Parameterizing Superquadrics

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Abstract

Superquadrics are well known and often used 3D surface objects in computer graphics. They are used for modelling parts of scenes that are then rendered using photorealistic image synthesis algorithms (e.g., ray tracing). For some techniques, like texturing, which are part of these rendering methods, the type of the parameterization of such a surface has to be chosen carefully and is not intuitively obvious at first sight. There are cases, where the straight forward extension of quadric parameterizations to superquadrics do not produce satisfying results. We therefore investigate a number of different parameterizations in combination with the corresponding formulas, and point out some significant differences between them.

Keywords: Conics, Superconics, Quadrics, Superquadrics, Parameterizations.

1 Introduction

Superquadrics are an extension of the basic quadric surfaces, which were introduced to the computer graphics community by Alan H. Barr in 1981 [Bar81a][Bar81b]. Such a superquadric surface is often defined as a spherical product of two parametric 2D curves, resulting in a parametric shape in 3D space.

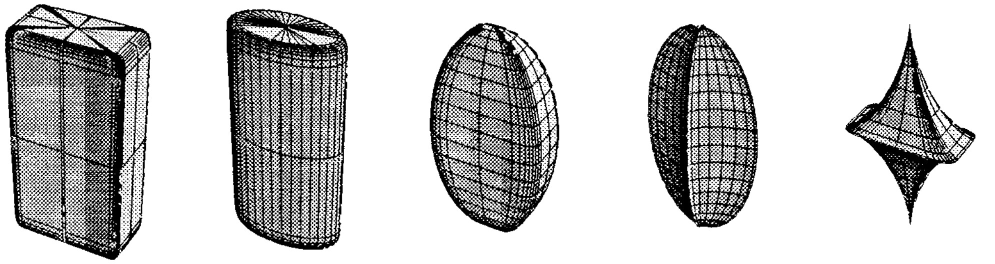

Figure 1-1: Some superellipsoids, which form a special class of superquadrics.

Such spherical products of superconic curves, which are an extension of conics, define well known and often used surfaces like superellipsoids or one or two sheeted superhyperboloids.

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Figure 1-1 illustrates the wide variety of possible shapes showing five superellipsoids with different shape parameters.

The main use of superquadrics is to represent solids for modelling in computer graphics. Algorithms to make these objects available for ray tracing were developed by Barr and Edwards [Barr86][Edwa82]. Intersecting a superquadric surface with a ray does not depend on the parameterization of the shape, but mapping 2D data, e.g. a texture, onto the surface brings up the need of an appropriate parameterization.

Another example of an application, where the choice of the parameterization is very important, is using such an object as projection surface for an extended camera specification [GrLo94].

The choice of an appropriate parameterization depends heavily on the needs of the application. It is therefore impossible to propose one of the huge number of possible parameterizations as the best solution for all the different purposes.

Because of the tight relationship (spherical product) between superquadrics and superconics, the problem of choosing a satisfying parameterization can be discussed for conics and superconics instead. A comparison between some of these parameterizations will be useful to determine, which one is the right choice for a given application. We therefore compiled a number of parameterization formulas and pointed out some significant differences between them.

2 The Spherical Product

Let us have a look at the spherical product first. If we have two parameterized curves $g(u)$ and $h(v)$, the spherical product $s(u,v) = g(u) \otimes h(v)$ defines a surface parameterized by u and v [Hanr89]. It is given by

$$\begin{pmatrix} s_x(u,v) \\ s_y(u,v) \\ s_z(u,v) \end{pmatrix} = \begin{pmatrix} g_x(u) \\ g_y(u) \end{pmatrix} \otimes \begin{pmatrix} h_x(v) \\ h_y(v) \end{pmatrix} = \begin{pmatrix} g_x(u) \cdot h_x(v) \\ g_y(u) \cdot h_x(v) \\ h_y(v) \end{pmatrix}.$$

For example, if $g(u)$ is the unit circle and $h(v)$ the unit semicircle with $h_x(v) \geq 0$, we get the unit sphere as the spherical product $s(u,v) = g(u) \otimes h(v)$. Thus, u is often called the longitude and v the latitude of $s(u,v)$.

