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Radial Basis Functions Interpolation and Applications: An Incremental Approach

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***Abstract*—**Radial Basis Functions (RBF) interpolation is primarily used for interpolation of scattered data in higher dimensions. The RBF interpolation is a non-separable interpolation which offers a smooth interpolation, generally in n-dimensional space.

We present a new method for RBF computation using an incremental approach. The proposed method is especially convenient in cases when larger data sets are randomly updated as the proposed method is of $O(N^2)$ computational complexity instead of $O(N^3)$ for insert / remove operations only and therefore it is much faster than the standard approach. If t-varying data or vector data are to be interpolated, the proposed method offers a significant speed-up as well.

***Keywords*—**Interpolation, computer graphics, Radial basis interpolation, incremental inverse matrix computation

I. INTRODUCTION

RADIAL basis functions (RBF) are widely used across of many fields solving technical and non-technical problems. RBF applications can be found in neural networks, fuzzy systems, pattern recognition, solvers of partial differential equations, computer graphics, data visualization, medical applications, reconstruction of corrupted images etc.

RBF interpolation is mostly used for interpolation of static scalar values, e.g. interpolation of potential fields. Nevertheless there are many applications where t-varying data or vector data are to be interpolated.

Interpolation using radial basis functions was introduced by Hardy [5]. As the interpolation is based on distances of unordered points generally in n-dimensional space, the RBF interpolation is not separable. This causes a higher computational complexity on one hand, but on the other hand data are considered as scattered across the given interval in the given n-dimensional space.

Typical example of data is a data set $\{\boldsymbol{x}_i, \boldsymbol{h}_i\}$, where $\boldsymbol{x}_i$ is a point n-dimensional space and $\boldsymbol{h}_i$ is an associated vector of values (temperature, humidity, speed, acceleration etc.). We want to compute a value $\boldsymbol{h}$ in the given point $\boldsymbol{x}$ and the interpolation is to be smooth.

Such requirements driven by different applications lead to large inverse matrix computation of $O(N^3)$ complexity.

The proposed approach is based on incremental computation of RBF, which is very effective especially for data insertion and removal operation over the given data set. The efficiency of the proposed approach is given by decreased computational complexity from $O(N^3)$ to $O(N^2)$.

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II. RADIAL BASIS FUNCTIONS

Radial basis functions interpolation was originally introduced by Hardy [5] by introduction of multiquadric method, which he called Radial Basis Function (RBF) method, which is based on interpolation formula

$$f(x) = \sum_{i=1}^{N} \lambda_i \, \phi(r_i)$$

where: $\phi(r_i) = \phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|)$ and $\boldsymbol{x}$ is generally n-dimensional vector and λ_i are weights. Since then many different RFBF interpolation schemes have been developed with some specific properties, e.g. Duchon [4] use $\phi(r) = r^2 lg\, r$ s, which is called Thin-Plate Spline (TPS), a function $\phi(r) = e^{-(\epsilon r)^2}$ was proposed by Shagen [9] and Wetland [12] introduced Compactly Supported RBF (CSRBF) as

$$\phi(r) = \begin{cases} (1-r)^q \, P(r), & 0 \le r \le 1 \\ 0, & r > 1 \end{cases} ,$$

where: $P(r)$ is a polynomial function and q is a parameter.

Theoretical problems with stability and solvability were solved by Micchelli [6] and Wright [13] and he has extended the RBF by adding a polynomial function $P_k(\boldsymbol{x})$ of degree k to the RBF that resulted to:

$$f(\boldsymbol{x}) = \sum_{i=1}^{N} \lambda_i \, \phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|) + P_k(\boldsymbol{x}) = \sum_{i=1}^{N} \lambda_i \, \phi_i(\boldsymbol{x}) + P_k(\boldsymbol{x})$$

and additional conditions were introduced:

$$\sum_{i=1}^{N} \lambda_i = 0 \qquad \sum_{i=1}^{N} \lambda_i \, \boldsymbol{x} = \boldsymbol{0}$$

