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Faculty of Applied Sciences
Department of Mathematics

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Basic properties of the p-trigonometric functions

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Lukáš Kotrla

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Fakulta aplikovaných věd

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Lukáš Kotrla

Declaration

I hereby declare that the entire bachelor thesis is my original work and that I have used only the cited sources.

Pilsen . . . .

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Abstract

This Bachelor thesis is devoted to study of properties of the function $\sin_p(x)$. It can be divided into two original research parts. The first part is devoted to study of the continuity of the n-th derivative of $\sin_p(x)$. We discuss several cases depending on the value of the parameter p and domains of interest in variable x. The second part focuses on a two ways we can express $\sin_p(x)$ in terms of power series. One way is to use Bell polynomials and the other is to use the general figure for the inverse series based on the Cauchy integral formula. Finally, we present a conjecture concerning the convergence of Taylor series representing $\sin_p(x)$. Solving this conjecture will significantly speed up numerical computations concerning $\sin_p(x)$ function.

Keywords

p-trigonometric functions, differentiability, continuity, the inversion of power series, Bell polynomials

Contents

1 Introduction

Let $p > 1, \Omega \subset \mathbb{R}$ is a bounded domain and $\lambda \in \mathbb{R}$ is a parameter. The Dirichlet problem for p-Laplacian operator (or generalized Laplace operator)

$$\begin{cases} \Delta_p u \overset{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u) & = & g(x,u;\lambda) & \text{for } x \in \Omega\,, \\ u & = & 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

arises from various applications in physics and engeneering. To name some of the most interesting applications (in our opinion), let us mention growing of sandpiles [2],[13] (for large values of $p \to +\infty$) or image processing [14] (for $p \to 1_+$). The p-Laplacian with other boundary conditions (e.g. Neumann, Robin etc.) appears for example in the context of mathematical models of climate (p=3), [4].

As an extensive bibliography (see e.g. [8] for further reference) shows, the existence results for (1) are closely related to the asymptotic properties of $g(x,u;\lambda)$ for $n \to +\infty$ and to the properties of (in general) nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u & = & \lambda |u|^{p-2}u & \text{in } \Omega\,, \\ u & = & 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2}$$

We say, that $\lambda \in \mathbb{R}$ is an eigenvalue of (2) if there is a nonzero function u that satisfy (2) (possibly in a weak sense or some generalised sense).

In any dimension $n \geq 2$, there are only few results concerning higher eigenvalues for (2), see e.g. Anane [1]. In dimension $n = 1$ the eigenvalue problem (2) is reduced to an ODE problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' - \lambda |u|^{p-2}u & = & 0 \quad \text{in } (0,\pi_p)\,, \\ u(0) & = & u(\pi_p) = 0\,, \end{cases} \tag{3}$$

where

$$\pi_p = 2\int_0^1 \frac{1}{(1-s^p)^{1/p}} ds = \frac{2\pi}{p\sin(\pi/p)}$$

Let us note, that the problem can be considered on any bounded open interval, but the choice $(0,\pi_p)$ significantly simplifies the calculations.

The eigenvalue problem has been studied in many papers, see e.g. Elbert[12], Lindqvist[16] and references therein. It follows from these works that the eigenvalues of (2) form a sequence

$$\lambda_k = k^p(p-1), k \in \mathbb{N}$$

and corresponding eigenfunctions are functions $\sin_p(kx)$.

Here the function $\sin_p x$ is defined as the solution [1] to

$$\left.\begin{aligned} -\left(|u'|^{p-2}u'\right)' - (p-1)|u|^{p-2}u &= 0\,, \\ u(0) &= 0\,, \\ u'(0) &= 1\,, \end{aligned}\right\} \tag{4}$$

[1] Note that (4) has a unique solution by [10]

which is equivalent to

$$\left.\begin{array}{rcl} u' & = & \varphi_{p'}(v)\,, \\ v' & = & -(p-1)|u|^{p-2}u\,, \end{array}\right\} \tag{5}$$

where

$$\varphi_p(z) = \begin{cases} |z|^{p-2}z & \text{for } z \neq 0\,, \\ 0 & \text{for } z = 0\,. \end{cases}$$

It means that u must be absolutly continuous by the Carathéodory definition of solution of (5). It was shown in [11] that $\sin_p(x)$ can be expressed on $[0, \frac{\pi_p}{2}]$ (using the first integral of (4)) as the inverse of

$$\operatorname{arcsin}_p(x) = \int_0^x (1-s^p)^{-\frac{1}{p}}\,ds\,, \tag{6}$$

which is extended to $[0, \pi_p]$ by reflection $\sin_p(x) = \sin_p(\pi_p - x)$ and to $[-\pi_p, \pi_p]$ as the odd function. Finally, it is extended to $\mathbb{R}$ as the $2\pi_p$-periodic function. For $p = 2$,

$$\operatorname{arcsin}_2(x) = \int_0^x \frac{1}{\sqrt{1-s^2}}dx = \arcsin(x)\,, \tag{7}$$

thus $\forall x \in \mathbb{R} : \sin_2(x) = \sin(x)$. The properties of functions $\sin_p(x)$ were studied extensively in the last 30 years. Indenpendenty, Elbert [12] and Lindqvist [16] discovered remarkable identity

$$\forall x \in \mathbb{R} : |\sin_p(x)|^p + |\sin_p'(x)|^p = 1 \tag{8}$$

valid for any $p > 1$ (for $p = 2$, it reduces to the famous trigonometric identity $\sin^2(x) + \cos^2(x) = 1$). For this reason, the following definition

$$\cos_p(x) \overset{\text{def}}{=} \sin_p'(x)$$

makes sence. It is interesting fact, that similar functions to $\sin_p$, $\cos_p$ were studied in a very different context in a work of Swedish mathematician Erik Lundberg in 1879. In [17] Lundberg defined a family of functions $y = S_{\frac{m}{n}}(x)$ that satisfy formula

$$x = \int_0^y \frac{1}{(1-y^n)^{\frac{m}{n}}}dy\,,$$

where $m, n \in \mathbb{N}$,$m < n$. He called these functions *sinualis* and studied series expansions.

2 Some facts of used theory

2.1 Power series

The power series are the first tool, which we will use in this work. They are given by following definition.