For many algorithms a normal vector is required at a surface point. As we are using two 2D curves $g(u)$ and $h(v)$ to form the spherical product, we can take, in case they exist, their tangent vectors $g_t(u) = (g_{tx}(u) \quad g_{ty}(u))^T$ and $h_t(v) = (h_{tx}(v) \quad h_{ty}(v))^T$ to calculate a normal vector at the point $s(u,v)$.

Calculating $\partial s(u,v)/\partial u$ we recognize, that $s_{tg}(u,v) = (g_{tx}(u) \quad g_{ty}(u) \quad 0)^T$ is a vector in the tangent plane of $s(u,v)$. Similarly, the calculation of $\partial s(u,v)/\partial v$ leads us to another vector in the tangent plane , which is given by $s_{th}(u,v) = (g_x(u) \cdot h_{tx}(v) \quad g_y(u) \cdot h_{tx}(v) \quad h_{ty}(v))^T$.

Thus the crossproduct $s_n(u,v) = s_{tg}(u,v) \times s_{th}(u,v)$ is a normal vector of the surface at the point $s(u,v)$ and may be defined in the following way:

$$\begin{pmatrix} s_{nx}(u,v) \\ s_{ny}(u,v) \\ s_{nz}(u,v) \end{pmatrix} = \begin{pmatrix} g_{tx}(u) \\ g_{ty}(u) \\ 0 \end{pmatrix} \times \begin{pmatrix} g_x(u)\cdot h_{tx}(v) \\ g_y(u)\cdot h_{tx}(v) \\ h_{ty}(v) \end{pmatrix} = \begin{pmatrix} g_{ty}(u)\cdot h_{ty}(v) \\ -g_{tx}(u)\cdot h_{ty}(v) \\ (g(u)^T \cdot g_t(u)^\perp)\cdot h_{tx}(v) \end{pmatrix}.$$

$g_t(u)^\perp$ .... $(-g_{ty}(u) \quad g_{tx}(u))^T$. $a^\perp$ is a rotated by $\pi/2$ [Hill94].

With this method we can easily define quadrics and superquadrics as spherical products of conics and superconics. Some examples are listed in table 2-1 [MaTh87]:

$g(u)$	$h(v)$	$s(u,v) = g(u) \otimes h(v)$
circle with radius r	circle with radius r, $h_x(v) \geq 0$	sphere with radius r
circle	line $x = x_{const} > 0$	cylinder
circle	line	cone
circle	ellipse with $h_x(v) \geq 0$	rotational (around z) ellipsoid
ellipse, parabola or hyperbola	line $x = x_{const} > 0$	elliptic, parabolic or hyperbolic cylinder
ellipse	line	elliptic cone
ellipse	ellipse with $h_x(v) \geq 0$	ellipsoid
ellipse	ellipse with $center_x > a_g$	toroid
ellipse	one sheeted hyperbola	one sheeted hyperboloid
hyperbola	one sheeted hyperbola	two sheeted hyperboloid
ellipse or hyperbola	parabola[2] with $h_x(v) \geq 0$	elliptic or hyperbolic paraboloid
superellipse	superellipse with $h_x(v) \geq 0$	superellipsoid
superellipse	superellipse with $center_x > a_g$	supertoroid
superellipse	one sheeted superhyperbola	one sheeted superhyperboloid
superhyperbola	one sheeted superhyperbola	two sheeted superhyperboloid

Table 2-1: Some quadrics and superquadrics defined as spherical product.

3 Conics and Superconics

When parameterizing superquadrics by using their spherical product definition, the usefulness heavily depends on the chosen parameterizations of the underlying curves. It is sufficient to discuss the parameterizations of conic and superconic curves, because they are composed to form superquadric surfaces by the spherical product, and advantages of superconic parameterizations yield similar advantages in the superquadric case.