Usually a linear polynomial is used, i.e. the polynomial $P_k(\boldsymbol{x})$ is taken as

$$P_k(\boldsymbol{x}) = a_0 + \boldsymbol{a}^T \boldsymbol{x}$$

As the values $f(\boldsymbol{x}_i)$ at points $\boldsymbol{x}_i$ are known, the equations above form a system of linear equations that has to be solved in order to determine coefficients λ_i and $a_0, \boldsymbol{a}$, i.e.

$$f(\boldsymbol{x}_j) = \sum_{i=1}^{N} \lambda_i \, \phi(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|) + P_k(\boldsymbol{x}_j) = \sum_{i=1}^{N} \lambda_i \, \phi_{i,j} + P_k(\boldsymbol{x}_j)$$

$$j = 1, \dots, n$$

It can be seen that for n-dimensional case and N points given a system of $(N + n + 1)$ has to be solved, where N is a number of points in the dataset and n is dimensionality of data.

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For $n=2$ vectors $\boldsymbol{x}_i$ and $\boldsymbol{a}$ are given as $\boldsymbol{x}_i = [x_i, y_i]^T$ and $\boldsymbol{a} = [a_x, a_y]^T$.

Using the matrix notation we can write for 2-dimensions:

$$\begin{bmatrix} \phi_{1,1} & .. & \phi_{1,N} & x_1 & y_1 & 1 \\ : & & : & : & : & : \\ \phi_{N,1} & .. & \phi_{N,N} & x_N & y_N & 1 \\ x_1 & .. & x_N & 0 & 0 & 0 \\ y_1 & .. & y_N & 0 & 0 & 0 \\ 1 & .. & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ : \\ \lambda_N \\ a_x \\ a_y \\ a_0 \end{bmatrix} = \begin{bmatrix} f_1 \\ : \\ f_N \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{P} \\ \boldsymbol{P}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix} \qquad \boldsymbol{Ax} = \boldsymbol{b}$$

$$\boldsymbol{a}^T \boldsymbol{x}_i + a_0 = a_x \, x_i + a_y \, y_i + a_0$$

It can be seen that for 2-dimensional case and N points given a system of $(N+3)$ linear equations has to be solved. If "global" functions, e.g. TPS ($\phi(r) = r^2 lg\, r$), are used the matrix $\boldsymbol{B}$ is "full", if CSRBF functions are used, the matrix $\boldsymbol{B}$ can be sparse.

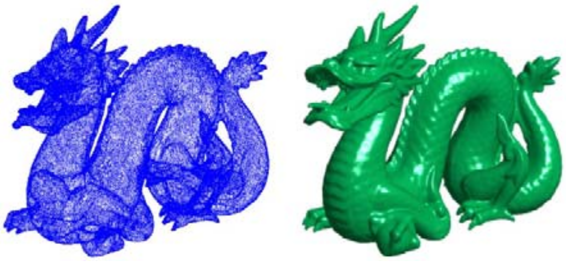

Figure 1: Surface reconstruction (438 000 points) Carr et al. [3]

Original image Bertalmio et al. [2] | Reconstructed image Uhlir et al. [10]

Figure 2: Reconstruction of inpainting

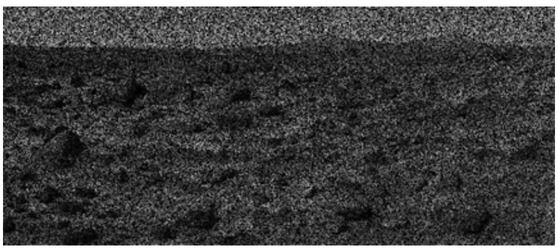

Figure 3a: Original image with 60% of damaged pixels

Figure 3b: Reconstructed image

Some interesting problems can be solved using RBF interpolation quite effectively, e.g. surface reconstruction from scattered data Carr et al. [3], Ohtake et al. [7], reconstruction of damaged images Uhlir et al. [10], Zapletal et al. [14], inpainting removal Bertalmio et al. [2], Wang et al. [11] etc.