Definition 2.1 *A power series is a function series of the form*

$$\sum_{n=0}^{+\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots\,,$$

where $x \in \mathbb{R}$ is the variable. $x_0\,,\ a_1\,,\ \ldots\,,\ a_n\,,\ \ldots$ are real numbers. Number x_0 is called the centre of the power series. Numbers a_n are called the coefficients of the power series.

In [19] Morse and Feshbach deal with the problem of inverting the power series. If $a_1 \neq 0$, then for the function

$$f(y) = f(0) + \sum_{n=1}^{+\infty} a_n(y-y_0)^n$$

the inverse function exists and it takes form

$$y(x) = y(0) + \sum_{n=1}^{+\infty} b_n(x-x_0)^n\,.$$

Coefficients

$$b_n = \frac{1}{n\cdot a_1^n}\sum_{s,t,u,\ldots}(-1)^{s+t+u+\ldots}\cdot\frac{n(n+1)\cdot\ldots\cdot(n-1+s+t+u+\ldots)}{s!t!u!\ldots}\left(\frac{a_2}{a_1}\right)^s\left(\frac{a_3}{a_1}\right)^t\ldots\,, \qquad (9)$$

where the sumation is over all $s\,,\ t\,,\ u\,,\ldots \in \mathbb{N}$ such that $s+2t+3u+\ldots = n-1$. This relationship is derived in [19] by using Cauchy's integral formula.

2.2 Bell polynomials

Here we define the Bell polynomials, which Bell introduce in his work [3]. They can be useful to inverting of power series and to finding domain of convergence of this inversion, but it is not topic of this work. The idea of using the Bell polynomials was suggested by Dr. Oleg Marichev [18] to Petr Girg during his visit at Wolfram research Inc. Dominici [9] deals with asymptotic behavior of Bell polynomials.

Definition 2.2 *The polynomials defined for $c_1, c_2, c_3, \ldots, c_{n-k+1} \in \mathbb{N}_0$ such that $c_1 + 2c_2 + 3c_3 + \ldots + (n-k+1)c_{n-k+1} = n$ and $c_1 + c_2 + c_3 + \ldots + c_{n-k+1} = k$ as the sum*

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \stackrel{\text{def}}{=} \sum \frac{n!}{c_1!c_2!\ldots c_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{c_{n-k+1}}, \tag{10}$$

are called Bell polynomials.

In Wheeler [24] there is given the theorem (p. 50, Theorem 6.), which shows how we can invert a formal Taylor series, i.e. Taylor series for which the problem of convergence is not considered and it has form

$$g(t) = \sum_{k=0}^{+\infty} g_k \frac{t^k}{k!},$$

where $g_k = g^{(k)}(t)$ at $t = 0$. Other information about formal power series can be found in [20]. Let us denote $[\cdot]_k$ a falling factorial function, i.e.

$$[x]_k = x \cdot (x-1) \ldots (x-k+1)$$

Theorem 2.1 *Let $g(t)$ be a formal Taylor series, with Taylor coefficients $\{g_n\}_{n=0}^{+\infty}$ such that $g_0 = 0$ and $g_1 \neq 0$, and let $g^{(-1)}(t)$ be its inverse series, then*

(i)

$$B_{n,k}\left(g^{(-1)}(t)\right) = \binom{n-1}{k-1} \sum_{i=0}^{n-k} [-n]_i \cdot g_1^{-n-i} B_{n-k,i}\left(\frac{g_2}{2}, \frac{g_3}{3}, \frac{g_4}{4}, \ldots\right)$$

(ii)

$$B_{n,k}\left(g^{(-1)}(t)\right) = \binom{n-1}{k-1} \sum_{i=0}^{n-k} (-1)^i \cdot g_1^{-n-i} B_{n-k+i,i}(0, g_2, g_3, g_4, \ldots)$$

Of course, the Taylor coefficients $g_n^{(-1)}$ of $g^{(-1)}(t)$ can be obtain from these formulas by setting $k = 1$.
Further information about Bell polynomials, especially many identities, can be found in[6], [7], [21], [22] or [23].

2.3 *p*-trigonometric functions

Many definitions and basic properties of p-trigonometric functions are given in introduction of this work. Here we bring out a few other properties, which are important in following sections.

Theorem 2.2 *The power series expansion of* $\operatorname{arcsin}_p(x)$ *have form*

$$\operatorname{arcsin}_p(x) = \sum_{n=0}^{+\infty} \frac{\Gamma(n+\frac{1}{p})}{\Gamma(\frac{1}{p})(n\cdot p+1)} \frac{x^{np+1}}{n!} \qquad \textit{for } x \in (0,1)\,. \tag{11}$$

Here Γ denotes the Euler Gamma function (see e.g. [19], p. 394). This formula is derived in [5] or [15].

Lemma 2.1 *Let* $p \in \mathbb{R}$, $p > 1$. *Functions* $\sin_p(x)$ *and* $\cos_p(x)$ *have following basic properties.*

1. $\sin_p(x) > 0$ *on* $(0, \pi_p)$, $\sin_p(0) = 0$ *and* $\sin_p(x) < 0$ *on* $(-\pi_p, 0)$.
2. $\sin_p(x)$ *is strictly increasing on* $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$.
3. $\cos_p(x) > 0$ *on* $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$, $\cos(-\frac{\pi_p}{2}) = \cos_p(\frac{\pi_p}{2}) = 0$ *and* $\cos_p(x) < 0$ *on* $(-\pi_p, -\frac{\pi_p}{2}) \cup (\frac{\pi_p}{2}, \pi_p)$.
4. *The* $(2n-1)$*-th derivative of* $\sin_p(x)$ *is even on* $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ *wherever it exists for* $n \in \mathbb{N}$ *given.*
5. *The* $2n$*-th derivative of* $\sin_p(x)$ *is odd on* $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ *wherever it exists for* $n \in \mathbb{N}$ *given.*

Proof: Lemma 2.1 contains basic facts that can be easily obtained and/or found in the literature. Therefore, the proof is ommited.