2 Note, that this parabola is rotated by $\pi/2$.

3.1 Conics, Definition and Parameterizations

Superconics are strongly related to conics and similar formulas are applied. We therefore discuss the conic shapes first. Although there are generalized conics apart from ellipse, parabola and hyperbola, we will concentrate on these best known types. They can be defined by their implicit forms as follows [Netz92]:

	Ellipse	Parabola	Hyperbola
Implicit Form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$2px-y^2=0$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Shape Parameters	a and b	p	a and b
Additional Shape Parameters	$a^2-b^2=e^2$, $\varepsilon=\frac{e}{a}<1$ $p=\frac{b^2}{a}$	$\varepsilon=1$	$a^2+b^2=e^2$, $\varepsilon=\frac{e}{a}>1$ $p=\frac{b^2}{a}$

Table 3-1: Conics, implicit form and shape parameters.

Although the above listed implicit forms together with the shape parameters[3] describe the curves completely, three additional shape parameters are listed in table 3-1, which are often used with conics. These are the linear eccentricity e, the numerical eccentricity ε and the semifocal chord p. For further details see [Netz92].

Now we can choose from a set of different parameterizations. Each one has different properties and none of them can be considered to be best in all circumstances:

"Standard" Parameterization
Hoffmann and Beach present these formulas as the standard parameterization of conics [Hoff89][Beac91]. Since this parameterization can be derived from the general implicit form $\tilde{a}x^2+2\tilde{b}xy+\tilde{c}y^2+2\tilde{d}x+2\tilde{e}y+\tilde{f}=0$ of a conic, it exists for all of them.

"Trigonometric" and "Hyperbolic" Parameterizations
A well known way to parameterize ellipse and hyperbola is using trigonometric or hyperbolic functions [Netz92][Hanr89].

"Angle, Center" and "Angle, Focal Point" Parameterizations
These two parameterizations use the angle α of a ray through the point $Curve(\alpha)$ as its parameter on the curve. They both are of the type $r(\alpha)\cdot(\cos\alpha \quad \sin\alpha)^{\mathrm{T}}$ with appropriate radius $r(\alpha)$. The difference between both parameterizations is the point, where the ray starts. This is the center of the conic for the first and one focal point for the second parameterization [Netz92].

Table 3-2 lists the formulas for these parameterizations. No "Trigonometric" parameterization for the parabola was found in literature, but as it could be useful, it has been derived by the

3 It is pointed out that these shape parameters a, b, e, ε and p should not be mixed up with the parameter t or α of the parameterizations. We therefore use the term shape parameter, whenever we want to depict these special parameters.

authors and listed in table 3-2 as well. Note, that since $\varepsilon = 1$ for the parabola, the "Angle, Focal Point" formulas are equal for all the conics.

	Ellipse	Parabola	Hyperbola
"Standard"	$\begin{pmatrix} a\frac{1-t^2}{1+t^2} \\ b\frac{2t}{1+t^2} \end{pmatrix}$	$\begin{pmatrix} \frac{t^2}{2p} \\ t \end{pmatrix}$	$\begin{pmatrix} a\frac{1+t^2}{1-t^2} \\ b\frac{2t}{1-t^2} \end{pmatrix}$
"Trigonometric"	$\begin{pmatrix} a\cos\alpha \\ b\sin\alpha \end{pmatrix}$	$\begin{pmatrix} \frac{\tan^2\alpha}{2p} \\ \tan\alpha \end{pmatrix}$	$\begin{pmatrix} a\sec\alpha \\ b\tan\alpha \end{pmatrix}$
"Hyperbolic"	—	—	$\begin{pmatrix} a\cosh\alpha \\ b\sinh\alpha \end{pmatrix}$
"Angle, Center"	$r(\alpha) = \frac{b}{\sqrt{1-\varepsilon^2\cos^2\alpha}}$	—	$r(\alpha) = \frac{b}{\sqrt{\varepsilon^2\cos^2\alpha - 1}}$
"Angle, Focal Point"	$r(\alpha) = \frac{p}{1+\varepsilon\cos\alpha}$	$r(\alpha) = \frac{p}{1+\cos\alpha}$	$r(\alpha) = \frac{p}{1+\varepsilon\cos\alpha}$

Table 3-2: Conics, some parameterizations.