All those applications of RBFs based interpolation have one significant disadvantage – the cost of computation. This is especially severe in applications where data are not static. There are actually two cases:

1. **Position of points is fixed**, but the value associated with a point is changed. In this case iterative methods are usually faster than explicit computation of an inverse matrix.
2. **Position of points is changed**. It means that the whole system of linear equations has to be form and recomputed which leads generally to $O(N^3)$ computational complexity and unacceptable time consuming computation.

In some applications a "*sliding window*" on data is required, especially in time related applications, when old data should not be used in the interpolation and new data should be included. This is typical situation in signal processing applications.

Considering facts above there is a question how to compute RBF incrementally with a lower computational complexity? This question will be answered in the following section.

III. Incremental RBF Computation

As the insert / remove operation is to be implemented as efficient as possible, we have to answer a question how to compute RBFs if a new data (point and value) is to be inserted into the given data set. As we are considering t-varying vector data the inverse matrix has to be computed. It means that we know $\mathbf{A}^{-1}$ matrix for interpolation of n values and we need to determine a matrix $\mathbf{M}^{-1}$ for $n+1$ values, original data plus inserted data.

The main question to be answered is:

Is it possible to use already computed RFB interpolation if a new point is included to the data set?

If the answer is positive it should lead to significant decrease of computational complexity. In the following we will present how a new point can be inserted, a selected point can be removed and also how to select the best candidate for a removal according to an error caused by this point removal.

Let us consider some operations with block matrices (we will assume that all operations are correct and matrices are non-singular in general etc.).

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$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} \\ -(\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}\boldsymbol{A}^{-1} & (\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} \end{bmatrix}$$

Let us consider a matrix $\boldsymbol{M}$ of *(n+1) × (n+1)* and a matrix $\boldsymbol{A}$ of $n \times n$ in the following block form:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^T & c \end{bmatrix}$$

Then the inverse of the matrix $\boldsymbol{M}$ applying the rule above can be written as:

$$\boldsymbol{M}^{-1} = \begin{bmatrix} \left(\boldsymbol{A} - \frac{1}{c}\boldsymbol{b}\boldsymbol{b}^T\right)^{-1} & -\frac{1}{k}\boldsymbol{A}^{-1}\boldsymbol{b} \\ -\frac{1}{k}\boldsymbol{b}^T\boldsymbol{A}^{-1} & \frac{1}{k} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{A}^{-1} + \frac{1}{k}\boldsymbol{A}^{-1}\boldsymbol{b}\boldsymbol{b}^T\boldsymbol{A}^{-1} & -\frac{1}{k}\boldsymbol{A}^{-1}\boldsymbol{b} \\ -\frac{1}{k}\boldsymbol{b}^T\boldsymbol{A}^{-1} & \frac{1}{k} \end{bmatrix}$$

where: $k = c - \boldsymbol{b}^T\boldsymbol{A}^{-1}\boldsymbol{b}$

We can easily simplify this equation if the matrix $\boldsymbol{A}$ is symmetrical as:

$$\boldsymbol{\xi} = \boldsymbol{A}^{-1}\boldsymbol{b} \quad k = c - \boldsymbol{\xi}^T\boldsymbol{b}$$

$$\boldsymbol{M}^{-1} = \frac{1}{k}\begin{bmatrix} k\boldsymbol{A}^{-1} + \boldsymbol{\xi}\otimes\boldsymbol{\xi}^T & -\boldsymbol{\xi} \\ -\boldsymbol{\xi}^T & 1 \end{bmatrix}$$

where: $\boldsymbol{\xi}\otimes\boldsymbol{\xi}^T$ means the tensor multiplication. It can be seen that all computations needed are of $O(N^2)$ computational complexity.

It means that we can compute an inverse matrix incrementally with $O(N^2)$ complexity instead of $O(N^3)$ complexity required originally in this specific case. It can be seen that the structure of the matrix $\boldsymbol{M}$ is "similar to the matrix of the RBF specification.