Lemma 2.2 *For all* $p \in \mathbb{R}$, $p > 1$

$$\sin_p''(x) = -\sin_p^{p-1}(x) {\sin_p'}^{2-p}(x) \qquad \textit{for } x \in (0, \frac{\pi_p}{2}) \tag{12}$$

$$\sin_p''(x) = \sin_p^{p-1}(x)\left(-\sin_p'(x)\right)^{2-p} \qquad \textit{for } x \in (\frac{\pi_p}{2}, \pi_p) \tag{13}$$

and

$$\sin_p''(x) = \sin_p^{p-1}(-x) {\sin_p'}^{2-p}(x) \qquad \textit{for } x \in (-\frac{\pi_p}{2}, 0) \tag{14}$$

Proof: For $x \in [0, \frac{\pi_p}{2}]$ the identity (8) has form

$$\sin_p(x)^p + \sin_p'(x)^p = 1. \tag{15}$$

Taking (15) into derivative we get

$$\sin_p^{p-1}(x)\sin_p'(x) + {\sin_p'}^{p-1}(x)\sin_p''(x) = 0\,.$$

Since $\sin_p'(x)$ is nonzero on $(0, \frac{\pi_p}{2})$

$$\sin_p''(x) = -\sin_p^{p-1}(x){\sin_p'}^{2-p}(x) \qquad \text{for } x \in (0, \frac{\pi_p}{2})\,.$$

For $x \in [\frac{\pi_p}{2}, \pi_p]$ we get from (8) following identity

$$\sin_p(x)^p + (-\sin_p'(x))^p = 1\,,$$

which gives by analogy as in the previous case ($\sin_p'(x)$ is nonzero on $(\frac{\pi_p}{2}, \pi_p)$ because of evenness)

$$\sin_p''(x) = \sin_p^{p-1}(x)(-\sin_p'(x))^{2-p} \qquad \text{for } x \in (\frac{\pi_p}{2}, \pi_p)\,.$$

At least for $x \in (-\frac{\pi_p}{2}, 0)$ we get from odness and (8)

$$|-1|^p|\sin_p(-x)|^p + |\sin_p'(x)|^p = \sin_p(-x) + \sin_p'^{\,p}(x) = 1\,,$$

which gives by similar arguments as above

$$\sin_p''(x) = \sin_p^{p-1}(-x)(\sin_p'(x))^{2-p} \qquad \text{for } x \in (-\frac{\pi_p}{2}, 0)\,,$$

and Lemma 2.2 is proved.

□

The statement of Lemma 2.2 for $x \in (0, \frac{\pi_p}{2})$ is also adduced in [5], [11] or [15], where many other properties of p-trigonometric functions can be found.

3 The existence and the continuity of the n-th derivative of $\sin_p(x)$

Theorem 3.1 *For all $p \in \mathbb{R}$, $p > 1$ the first derivation of function $\sin_p(x)$ is continuous on $\mathbb{R}$.*

Proof: The function $\sin_p(x)$ satisfies the identity

$$|\sin_p(x)|^p + |\sin_p'(x)|^p = 1 \qquad \text{on } \mathbb{R}\,. \tag{16}$$

Considering Lemma 2.1

$$\sin_p'(x) = \sqrt[p]{1 - |\sin_p(x)|^p} \qquad \text{on } \left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right)$$

and

$$\sin_p'(x) = -\sqrt[p]{1 - |\sin_p(x)|^p} \qquad \text{on } \left(-\pi_p, -\frac{\pi_p}{2}\right) \cup \left(\frac{\pi_p}{2}, \pi_p\right)\,.$$

By the continuity of $\sin_p(x)$ on $\mathbb{R}$, the fact that range of the $|\sin_p(x)|$ is $[0,1]$ and the continuity of $z \mapsto \sqrt[p]{1 - z^p}$ on $[0,1]$ for $p > 1$ we find that

$$\sin_p'(\cdot) \in C\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right)\,,$$

$$\sin_p'(\cdot) \in C\left(-\pi_p, -\frac{\pi_p}{2}\right)\,,$$

and

$$\sin_p'(\cdot) \in C\left(\frac{\pi_p}{2}, \pi_p\right)\,.$$

Since $\sin_p(x)$ is continuous on $\mathbb{R}$ and $|\sin_p(-\frac{\pi_p}{2})| = |\sin_p(\frac{\pi_p}{2})| = 1$ we get from (16)

$$\lim_{x \to -\frac{\pi_p}{2}^-} \sin_p'(x) = \lim_{x \to \frac{\pi_p}{2}^-} \sin_p'(x) = \lim_{x \to \frac{\pi_p}{2}^+} \sin_p'(x) = \lim_{x \to -\frac{\pi_p}{2}^+} \sin_p'(x) = 0\,.$$

Hence $\sin_p'(x)$ is continuous on $(-\pi_p, \pi_p)$. From (16), the continuity of $\sin_p(x)$ and

$$\sin_p(-\pi_p) = \sin_p(\pi_p) = 0$$

we obtain that

$$\lim_{x \to -\pi_p^-} \sin_p'(x) = \lim_{x \to \pi_p^+} \sin_p'(x) = -1\,,$$

which implies (considering $2\pi_p$-periodicity) the continuity of $\sin_p'(x)$ on $\mathbb{R}$.

$\square$

Theorem 3.2 *Let $p \in \mathbb{R} \setminus \{2\}$ such that $p > 1$.*

1. *If $p > 2$, then the function $\sin_p(\cdot) \in C^1(\mathbb{R})$ and $\sin_p(\cdot) \notin C^2(\mathbb{R})$.*

2. *If $p \in (1,2)$, then the function $\sin_p(\cdot) \in C^2(\mathbb{R})$ and $\sin_p(\cdot) \notin C^3(\mathbb{R})$.*

Proof: Since by definition $\sin_p(\cdot) \in C(\mathbb{R})$ and by Theorem 3.1 $\sin_p'(x)$ is continuous on $\mathbb{R}$, $\sin_p(\cdot) \in C^1(\mathbb{R})$ for all given p. For $x \in [0, \frac{\pi_p}{2}]$ the identity (8) has form

$$\sin_p(x)^p + \sin_p'(x)^p = 1\,. \tag{17}$$

Taking into derivative equation (17) we get

$$\sin_p^{p-1}(x)\sin_p'(x) + \sin_p'^{\,p-1}(x)\sin_p''(x) = 0\,.$$

It is evident that if $\sin_p'^{\,p-1}(x) \neq 0$

$$\sin_p''(x) = -\sin_p^{p-1}(x)\sin_p'^{\,2-p}(x)\,. \tag{18}$$

From Lemma 2.1 $\cos_p(x) > 0$ on $(0, \frac{\pi_p}{2})$ and there is a difficulty at $x = \frac{\pi_p}{2}$. Due to

$$\lim_{x \to \frac{\pi_p}{2}^-} \sin_p''(x) = -\infty \qquad \text{for } p > 2\,,$$

the continuity of $\sin_p''(x)$ falls at $x = \frac{\pi_p}{2}$ for $p > 2$.
Vice versa for $p \in (1,2)$

$$\lim_{x \to \frac{\pi_p}{2}^-} \sin_p''(x) = 0 \qquad \text{for } p \in (1,2)\,. \tag{19}$$