Since we need tangent vectors for the calculation of the normal vector at a point of a spherical product surface, the formulas in table 3-3 were derived by differentiating the parameterizations in table 3-2 and simplifying these formulas by factoring out common terms. Note, that the formula for the "Angle, Focal Point" tangent vector of the parabola degenerates to the null vector for any $\alpha = (1+2k)\pi$. Since this parameterization yields parabolas open to the negative x-axis these parameter values make no sense anyhow (for these values no finite points on the parabola are specified).

	Ellipse	Parabola	Hyperbola
"Standard"	$\begin{pmatrix} -2at \\ b(1-t^2) \end{pmatrix}$	$\begin{pmatrix} t \\ p \end{pmatrix}$	$\begin{pmatrix} 2at \\ b(1+t^2) \end{pmatrix}$
"Trigonometric"	$\begin{pmatrix} -a\sin\alpha \\ b\cos\alpha \end{pmatrix}$	$\begin{pmatrix} \tan\alpha \\ p \end{pmatrix}$	$\begin{pmatrix} a\tan\alpha \\ b\sec\alpha \end{pmatrix}$
"Hyperbolic"	—	—	$\begin{pmatrix} a\sinh\alpha \\ b\cosh\alpha \end{pmatrix}$
"Angle, Center"	$\begin{pmatrix} -a^2\sin\alpha \\ b^2\cos\alpha \end{pmatrix}$	—	$\begin{pmatrix} a^2\sin\alpha \\ b^2\cos\alpha \end{pmatrix}$
"Angle, Focal Point"	$\begin{pmatrix} -\sin\alpha \\ \varepsilon+\cos\alpha \end{pmatrix}$	$\begin{pmatrix} -\sin\alpha \\ 1+\cos\alpha \end{pmatrix}$	$\begin{pmatrix} -\sin\alpha \\ \varepsilon+\cos\alpha \end{pmatrix}$

Table 3-3: Parameterized tangent vectors of conics.

3.2 Superconics, Definition and Parameterizations

Again these curves can be defined by their implicit forms, which are extensions of related conic forms. Although the superparabola was not found in literature, it seems to be useful and is therefore defined here. The formula was derived from extensions of the parabola in [Lori02].

	Superellipse	Superparabola	Superhyperbola
Implicit Form	$\left(\frac{x^2}{a^2}\right)^{\frac{1}{\gamma}} + \left(\frac{y^2}{b^2}\right)^{\frac{1}{\gamma}} = 1$	$(2p)^{\frac{2}{\gamma}-1} x - (y^2)^{\frac{1}{\gamma}} = 0$	$\left(\frac{x^2}{a^2}\right)^{\frac{1}{\gamma}} - \left(\frac{y^2}{b^2}\right)^{\frac{1}{\gamma}} = 1$
Shape parameters	a, b and γ	p and γ	a, b and γ

Table 3-4: Superconics, implicit form and shape parameters.

One additional shape parameter γ appears in the formulas listed in table 3-4. If this superconic shape parameter is less than 1, the curve becomes square shaped with rounded corners. With $\gamma \approx 1$ the superconic looks like its related conic. Having $\gamma \approx 2$, the superconic becomes flat with pointed corners. γ values greater than 2 result in pinched superconics with concave surfaces.

We listed up to five parameterizations for each conic. When we want to have these parameterizations for superconics as well, we recognize that they become a little bit more complicated. In fact, the "Standard" and "Angle, Focal Point" parameterizations seem not to be available explicitly for superconics. Table 3-5 lists the formulas for the other parameterizations. Because they were not found in literature, the "Trigonometric" formula for the superparabola and the "Angle, Center" parameterization have been derived by the authors.