Now, there is a question how the incremental computation of an inverse matrix can be used for RBF interpolation?

We know that the matrix **A** in the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is symmetrical and non-singular if appropriate rules for RBFs are kept.

A. Point Insertion

Let us consider RBF interpolation for *N+1* points and the following system of equations is obtained:

$$\begin{bmatrix} \phi_{1,1} & \cdot\cdot & \phi_{1,N} & \phi_{1,N+1} & x_1 & y_1 & 1 \\ \vdots & & \cdot\cdot & \vdots & \vdots & \vdots & 1 \\ \phi_{N,1} & \vdots & \phi_{N,N} & \phi_{N,N+1} & x_N & y_N & 1 \\ \phi_{N+1,1} & & \phi_{N+1,N} & \phi_{N+1,N+1} & x_{N+1} & y_{N+1} & 1 \\ x_1 & \cdot\cdot & x_N & x_{N+1} & 0 & 0 & 0 \\ y_1 & \cdot\cdot & y_N & y_N & 0 & 0 & 0 \\ 1 & \cdot\cdot & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \\ \lambda_{N+1} \\ a_x \\ a_y \\ a_0 \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \\ f_{N+1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where: $\phi_{i,j} = \phi_{j,i}$

Let us imagine a simple situation. We have already computed the interpolation for *N* points and we need to include a new point into the given data set. A brute force approach of full RBF computation on the new data set can be used with $O(N^3)$ complexity computation.

Reordering the equations above we get:

$$\begin{bmatrix} 0 & 0 & 0 & x_1 & \cdot\cdot & x_N & x_{N+1} \\ 0 & 0 & 0 & y_1 & \cdot\cdot & y_N & y_{N+1} \\ 0 & 0 & 0 & 1 & \cdot\cdot & 1 & 1 \\ x_1 & y_1 & 1 & \phi_{1,1} & \cdot\cdot & \phi_{1,N} & \phi_{1,N+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_N & y_N & 1 & \phi_{N,1} & \cdot\cdot & \phi_{N,N} & \phi_{N,N+1} \\ x_{N+1} & y_{N+1} & 1 & \phi_{N+1,1} & \cdot\cdot & \phi_{N+1,N} & \phi_{N+1,N+1} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_0 \\ \lambda_1 \\ \vdots \\ \lambda_N \\ \lambda_{N+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_1 \\ \vdots \\ f_N \\ f_{N+1} \end{bmatrix}$$

We can see that last row and last column is "inserted". As RBF functions are symmetrical the recently derived formula for iterative computation of the inverse function can be used. So the RBF interpolation is given by the matrix $\boldsymbol{M}$ as

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^T & c \end{bmatrix}$$

where the matrix $\boldsymbol{A}$ is the RBF matrix *(N+3) × (N+3)* and the vector $\boldsymbol{b}$ *(N+3)* and scalar value *c* are defined as:

$$\boldsymbol{b} = [x_{N+1} \quad y_{N+1} \quad 1 \quad \phi_{1,N+1} \quad \cdot\cdot \quad \phi_{N,N+1}]^T$$

$$c = \phi_{N+1,N+1}$$

It means that we know how to compute the matrix $\boldsymbol{M}^{-1}$ if the matrix $\boldsymbol{A}^{-1}$ is known.

That is exactly what we wanted!

Recently we have proved that iterative computation of inverse function is of $O(N^2)$complexity, that offers a significant performance improvement for points insertion. It should be noted that some operations can be implemented more effectively, especially $\boldsymbol{\xi}\otimes\boldsymbol{\xi}^T = \boldsymbol{A}^{-1}\boldsymbol{b}\boldsymbol{b}^T\boldsymbol{A}^{-1}$ as the matrix $\boldsymbol{A}^{-1}$ is symmetrical etc.