For $x \in [\frac{\pi_p}{2}, \pi_p]$ we get from (8) following identity

$$\sin_p(x)^p + (-\sin_p'(x))^p = 1\,,$$

which by analogy to the case $p > 2$ gives

$$\sin_p''(x) = \sin_p^{p-1}(x)(-\sin_p'(x))^{2-p}$$

and

$$\lim_{x \to \frac{\pi_p}{2}^+} \sin_p''(x) = 0 \qquad \text{for } p \in (1,2)\,. \tag{20}$$

From (19) and (20) we can define $\sin_p''(\frac{\pi_p}{2}) = 0$ and for $p \in (1,2)$ the function

$$\sin_p''(x) = \begin{cases} -\sin_p^{p-1}(x)\sin_p'^{\,2-p}(x) & x \in [0, \frac{\pi_p}{2})\,, \\ 0 & x = \frac{\pi_p}{2}\,, \\ \sin_p^{p-1}(x) - \sin_p'^{\,2-p}(x) & x \in (\frac{\pi_p}{2}, \pi_p]\,, \end{cases}$$

is continuous on $[0, \pi_p]$. Due to the 2π-periodicity and the odness of $\sin_p''(x)$ (by Lemma 2.1), which also implies the fact that

$$-\sin_p''(-\pi) = 0 = \sin_p''(\pi)\,,$$

we get continuity on $\mathbb{R}$.
Taking into derivative (18) we obtain for $x \in [0, \frac{\pi_p}{2})$

$$\sin_p'''(x) = -(p-1)\sin_p^{p-2}(x){\sin_p'}^{3-p}(x) + (2-p)\sin_p^{p-1}(x){\sin_p'}^{1-p}(x)\sin_p''(x)\,.$$

By substitution (18) for $\sin_p''(x)$ we get

$$\sin_p'''(x) = -(p-1)\sin_p^{p-2}(x){\sin_p'}^{3-p}(x) + (2-p)\sin_p^{2p-2}(x){\sin_p'}^{3-2p}(x)\,.$$

Due to $\sin_p(0) = 0$ which implies $\cos_p(0) = 1$

$$\lim_{x\to 0^+}\sin_p'''(x) = -\infty + 0 = -\infty \qquad \text{for } p \in (1,2)\,.$$

It follows $\sin_p(\cdot) \notin C^3(\mathbb{R})$ for $p \in (1,2)$ and the proof is complete.

□

Lemma 3.1 *Let $p \in \mathbb{R}$ such that $p > 1$ and let $k \in \mathbb{N}$. For all $k = 1\,,\,2\,,\,3\,,\,4\,,\ldots,\,k_0$, $r_k\,,\,q_k \in \mathbb{R}$*

$$\sum_{k=1}^{k_0} a_k \sin_p^{q_k}(x)\cos_p^{r_k}(x)$$

is continuous on $(0, \frac{\pi_p}{2})$.

Proof: If any function $f(x)$ is continuous, then for all real constant α function $\alpha \cdot f(x)$ is also continuous. Thus we can consider simplifying assumption $a_k = 1$. We choose any $q\,,\,r \in \mathbb{R}$. By the continuity of $\sin_p(x)$ on $(0, \frac{\pi_p}{2})$, the fact that $\sin_p(x) \in (0,1)$ for $x \in (0, \frac{\pi_p}{2})$ and the continuity of $z \to z^q$ on $(0,1)$ we have the continuity of $\sin_p^q(x)$ on $(0, \frac{\pi_p}{2})$.
Similarly by the continuity of $\cos_p(x)$ on $(0, \frac{\pi_p}{2})$, the fact that $\cos_p(x) \in (0,1)$ on $(0, \frac{\pi_p}{2})$ and the continuity of $z \to z^r$ on $(0,1)$ we find that $\cos_p^r(x)$ is continuous on $(0, \frac{\pi_p}{2})$. The proof is completed by notation that sum or product of finite number of continuous functions is continuous function, too.

□

Lemma 3.2 *Let $p \in \mathbb{R}$ such that $p > 1$ and let $k_0 \in \mathbb{N}$, $k = 1\,,\,2\,,\,3\,,\,4\,,\,5\,,\ldots,k_0$, $r_k \in \mathbb{R}$, $q_k > 0$, $a_k\,,\,b_k \in \mathbb{R}$.*
For

$$f_1(x) = \sum_{k=1}^{k_0} a_k \sin_p^{q_k}(x)\cos_p^{r_k}(x)$$

defined on $(0, \frac{\pi_p}{2})$ *and*

$$f_2(x) = \sum_{k=1}^{k_0} b_k \sin_p^{q_k}(-x) \cos_p^{r_k}(x)$$

defined on $(-\frac{\pi_p}{2}, 0)$ *the function*

$$f(x) = \begin{cases} f_1(x) & x \in (0, \frac{\pi_p}{2}), \\ f_2(x) & x \in (-\frac{\pi_p}{2}, 0) \end{cases}$$

is continuous on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$.

Proof: By Lemma 3.1 the function $f_1(x)$ is continuous on $(0, \frac{\pi_p}{2})$. Since $(-1) \cdot \sin_p(-x)$ for $x \in (-\frac{\pi_p}{2}, 0)$ is equal to $\sin_p(x)$ for $x \in (0, \frac{\pi_p}{2})$, the function $f_2(x)$ is also continuous by Lemma 3.2 and since $\sin_p^q(0) = 0$ for any $q > 0$ and $\cos_p^r(0) = 1$ for all $r \in \mathbb{R}$

$$0 = \lim_{x \to 0^-} f_2(x) = \lim_{x \to 0^+} f_1(x) = 0,$$

we get desired continuity on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$, which conclude the proof.

