}_2^2\big) - 2\tilde{\xi}_1 + 2\tilde{\xi}_2 atan\Big(\frac{\tilde{\xi}_1}{\tilde{\xi}_2}\Big)\Big]\,; \tag{10}$$

$$I_5 = -2L_1\big(J_2 + \tilde{\xi}_2\big)\,; \tag{11}$$

and

$$J_1 = L_1 - \tilde{\xi}_1\,; \\ J_2 = L_2 - \tilde{\xi}_2\,. \tag{12}$$

This formulas can be used in assembling procedure of linear equation system. In postprocessing its usage is limited to the points, which does not placed on the border of domain elements. The reason is that, the limit for $atan(x/y)$ with $x \to 0$, $y \to 0$ does not exist. Thus coordinate transformation K has to meet some additional requirements. Let V be set of vertexes of domain element and B is a boundary of domain element. Then, additional requirements are:

1) if $\xi \in V$ then $\tilde{\xi} = (0, 0)$;
2) if $\xi \in B/V$ then $\tilde{\xi}_1 = 0$, or, optionally, $\tilde{\xi}_2 = 0$.

In first case, when $\tilde{\xi} = (0, 0)$, integral (6) reduces to

$$I = -\frac{b}{24\pi}\int_{-L_1}^{L_1}\int_{-L_2}^{L_2} ln\big(\tilde{x}_1^2 + \tilde{x}_2^2\big) d\tilde{x}_1 d\tilde{x}_2 = \\ = -\frac{b}{8\pi}\Big[L_1 L_2 \cdot ln\big(L_1^2 + L_2^2\big) + L_1^2 atan\Big(\frac{L_2}{L_1}\Big) + \\ + L_2^2 atan\Big(\frac{L_1}{L_2}\Big) - 3L_1 L_2\Big]\,. \tag{13}$$

It should be noted, that integral (13) does not depend on coordinates of point ξ. This means, that no coordinate transformation is needed, when ξ coincides with corner of some rectangle domain element. In second case, when $\xi \in B/V$, integration can be carried around extended domain $\tilde{D} = D \cup D1$, where $D1$ is domain element D, rotated around point $\tilde{\xi}$ on angle $\pm\pi$. Thus point $\tilde{\xi}$ is not on $\tilde{D}$ domain element boundary. Because of simmetricity of Green function, integral (3) becomes

$$I = \frac{1}{2}\tilde{I}, \\ \tilde{I} = \int_{\tilde{D}} G(\xi, x) d\Omega\,, \tag{14}$$

where $\tilde{I}$ can be evaluated using transformation K and formulas (6)-(12) repeatedly.

IV. Conclusion

In present work some technique for domain element integration is proposed. The technique proposed can be used in boundary element method for two-dimensional potential problems. Rectangle domain element with constant value is considered. Approach of work [4] was used. Some additional requirements were formulated for coordinate transform, proposed in [4]. It is shown also, that in some particular cases no coordinate transformation needed. An implementation of coordinate transform is subject for future work.

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