B. Point Removal

In some cases it is necessary to remove a point from the given data set. It is actually an inverse operation to the insertion operation described above. Let us consider a matrix $\boldsymbol{M}$ of the size *(N+1) × (N+1)* as

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^T & c \end{bmatrix}$$

Now, the inverse matrix $\boldsymbol{M}^{-1}$ is known and we want to compute matrix $\boldsymbol{A}^{-1}$, which is of the size $N \times N$.

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Recently we derived opposite rule:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^T & c \end{bmatrix}$$

$$\boldsymbol{\xi} = \boldsymbol{A}^{-1}\boldsymbol{b} \quad k = c - \boldsymbol{\xi}^T\boldsymbol{b}$$

$$\boldsymbol{M}^{-1} = \begin{bmatrix} \boldsymbol{A}^{-1} + \frac{1}{k}\boldsymbol{\xi}\otimes\boldsymbol{\xi}^T & -\frac{1}{k}\boldsymbol{\xi} \\ -\frac{1}{k}\boldsymbol{\xi}^T & \frac{1}{k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{bmatrix}$$

It can be seen that

$$\boldsymbol{Q}_{11} = \boldsymbol{A}^{-1} + \frac{1}{k}\boldsymbol{\xi}\otimes\boldsymbol{\xi}^T$$

and therefore

$$\boldsymbol{A}^{-1} = \boldsymbol{Q}_{11} - \frac{1}{k}\boldsymbol{\xi}\otimes\boldsymbol{\xi}^T$$

Now we have both operations, i.e. insertion and removal, with effective computation of *O*(N^2) computational complexity instead of *O*(N^3). It should be noted that vectors related to the point assigned for a removal must be in the last row and last column of the matrix $\boldsymbol{M}^{-1}$.

C. Point selection

As the number of points within the given data set could be high, the point removal might be driven by a requirement of removing a point which causes a *minimal error of the interpolation*. This is a tricky requirement as there is probably no general answer. The requirement should include additional information which interval of $\boldsymbol{x}$ is to be considered.

Generally we have a function

$$f(\boldsymbol{x}) = \sum_{i=1}^{N} \lambda_i\, \phi_i(\boldsymbol{x}) + P_k(\boldsymbol{x})$$

and we want to remove a point $\boldsymbol{x}_j$ which causes a minimal error ε_j of interpolation, i.e.

$$f_j(\boldsymbol{x}) = \sum_{i=1, i\neq j}^{N} \lambda_i\, \phi_i(\boldsymbol{x}) + P_k(\boldsymbol{x})$$

and we want to minimize

$$\varepsilon_j = \int_{\Omega} \left| f(\boldsymbol{x}) - f_j(\boldsymbol{x}) \right| d\boldsymbol{x}$$

where Ω is the interval on which the interpolation is to be made. It means that if the point $\boldsymbol{x}_j$ is removed the error ε_j is determined as:

$$\varepsilon_j = \lambda_j \int_{\Omega} \phi\left(\left\|\boldsymbol{x} - \boldsymbol{x}_j\right\|\right) d\boldsymbol{x}$$

As we know the interval Ω on which the interpolation is to be used, we can compute or estimate the error ε_j for each point $\boldsymbol{x}_j$ in the given data set and select the best one. For many functions ϕ the error ε_j can be computed or estimated analytically as the evaluation of ε_j is simple for many functions, e.g.

$$\int r^m \ln dr = r^{m+1}\frac{\ln r}{m+1} - \frac{1}{(m+1)^2}$$

It means that for TPS function $r^2 \ln r$ the error ε_k is easy to evaluate. In the case of CSRBF the estimation is even simpler as they have a limited influence, so generally λ_j determines the error ε_j.

It should be noted, that a selection of a point with the lowest influence to the interpolation precision in the given interval Ω is of *O*(*N*) complexity only.

We have shown a novel approach to RBF computation which is convenient for larger data sets. It is especially convenient for t-varying data and for applications, where a "sliding window" is used. Basic operations – point insertion and point removal – have been introduced. These operations have *O*(N^2) computational complexity only, which makes a significant difference from the original approach used for RBFs computation.