□

Theorem 3.3 *Let* $p \in \mathbb{R}$, $p > 1$, $n = 2, 3, 4, \ldots$ *then* $\sin_p^{(n)}(x)$ *exists on* $(0, \frac{\pi_p}{2})$, *it is continuous and there exists* $k = 1, 2, \ldots, 2^{n-2}$, $a_k \in \mathbb{R}$, l_k, $m_k \in \mathbb{Z}$ *such that*

$$\sin_p^{(n)}(x) = \sum_{k=1}^{2^{n-2}} a_k \sin_p^{l_k p + m_k}(x) \cos_p^{1 - l_k p - m_k}(x). \tag{21}$$

Proof: We proceed by induction.

Step 1: By Theorem 3.2 $\sin_p(\cdot) \in C^1(\mathbb{R})$ and thus $\sin_p(\cdot) \in C^1(0, \frac{\pi_p}{2})$. By Lemma 2.2

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cos_p^{2-p}(x) \qquad \text{for } x \in (0, \frac{\pi_p}{2}) \tag{22}$$

It is clear that $k = 1$, $a_1 = -1$, $l_1 = 1$, $m_1 = 1$. By Lemma 3.1 $\sin_p^{p-1}(x) \cos_p^{2-p}(x)$ is continuous on $(0, \frac{\pi_p}{2})$ for given all p and the statement of Theorem 3.3 is true for $n = 2$.

Step 2: Let us assume that statement of Theorem 3.3 is true for n, i.e.

$$\sin_p^{(n)}(x) = \sum_{k=1}^{2^{n-2}} a_k \sin_p^{l_k p + m_k}(x) \cos_p^{1 - l_k p - m_k}(x)$$

Due to the fact that the function $z \mapsto z^q$, $z > 0$, $q \in \mathbb{R}$ belongs to $C^\infty(0, +\infty)$ considering Lemma 2.1 and Theorem 3.2 is $\sin_p^{(n)}(x)$ differentiable function. The chain rule and the product rule we apply and we get from (21)

$$\begin{aligned} \sin_p^{(n+1)}(x) &= \sum_{k=1}^{2^{n-2}} a_k (l_k p + m_k) \sin_p^{l_k p + m_k - 1}(x) \cos_p(x) \cos_p^{1 - l_k p - m_k}(x) + \\ &+ \; a_k (1 - l_k p - m_k) \sin_p^{l_k p + m_k}(x) \cos_p^{1 - l_k p - m_k - 1}(x) \sin_p''(x). \end{aligned}$$

Subtitue (22) for $\sin_p''(x)$ from

$$\begin{aligned}\sin_p^{(n+1)}(x) &= \sum_{k=1}^{2^{n-2}} a_k(l_k p + m_k)\sin_p^{l_k p + m_k - 1}(x)\cos_p^{1 - l_k p - m_k + 1}(x) - \\ &- a_k(1 - l_k p - m_k)\sin_p^{(l_k+1)p + m_k - 1}(x)\cos_p^{1-(l_k+1)p - m_k + 1}(x)\,.\end{aligned}$$

By the fact that the function $z \mapsto z^q$, $z > 0$, $q \in \mathbb{R}$ belongs to $C^\infty(0, +\infty)$ and Lemma 2.1 again, $\sin_p^{(n+1)}(x)$ is continuous.
Denoting

$$\begin{aligned}\bar{a}_{2k-1} &= a_k(l_k p + m_k)\,, \\ \bar{a}_{2k} &= -a_k(1 - l_k p - m_k)\,, \\ \bar{l}_{2k-1} &= l_k\,, && (23) \\ \bar{m}_{2k-1} &= m_k - 1\,, && (24) \\ \bar{l}_{2k} &= l_k + 1\,, && (25) \\ \bar{m}_{2k} &= m_{2k} - 1\,. && (26)\end{aligned}$$

Thus we have

$$\sin_p^{(n+1)}(x) = \sum_{k=1}^{2^{n-1}} \bar{a}_k \sin_p^{\bar{l}_k p + \bar{m}_k}(x)\cos_p^{\bar{l}_k p + \bar{m}_k}(x)\,,$$

which concludes the proof by induction.

□

Let us mention that the similar statement works for $x \in (-\frac{\pi_p}{2}, 0)$ and $x \in (\frac{\pi_p}{2}, \pi_p)$, but the series have form

$$\sin_p^{(n)}(x) = \sum_{k=1}^{2^{n-2}} b_k \sin_p^{l_k p + m_k}(-x)\cos_p^{1 - l_k p - m_k}(x)\,, \tag{27}$$

and

$$\sin_p^{(n)}(x) = \sum_{k=1}^{2^{n-2}} b_k \sin_p^{l_k p + m_k}(x)\left(-\cos_p(x)\right)^{1 - l_k p - m_k}\,. \tag{28}$$

The proofs are analogous, if the appropriate form of (8) from Lemma 2.2 is used, i.e. equation (14) and (13). The other important fact is that (in general) $a_k \neq b_k$, but $|a_k| = |b_k|$. It is easy corollary of the oddness of $\sin_p(x)$.
Let us establish the following notation:

$$\begin{aligned}\sin_p(x) &= S_p\,, \\ \sin_p'(x) &= C_p\,.\end{aligned}$$

Remark 3.1 *We express first four derivatives of* $\sin_p(x)$ *for* $p \in \mathbb{N}$, $p > 1$ *and* $x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. *From the oddness of* $\sin_p(x)$ *(by definition) for* $x < 0$

$$|\sin_p(x)|^p = |-\sin_p(-x)|^p = |-1|^p \cdot |\sin_p(-x)|^p = \sin_p^p(-x)$$

and we can rewrite identity (8) for x *on* $(-\frac{\pi_p}{2}, 0)$

$$\sin_p^p(-x) + \cos_p^p(x) = 1\,.$$

Moreover by the evenness of $\cos_p(x)$ *on* $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$

$$\cos_p(-x) = \cos_p(x)\,.$$

If $p \geq n$, *then for* n*-th derivative of* $\sin_p(x)$ $(n = 1\,, 2\,, 3\,, 4)$ *holds*

