Global solution of a class of interval parameter optimization problems

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***Abstract*—A class of parametric optimization problems is considered which arises in the analysis of linear electric circuits whose parameters are uncertain and given as intervals. This class involves such problems as tolerance analysis, stability analysis and power consumption analysis. A unified method for globally solving the resulting interval parameter optimization problems is suggested. It is based on the use of so-called modified monotonicity conditions.**

Keywords—linear circuits; interval parameters; global optimization; unified solution method; modified monotonicity.

I. INTRODUCTION

Let $p=(p_1,..,p_m)$ be a real m-dimensional vector belonging to a given interval vector $\boldsymbol{p}=(\boldsymbol{p}_1,..,\boldsymbol{p}_m)$. Also, let $A(p)$ and $b(p)$ be a real $(n\times n)$ matrix and a n-dimensional vector $b(p)$ whose elements depend on *p*. As is well known, a (real) linear interval parameter (LIP) system is defined as the family of linear algebraic systems

$$A(p)\,x=b(p)\,, \tag{1}$$

$$a_{ij}(p)=a_{ij}(p_1,...,p_m),\quad b_i(p)=b_i(p_1,...,p_m) \tag{1a}$$

where $a_{ij}(p)=a_{ij}(p_1,...,p_m)$ and $b_i(p)=b_i(p_1,...,p_m)$ are given nonlinear (in the general case) functions,

$$p_\mu\in\boldsymbol{p}_\mu\,,\quad \mu=1,...,m\,. \tag{1b}$$

Systems of this type describe various analysis problems such as *tolerance analysis* [1], [2], [5], [7,] *stability analysis* [1] [4], *power consumption analysis* [3], [6] etc. in *linear electric circuits whose parameters are uncertain and given as intervals*. In this paper, we show that all these problems can be stated using a *general formulation*. According to this approach any specific problem is reduced to solving several times (up to 2*n*) the following optimization problem: find the global minimum

$$g_l^*=\min g_l(x,p) \tag{2}$$

subject to the constraint (1) where g_l is, in the general case, a nonlinear function. It should be stressed that the global optimization problem (2), (1) is NP-hard (its numerical complexity grows exponentially with *m* and *n*). In this paper, a much simpler unified method of polynomial complexity for solving (2), (1) is suggested which is based on the use of so-called *modified monotonicity conditions.*

II. PROBLEM FORMULATION

To present *the general formulation* suggested in this paper, we first introduce an additional n'-dimensional *output variable vector* $(n'\le n)$

$$y=f(x,p)\,. \tag{3}$$

Let $y_k=f_k(x,p)$ be the *k*th component of *y*. We now consider the pair of global optimization problems

$$y_{kl}^*=\min\{f_k(x,p):\ A(p)x=b(p),\ p\in\boldsymbol{p}\}, \tag{4a}$$

$$\begin{aligned} y_{ku}^* &=\max\{f_k(x,p):\ A(p)x=b(p),\ p\in\boldsymbol{p}\}\\ &=-\min\{-f_k(x,p):\ A(p)x=b(p),\ p\in\boldsymbol{p}\}\end{aligned}. \tag{4b}$$

Obviously, the interval $\boldsymbol{y}_k^*=\left[y_{k-}^*,\,y_{k+}^*\right]$ defines the range of y_k over $\boldsymbol{p}$ for the linear constraint (1). The interval vector $\boldsymbol{y}^*=\left(\boldsymbol{y}_1^*,...,\boldsymbol{y}_{n'}^*\right)$ will be called the *exact* (within round-off errors) *interval* (EI) solution of (3), (1). Thus, the general formulation sought can be stated as follows: given the triple $\{A(p),\,b(p),\,f\}$ and the interval vector $\boldsymbol{p}$, find the corresponding EI solution $\boldsymbol{y}^*$. As is seen from (4), solving the general formulation problem reduces indeed to solving $2n'$ problems of the type (2).

The above general formulation covers a large class of known range determination problems. This will be illustrated by the following example.

Example. Tolerance analysis of DC circuits. In this case, *f* is independent of *p* and $f=E$ (*E* is the identity matrix). Thus, the problem is to determine the EI solution $\boldsymbol{x}^*$ related only to (1). From (4) it can be seen that each component $\boldsymbol{x}_k^*$ is obtained by solving two *parametric linear programming problems.*

In the case of AC circuits, consider the problem of determining the square of the magnitude *v* of a single complex variable V_k involved in a $(n\times n)$ complex-valued system [1]

$$GV=J\,. \tag{5}$$

The latter system can be rewritten equivalently as a $(2n \times 2n)$ real-valued LIP system by introducing a $2n$-dimensional real state vector x. In x, the first n components correspond to the respective real parts of V_j while the next n components correspond to the respective imaginary parts of V_j. Thus, for this example [2]

$$y = x_k^2 + x_{k+n}^2 . \quad (6)$$

III. UNIFIED METHOD

An iterative method for solving (4) is suggested in this section. For simplicity, it will be presented for the simpler case of $f = E$, i.e. when the problem is to compute the EI solution $\boldsymbol{x}^*$ related only to (1). The new method is based on individually finding each interval component $\boldsymbol{x}_k^*$ of $\boldsymbol{x}^*$. Each component $\boldsymbol{x}_k^* = [x_{k-}^*, x_{k+}^*]$ is, in its turn, found by separately determining the lower end-point x_{k-}^* and upper end-point x_{k+}^* of $\boldsymbol{x}_k^*$, respectively. The lower end-point x_{k-}^* is located as the solution of the following global optimization problem