IV. Experimental Results

The proposed method for incremental RBF computation was tested especially as far as the speed-up is concerned. The tests carried out proved the theoretical expectations.

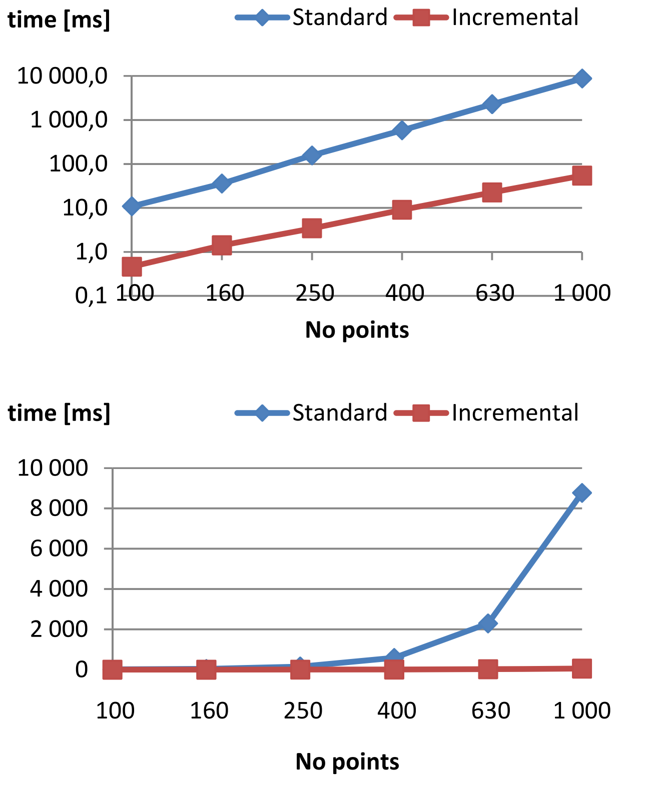

Figure 4: Computational time - comparison

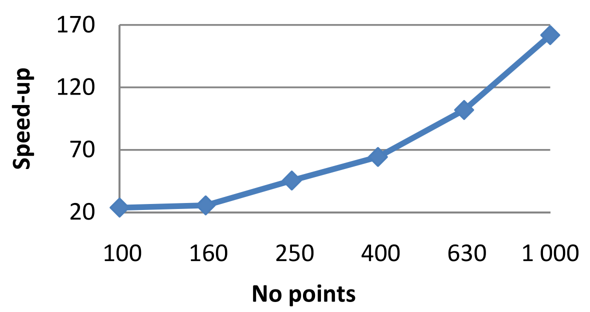

Figure 5: Speed-up

It can be seen that the speed-up grows significantly with the number of interpolated data. This is due to the change of computational complexity from $O(n^3)$ to $O(n^2)$. Figure 4 and Figure 5 presents computational time and speed-up for small number of points (data set size) just to show that even for small data sets the speed-up is significant and growing significantly with number of points interpolated.

V. Conclusion

The proposed *Incremental approach to RBF computation* has advantages over the standard techniques based RBF interpolation due to possibility to insert / remove points with decreased computational complexity from $O(N^3)$ to $O(N^2)$. This enables to apply this approach in applications when interpolation or rendering of data in a "sliding window" and / or t-varying interpolation data are required; in applications when some data are becoming invalid and new data are acquired and need to be included into the interpolated data set. Due to lower computational complexity it is also possible to handle data sets in which scalar values associated with t-varying points, i.e. it is possible to handle non-static data as well.

It is expected that the presented approach can lead to development of new algorithms especially in surface reconstruction of 3D objects. As the proposed *Progressive RBF Interpolation* uses vector / matrix operations exclusively it is suitable for GPU / Larabee architectures as well.

Future work will be devoted to development of methods minimizing the error of interpolation with maximization of number of points removed from the dataset. This should lead to data compression techniques based on RBF representation. Also special data structures should increase additional speed-up due to better memory management.

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