x	$\geq$	0 *or* p *even for* $x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2})$	x	$<$	0
1	$=$	$S_p^p(x) + C_p^p(x)$	1	$=$	$S_p^p(-x) + C_p(x)^p$
0	$=$	$S_p^{p-1}(x)C_p(x)$	0	$=$	$-S_p^{p-1}(-x)C_p(x)$
	$+$	$C_p^{p-1}(x)S_p''(x)$		$+$	$C_p^{p-1}(x)S_p''(x)$
S_p''	$=$	$-S_p^{p-1}(x)C_p^{2-p}(x)$	S_p''	$=$	$S_p^{p-1}(-x)C_p^{2-p}(x)$
S_p'''	$=$	$-(p-1)S_p^{p-2}(x)C_p^{3-p}(x)$	S_p'''	$=$	$(p-1)S_p^{p-2}(-x)C_p^{3-p}(x)$
	$+$	$(2-p)S_p^{p-1}(x)C_p^{1-p}(x)S_p''(x)$		$+$	$(2-p)S_p^{p-1}(-x)C_p^{1-p}(x)S_p''(x)$
	$=$	$-(p-1)S_p^{p-2}(x)C_p^{3-p}(x)$		$=$	$(p-1)S_p^{p-2}(-x)C_p^{3-p}(x)$
	$+$	$(2-p)S_p^{2p-2}(x)C_p^{3-2p}(x)$		$+$	$(2-p)S_p^{2p-2}(-x)C_p^{3-2p}(x)$
S_p^{IV}	$=$	$-(p-1)(p-2)S_p^{p-3}(x)C_p^{4-p}(x)$	S_p^{IV}	$=$	$(p-1)(p-2)S_p^{p-3}(-x)C_p^{4-p}(x)$
	$-$	$(p-1)(3-p)S_p^{p-2}(x)C_p^{2-p}(x)S_p''(x)$		$+$	$(p-1)(3-p)S_p^{p-2}(-x)C_p^{2-p}(x)S_p''(x)$
	$+$	$(2-p)(2p-2)S_p^{2p-3}(x)C_p^{4-2p}(x)$		$+$	$(2-p)(2p-2)S_p^{2p-3}(-x)C_p^{4-2p}(x)$
	$+$	$(2-p)(3-2p)S_p^{2p-2}(x)C_p^{2-2p}(x)S_p''(x)$		$+$	$(2-p)(3-2p)S_p^{2p-2}(-x)C_p^{2-2p}(x)S_p''(x)$
	$=$	$-(p-1)(p-2)S_p^{p-3}(x)C_p^{4-p}(x)$		$=$	$(p-1)(p-2)S_p^{p-3}(-x)C_p^{4-p}(x)$
	$+$	$(p-1)(3-p)S_p^{2p-3}(x)C_p^{4-2p}(x)$		$+$	$(p-1)(3-p)S_p^{2p-3}(-x)C_p^{4-2p}(x)$
	$+$	$(2-p)(2p-2)S_p^{2p-3}(x)C_p^{4-2p}(x)$		$+$	$(2-p)(2p-2)S_p^{2p-3}(-x)C_p^{4-2p}(x)$
	$-$	$(2-p)(3-2p)S_p^{3p-3}(x)C_p^{4-3p}(x)$		$+$	$(2-p)(3-2p)S_p^{3p-3}(-x)C_p^{4-3p}(x)$

We consider the case $p = 3$. *By Theorem 3.2 we know that* $\sin_3(x)$ *belongs* $C^1(-\frac{\pi_3}{2}, \frac{\pi_3}{2})$. *For second derivate we have continuity on both intervals by Lemma 3.1 (*$q = 2$, $r = -1$*)*

and for $x=0$ we have

$$\begin{aligned}
\sin_3^2(0)\cdot \sin_3'^{-1}(0) = 0\cdot 1 &= 0\,,\\
\lim_{x\to 0^-} \sin_3^2(x)\cdot \sin_p'^{-1}(x) &= 0\,,
\end{aligned}$$

and it follows that $\sin_3(\cdot)\in C^2(-\frac{\pi_3}{2},\frac{\pi_3}{2})$. *Likewise all addends of the third derivative on both intervals satisfies the conditions of Lemma 3.1 (so it is sum of continuous functions) and*

$$\begin{aligned}
(-2)\sin_3(0)\cdot 1 + (-1)\sin_3^4(0)\cdot \sin_3'^{-3}(0) &= 0\,,\\
\lim_{x\to 0^-}\Big(2\sin_3(x)\cdot 1 + (-1)\sin_3^4(x)\cdot \sin_3'^{-3}(x)\Big) &= 0\,.
\end{aligned}$$

Hence $\sin_3'''(x)$ *is also continuous on* $(-\frac{\pi_3}{2},\frac{\pi_3}{2})$. *On the other hand the fourth derivative is not continuous. We can obtain* $\sin_3^{IV}(x)$ *easily taking* $\sin_3'''(x)$ *into derivative and we get*

$$\begin{aligned}
\sin_3^{IV}(x) &= -2\sin_3'(x) - 4\sin_3^3(x)\cdot\sin_3'^{-2}(x) - 3\sin_3^6(x)\cdot \sin_3'^{-5}(x) \quad x\in(0,\tfrac{\pi_3}{2})\,,\\
\sin_3^{IV}(x) &= 2\sin_3'(x) - 4\sin_3^3(x)\cdot\sin_3'^{-2}(x) + 3\sin_3^6(x)\cdot \sin_3'^{-5}(x) \quad x\in(-\tfrac{\pi_3}{2},0)\,.
\end{aligned}$$

The conditions of Lemma 3.1 holds for all addens of the fourth derivative on both intervals except the first members. There are members including $\sin_3^0(x)$ *which is on* $(-\frac{\pi_3}{2},0)\cup(0,\frac{\pi_3}{2})$ *equal to one and here Lemma 3.1 for* $\sin_3'(x)$ *holds. It remains the task of continuity at* $x=0$. *We evaluate following limits*

$$\begin{aligned}
\lim_{x\to 0^+}\Big(-2\sin_3'(x) - 4\sin_3^3(x)\cdot\sin_3'^{-2}(x) - 3\sin_3^6(x)\cdot \sin_3'^{-5}(x)\Big) &= -2\,,\\
\lim_{x\to 0^-}\Big(2\sin_3'(x) - 4\sin_3^3(x)\cdot\sin_3'^{-2}(x) + 3\sin_3^6(x)\cdot \sin_3'^{-5}(x)\Big) &= 2\,,
\end{aligned}$$

which implies that $\sin_3(\cdot)\notin C^4(-\frac{\pi_3}{2},\frac{\pi_3}{2})$.

Main results of this section are Theorem 3.4 and Theorem 3.5.