$$x_{k-}^* = \min e_k^T x \quad (7a)$$

(e_k^T is the kth column of E) subject to the constraint

$$A(p)x = b(p), \quad p \in \boldsymbol{p} . \quad (7b)$$

The solution of (7) is found by an iterative method which, at each iteration, makes use of a respective outer solution $\boldsymbol{x}$ and an upper bound x_k^u on x_{k-}^*. The latter bound is determined using a simple local minimization technique requiring a polynomial amount of computation with respect to n and m. The upper end-point x_{k+}^* is located in a similar manner using relevant outer solutions $\boldsymbol{x}$ and lower bounds x_k^l on x_{k+}^*. In both cases, appropriate modified monotonicity conditions are checked and used. Such an approach results in a better performance as compared to other similar methods employing, however, standard monotonicity conditions.

The derivation of the modified monotonicity conditions is shown for the case of determining the lower end-point x_{k-}^*. After differentiating (1) with respect to p_l for a given $\boldsymbol{p}$ we get the system

$$\sum_{j=1}^{n} a_{ij}(p) \frac{\partial x_j}{\partial p_l} = \frac{\partial b_i(p)}{\partial p_l} - \sum_{j=1}^{n} \frac{\partial a_{ij}(p)}{\partial p_l} x_j , \quad i = 1, ..., n , \quad (8a)$$

which can be written equivalently as

$$A(p) d_l = \gamma_l(p) - \eta_l(p) x , \quad p \in \boldsymbol{p} , \quad x \in \boldsymbol{x} \quad (8b)$$

where $\gamma_l(p)$ is a column vector and $\eta_l(p)$ is a matrix. Let $\boldsymbol{d}_l$ denote an outer solution to (7b). Thus, the kth component $\boldsymbol{d}_{lk}$ of $\boldsymbol{d}_l$ is an enclosure for the derivative $\partial x_k / \partial p_l$ of x_k with respect to p_l for a given $\boldsymbol{p}$. The requirement $0 \notin \boldsymbol{d}_{lk}$ called *global monotonicity condition* has been used in [2]. If $0 \notin \boldsymbol{d}_{lk}$, we can reduce the interval $\boldsymbol{p}_l$ to an end-point p_{l-} or p_{l+} depending on whether $\boldsymbol{d}_{lk} \geq 0$ or $\boldsymbol{d}_{lk} \leq 0$. In this paper, *a better monotonicity condition* $0 \notin \boldsymbol{d}'_{lk}$ *called modified* is suggested. The interval $\boldsymbol{d}'_{lk}$ is defined as follows. We introduce a modified outer solution vector $\tilde{\boldsymbol{x}}$ with components

$$\tilde{\boldsymbol{x}}_i = \begin{cases} \boldsymbol{x}_i , & \text{if} \quad i \neq k \\ \tilde{\boldsymbol{x}}_k = [x_{k-}, x_k^u] & \text{if} \quad i = k \end{cases} . \quad (9)$$

The new approach consists in replacing (8b) with the following system:

$$A(p) d_l = \gamma_l(p) - \eta_l(p) x , \quad p \in \boldsymbol{p} , \quad x \in \tilde{\boldsymbol{x}} . \quad (10)$$

The interval $\boldsymbol{d}'_{lk}$ is computed as the kth component of the outer solution $\boldsymbol{d}'_l$ of (10). Again, if $0 \notin \boldsymbol{d}'_{lk}$, the interval $\boldsymbol{p}_l$ is reduced to an end-point p_{l-} or p_{l+} if $\boldsymbol{d}'_{lk} \geq 0$ or $\boldsymbol{d}'_{lk} \leq 0$, respectively. Since $\tilde{\boldsymbol{x}}_k \subset \boldsymbol{x}_k$, $\tilde{\boldsymbol{x}} \subset \boldsymbol{x}$, which entails $\boldsymbol{d}'_{lk} \subset \boldsymbol{d}_{lk}$ (the interval operations are known to be inclusive monotonic). Hence, the *modified monotonicity conditions are less restrictive as compared to the previously used global monotonicity conditions***.** It is shown that the new method converges in at most m number of iterations (m is the size of the parameter vector p) and that its numerical complexity is polynomial with respect to n and m.

The same conclusions remain valid for arbitrary function f. We now differentiate $y_k = f_k(x, p)$ to get

$$\frac{\partial y_k}{\partial p_l}(p) = \frac{\partial f_k}{\partial p_l}(x, p) + \sum_{j=1}^{n} \frac{\partial f_k}{\partial p_l}(x, p) \frac{\partial x_j}{\partial p_l}(x, p) . \quad (11)$$

The modified monotonicity conditions now refer to the derivative $\partial y_k / \partial p_l$

IV. APPLICATIONS

In this section, the above *general formulation* and *unified method* are applied to solve the following specific problems: (i) tolerance analysis of linear electric circuits described by nodal analysis equations, (ii) determination of the stability margin of linear electric circuits described by DAE equations [4] and (iii) determination of the range of the power consumed in linear electric circuits [3] described by nodal analysis equations. Comparisons with previous solutions are also provided.

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