	Superellipse	Superparabola	Superhyperbola
"Trigonometric"	$\begin{pmatrix} a\cos^{\gamma}\alpha \\ b\sin^{\gamma}\alpha \end{pmatrix}$	$\begin{pmatrix} (2p)^{1-\frac{2}{\gamma}}\tan^2\alpha \\ \tan^{\gamma}\alpha \end{pmatrix}$	$\begin{pmatrix} a\sec^{\gamma}\alpha \\ b\tan^{\gamma}\alpha \end{pmatrix}$
"Hyperbolic"	—	—	$\begin{pmatrix} a\cosh^{\gamma}\alpha \\ b\sinh^{\gamma}\alpha \end{pmatrix}$
"Angle, Center"	$r(\alpha) = \left(\left(\frac{\cos^2\alpha}{a^2}\right)^{\frac{1}{\gamma}} + \left(\frac{\sin^2\alpha}{b^2}\right)^{\frac{1}{\gamma}}\right)^{-\frac{\gamma}{2}}$	—	$r(\alpha) = \left(\left(\frac{\cos^2\alpha}{a^2}\right)^{\frac{1}{\gamma}} - \left(\frac{\sin^2\alpha}{b^2}\right)^{\frac{1}{\gamma}}\right)^{-\frac{\gamma}{2}}$

Table 3-5: Superconics, parameterizations.

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Again we are interested in the corresponding tangent vectors. Table 3-6 lists these formulas:

	Superellipse	Superparabola	Superhyperbola
"Trigonometric"	$\begin{pmatrix} -a\sin^{2-\gamma}\alpha \\ b\cos^{2-\gamma}\alpha \end{pmatrix}$	$\begin{pmatrix} (2p)^{2-\frac{2}{\gamma}}\tan^{2-\gamma}\alpha \\ p\gamma \end{pmatrix}$	$\begin{pmatrix} a\tan^{2-\gamma}\alpha \\ b\sec^{2-\gamma}\alpha \end{pmatrix}$
"Hyperbolic"	—	—	$\begin{pmatrix} a\sinh^{2-\gamma}\alpha \\ b\cosh^{2-\gamma}\alpha \end{pmatrix}$
"Angle, Center"	$\begin{pmatrix} -\left(\frac{\sin^{2-\gamma}\alpha}{b^2}\right)^{\frac{1}{\gamma}} \\ \left(\frac{\cos^{2-\gamma}\alpha}{a^2}\right)^{\frac{1}{\gamma}} \end{pmatrix}$	—	$\begin{pmatrix} \left(\frac{\sin^{2-\gamma}\alpha}{b^2}\right)^{\frac{1}{\gamma}} \\ \left(\frac{\cos^{2-\gamma}\alpha}{a^2}\right)^{\frac{1}{\gamma}} \end{pmatrix}$

Table 3-6: Parameterized tangent vectors of superconics.

4 A Comparison of the Parameterizations

We have presented a set of parameterizations of conics and superconics. We now want to compare these formulas and discuss some of their properties:

4.1 Symmetry

The "Trigonometric", "Hyperbolic" and "Angle, Center" parameterizations for ellipse, hyperbola and their related superconics are symmetric to both the x- and y-axis, and therefore also to the origin. For each combination the following holds:

$$\begin{pmatrix} Curve_x(\alpha) \\ Curve_y(\alpha) \end{pmatrix} = \begin{pmatrix} Curve_x(-\alpha) \\ -Curve_y(-\alpha) \end{pmatrix} = \begin{pmatrix} -Curve_x(\pi-\alpha) \\ Curve_y(\pi-\alpha) \end{pmatrix} = \begin{pmatrix} -Curve_x(-\pi+\alpha) \\ -Curve_y(-\pi+\alpha) \end{pmatrix}$$

The "Standard" and "Angle, Focal Point" parameterizations for conics and the "Trigonometric" formulas for parabola and superparabola are symmetric only to the x-axis.