Theorem 3.4 *Let* $p=2(i+1)$, $i\in\mathbb{N}$. *Then*

$$\sin_p(\cdot)\in C^\infty(-\frac{\pi_p}{2},\frac{\pi_p}{2}).$$

Proof: By Theorem 3.2 it is clear that $\sin_p(\cdot)\in C^1(-\frac{\pi_p}{2},\frac{\pi_p}{2})$ for $p>1$. From Theorem 3.3 and by equation (27) we get for any $n\in\mathbb{N}$, $n\geq 2$

$$\sin_p^{(n)}(x) = \begin{cases} \sum_{k=1}^{2^{n-2}} b_{k,n}\sin_p^{l_{k,n}p+m_{k,n}}(-x)\cos_p^{1-l_{k,n}p+m_{k,n}}(x) & \text{for } x\in(-\frac{\pi_p}{2},0)\,,\\ \sum_{k=1}^{2^{n-2}} a_{k,n}\sin_p^{l_{k,n}p+m_{k,n}}(x)\cos_p^{1-l_{k,n}p+m_{k,n}}(x) & \text{for } x\in(0,\frac{\pi_p}{2})\,. \end{cases} \tag{29}$$

To make the idea of proof more apparent, we introduce the following substitution

$$q_{k,n} = l_{k,n} p + m_{k,n} \,. \tag{30}$$

Since for all k and n, are $l_{k,n}$, $m_{k,n} \in \mathbb{Z}$ and $p \in \mathbb{N}$ exponent $q_{k,n} \in \mathbb{Z}$. We also define $\sin_p^0(x) \stackrel{\text{def}}{=} 0$ for all x on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. Our proof proceeds by induction in n.

Step 1: For $n = 2$ the continuity of $\sin_p''(x)$ is simple application of Lemma 3.2 on (29), because $q_{1,2} = p - 1 \geq 3$.

Step 2: Let us assume that for n $\sin_p^{(n)}(x)$ is continuous on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. Since for $q < 0$, $\sin_p^q(0)$ is not defined, for $k = 1\,, 2\,, 3\,, 4\,, \dots, 2^{n-2}$ in (29)

$$q_{k,n} \geq 0 \,.$$

By equations (23)-(26) and by

$$(\sum_{j=1}^{N} u_j \cdot v_j)' = \sum_{j=1}^{N} u_j' \cdot v_j + \sum_{j=1}^{N} u_j \cdot v_j'$$

for derivative of (29) (considering the subtitution (30))

$$\left.\begin{array}{rcll} q_{2k-1,n+1} &=& q_{k,n} - 1 & \text{from } \frac{d}{dx} \sin_p^{q_{k,n}}(x)\,, \\ q_{2k,n+1} &=& q_{k,n} + p - 1 & \text{from } \frac{d}{dx} \cos_p^{1-q_{k,n}}(x)\,, \end{array}\right\} \tag{31}$$

Since $p > 1$ in the case that $q_{k,n} \geq 2$ for all $k = 1\,, 2\,, 3\,, 4\,, \dots, 2^{n-2}$, $q_{k,n+1} > 0$ for all $k = 1\,, 2\,, 3\,, 4, \dots, 2^{n-1}$ and $\sin_p^{(n+1)}(x)$ is well-defined and continuous by Lemma 3.2.
In the next case there is at least one k_0 for which $q_{k_0,n} = 1$. Corresponding members ($\cos_p(x) \neq 0$ on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$) are

$$\left.\begin{array}{rcll} a_{k_0,n} \sin_p^{q_{k_0,n}}(x) \cos_p^{1-q_{k_0,n}}(x) &=& a_{k_0,n} \sin_p^1(x) \cdot 1 & \text{for } x \in (0, \frac{\pi_p}{2})\,, \\ b_{k_0,n} \sin_p^{q_{k_0,n}}(-x) \cos_p^{1-q_{k_0,n}}(x) &=& b_{k_0,n} \sin_p^1(-x) \cdot 1 & \text{for } x \in (-\frac{\pi_p}{2}, 0)\,. \end{array}\right\} \tag{32}$$

in the series for n-th derivative of $\sin_p(x)$. The derivatives of members (32)

$$\left.\begin{array}{ll} \frac{d}{dx}\left(a_{k_0,n} \sin_p^1(x) \cdot 1\right) = a_{k_0,n} \cos_p(x) \cdot 1 + a_{k_0,n} \sin_p(x) \cdot 0 & \text{for } x \in (0, \frac{\pi_p}{2})\,, \\ \frac{d}{dx}\left(b_{k_0,n} \sin_p^1(-x) \cdot 1\right) = -b_{k_0,n} \cos_p(x) \cdot 1 + b_{k_0,n} \sin_p(-x) \cdot 0 & \text{for } x \in (-\frac{\pi_p}{2}, 0)\,, \end{array}\right\} \tag{33}$$

appears the series for $(n+1)$-th derivative of $\sin_p(x)$. Thus these members are well-defined on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. Setting

$$\begin{array}{rcl} a_{2k_0-1,n+1} &:=& a_{k_0,n}\,, \\ a_{2k_0,n+1} &:=& 0 \cdot a_{k_0,n} = 0\,, \\ b_{2k_0-1,n+1} &:=& -b_{k_0,n}\,, \\ b_{2k_0,n+1} &:=& 0 \cdot b_{k_0,n} = 0\,, \end{array}$$

and observing that $\cos_p(x)$ is even, we can rewrite (33) as

$$\begin{array}{rcl} \frac{d}{dx}\left(a_{k_0,n}\sin_p^1(x)\cdot 1\right) & = & a_{2k_0-1,n+1}\sin_p^0(x)\cos_p(x)+a_{2k_0,n+1}\sin_p^1(x)\cos_p^0(x)\,, \\ \frac{d}{dx}\left(b_{k_0,n}\sin_p^1(-x)\cdot 1\right) & = & b_{2k_0-1,n+1}\sin_p^0(-x)\cos_p(x)+b_{k_0,n+1}\sin_p(-x)\cos_p^0(x)\,, \end{array}$$

which formally satisfies (29).
In the last case there is at least one k_0 for which $q_{k_0,n}=0$. From the previous cases, the corresponding members of (29) are

$$\begin{array}{ll} a_{k_0,n}\cdot 1\cdot\cos_p(x) & \text{for } x\in(0,\frac{\pi_p}{2})\,, \\ b_{k_0,n}\cdot 1\cdot\cos_p(x) & \text{for } x\in(-\frac{\pi_p}{2},0)\,, \end{array}$$