4.2 Distortions

In many cases a regular distribution of the parameter lines over the surface is required. Unfortunately the density of these parameter lines extremely varies with some of the parameterizations. In general, distortions of this kind are less a problem with conics than with superconic formulas.

For instance, the "Angle, Center" parameterization is quite similar to the "Trigonometric" one for ellipse and hyperbola, but comparing these two for superellipse and superhyperbola shows significant differences, that have to be taken into account. Unfortunately both parameterizations show bad behavior near the asymptotes of the superhyperbola. If these ranges of the curve are very important, the "Hyperbolic" formulas yield better results.

Figures 4-1 and 4-2 show that using the "Angle, Center" parameterization instead of the "Trigonometric" version yields better results for the superellipse with respect to the density of the parameter lines.

Figure 4-1: "Trigonometric" and "Angle, Center" parameterization of the ellipse.

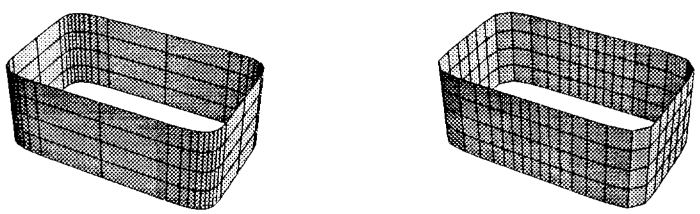

Figure 4-2: "Trigonometric" and "Angle, Center" parameterization of the superellipse.

Figures 4-3 through 4-5 show the same fact for the superhyperbola. By having only little differences with the hyperbola, severe distortions occur with the superhyperbola. Depending on which part of the superhyperbola is most important for the application, the "Angle, Center" or "Hyperbolic" parameterization can be better.

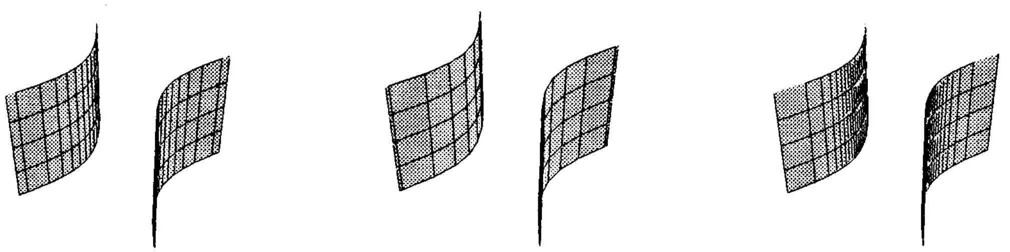

Figure 4-3: "Trigonometric", "Hyperbolic" and "Angle, Center" parameterization of the hyperbola.

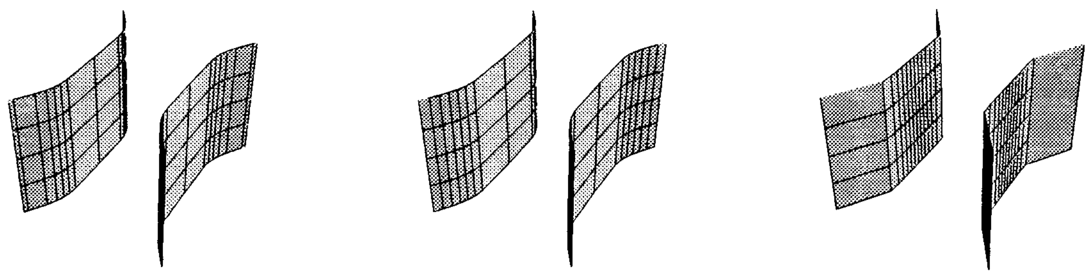

Figure 4-4: "Trigonometric", "Hyperbolic" and "Angle, Center" parameterization of the superhyperbola.

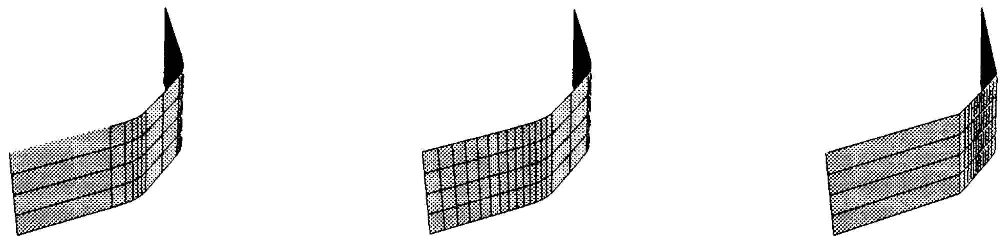

Figure 4-5: The parameterizations of figure 4-4 at the asymptotes of the superhyperbola.

4.3 Parameter Range

The "Standard" parameterization of conics and the "Hyperbolic" parameterization of hyperbola and superhyperbola map $\Re$ once onto the curve. This might be a useful property. But on the other hand, there will always be a range of the curve, which is not addressable in practice, since only a small range out of $\Re$ can be represented with floating point numbers on a computer. See figure 4-6 for an illustration.

All the other parameterizations map a finite interval of $\Re$ onto the whole curve, which is periodically replicated on the rest of $\Re$. Most of these periods are of the length 2π, only the "Trigonometric" formula for the parabola and superparabola has a period of length π.

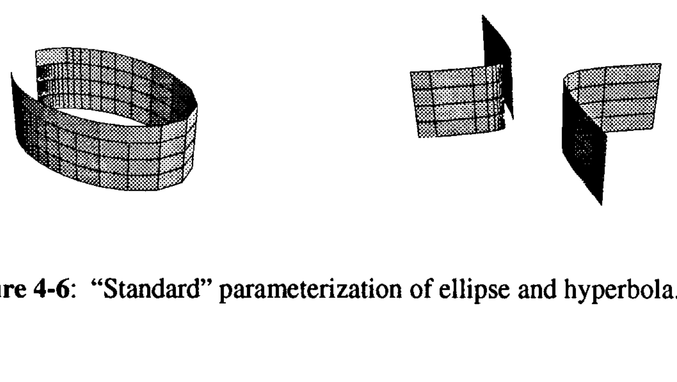

Figure 4-6: "Standard" parameterization of ellipse and hyperbola.

Another problem with the "Angle, Center" parameterization is, that there are parameter ranges, where no point and tangent vector of the curve is specified. These parameter ranges are $[(2k+1)\cdot\pi/2-\varphi,(2k+1)\cdot\pi/2+\varphi]$ with $\sin\varphi=1/\varepsilon$.

5 Conclusion

The impressing variety of possible shapes and the simplicity of the formulas cause superquadrics to be well known and often used objects in computer graphics. For several rendering purposes, e.g., texturing, the implicit form of such a surface is not sufficient and a parameterization must be used. We demonstrated that this choice has to be taken carefully.

We addressed the problem of choosing an appropriate parameterization and discussed it for conics and superconics. The results of the comparison can be directly used with a superquadric surface, if its definition as a spherical product is used.

We listed a number of parameterization formulas for conic and superconic curves and the corresponding tangent vectors. Most of them were taken from literature and compiled to a brief summary of these methods, but some of them had to be derived by the authors themselves, since they could not be found elsewhere. The parameterizations were compared due to their advantages and disadvantages with respect to given needs.

We kept to the simple and explicit formulas for parameterizing conics and superconics and therefore did not discuss more complex methods. Actually this does not mean, that these parameterizations, which were left out of this discussion, are not realizable or do not yield satisfying results. Thus further work has to be done to find solutions in these cases.

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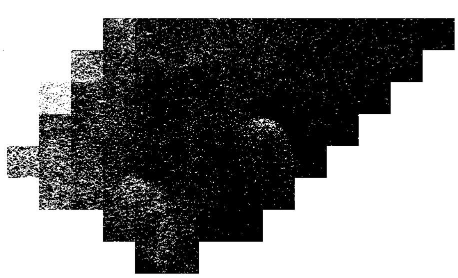

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