The derivatives of these members for x on $(0,\frac{\pi_p}{2})$ and x on $(-\frac{\pi_p}{2},0)$ are, respectively

$$\begin{array}{rcl} \frac{d}{dx}\left(a_{k_0,n}\cdot 1\cdot\cos_p(x)\right) & = & a_{k_0,n}\cdot 0\cdot\cos_p(x)+a_{k_0,n}\cdot 1\cdot\sin_p''(x) \\ & = & a_{k_0,n}\cdot 0\cdot\sin_p^0(x)\cos_p(x)-a_{k_0,n}\sin_p^{p-1}(x)\cos_p^{2-p}(x)\,, \\ \frac{d}{dx}\left(b_{k_0,n}\cdot 1\cdot\cos_p(x)\right) & = & b_{k_0,n}\cdot 0\cdot\cos_p(x)+b_{k_0,n}\cdot 1\cdot\sin_p''(x) \\ & = & b_{k_0,n}\cdot 0\cdot\sin_p^0(x)\cos_p(x)+b_{k_0,n}\sin_p^{p-1}(-x)\cos_p^{2-p}(x)\,, \end{array}$$

where we substitue (12) for $\sin_p''(x)$ on $(0,\frac{\pi_p}{2})$ and (14) for $\sin_p''(x)$ on $(-\frac{\pi_p}{2},0)$. It is obvious that these members are well defined on $(-\frac{\pi_p}{2},\frac{\pi_p}{2})$.
Setting

$$\begin{array}{rcl} a_{2k_0-1,n+1} & := & 0\cdot a_{k_0,n}=0\,, \\ a_{2k_0,n+1} & := & -a_{k_0,n}\,, \\ b_{2k_0-1,n+1} & := & 0\cdot b_{k_0,n}=0\,, \\ b_{2k_0,n+1} & := & b_{k_0,n}\,, \end{array}$$

we get

$$\begin{array}{rcl} \frac{d}{dx}\left(a_{k_0,n}\cdot 1\cdot\cos_p(x)\right) & = & a_{2k_0-1,n+1}\sin_p^0(x)\cos_p^1(x)+a_{2k_0,n+1}\sin_p^{p-1}(x)\cos_p^{2-p}(x)\,, \\ \frac{d}{dx}\left(b_{k_0,n}\cdot 1\cdot\cos_p(x)\right) & = & b_{2k_0-1,n+1}\sin_p^0(x)\cos_p^1(x)+b_{2k_0,n+1}\sin_p^{p-1}(-x)\cos_p^{2-p}(x)\,, \end{array}$$

which formally satisfies (29).
Considering these three cases and Lemma 3.2 if $q_{k,n}\neq 1$ the $(n+1)$-th derivative of $\sin_p(x)$ is continuous on $(-\frac{\pi_p}{2},\frac{\pi_p}{2})$.
It remains a question of continuity in the case there is at least one $q_{k_0,n}=1$. Without loss of generality we may assume there is exactly one such k_0, because if there is more then one, we can add all coefficients $a_{k_0,n}$. The members corresponding to $(n+1)$-th derivative

$$\left.\begin{array}{ll} a_{2k_0-1,n+1}\cos_p(x) & \text{for } x\in(0,\frac{\pi_p}{2})\,, \\ b_{2k_0-1,n+1}\cos_p(x) & \text{for } x\in(-\frac{\pi_p}{2},0)\,, \end{array}\right\} \tag{34}$$

are formed by taking $\sin_p^q(x)$ (or $\cos_p^r(x)$) into derivative finitely many times. By (31) the appropriate $q_{2k_0-1,n+1}=0$. Let $j\in\mathbb{N}$ is the number of derivatives of $\cos_p^r(x)$ needed to obtain (34) from $\sin_p''(x)$. Considering (31) exponent $q_{k,n}$ depens on j

$$q_{2k_0,n+1}=j(p-1)+(n+1-2-j)(-1)+q_{1,2}=(j+1)p-n\,. \tag{35}$$

It follows that for p even and $q_{2k_0-1,n+1} = 0$

$$n = (j+1)p$$

is also even and $\sin_p^{(n+1)}(x)$ is even function by Lemma 2.1. Due to evenness and the facts that

$$\lim_{x\to 0} a_{k,n+1} \sin_p^{q_{k,n+1}}(x) \cos_p^{1-q_{k,n+1}}(x) = 0$$

for all $k \neq 2k_0 - 1$ and

$$b_{2k_0-1,n+1} = \lim_{x\to 0^-} \sin_p^{(n+1)}(x) = \lim_{x\to 0^+} \sin_p^{(n+1)}(x) = a_{2k_0-1,n+1}$$

function $\sin_p^{(n+1)}(x)$ is continuous on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. The proof is completed.

□

Theorem 3.4 allows us develop $\sin_p(x)$ in the Taylor series for even p, but computing of its coefficients is left as open problem.

Lemma 3.3 *Let $f(x)$ and $g(x)$ belongs $C^m(a,b)$. Then*

$$(f\cdot g)^{(m)} = \sum_{j=0}^{m} \binom{m}{j} f^{(j)} \cdot g^{(m-j)}\,. \tag{36}$$

Proof: We proof this lemma by recursion on n.

Step 1: We use the well-known formula $(uv)' = u'v + uv'$ for the first derivative and we get

$$(f\cdot g)' = f'\cdot g + f\cdot g' = \sum_{j=0}^{1} \binom{1}{j} f^{(j)} \cdot g^{(1-j)}$$

Thus the statement of Lemma 3.3 is true for $n = 1$.

Step 2: Assuming (36) to hold for n, we will prove it for $n+1$.
From (36) and the linearity of derivative ($(u+v)' = u' + v'$) we have

$$\frac{d}{dx}(f\cdot g)^{(n)} = \sum_{j=0}^{n} \frac{d}{dx} \binom{n}{j} f^{(j)} \cdot g^{(n-j)}.$$

Derivative of the j-th and $(j+1)$-th member of the series are

$$\frac{d}{dx}\binom{n}{j} f^{(j)} \cdot g^{(n-j)} = \binom{n}{j} f^{(j+1)} \cdot g^{(n-j)} + \binom{n}{j} f^{(j)} \cdot g^{(n-j+1)}$$

and

$$\frac{d}{dx}\binom{n}{j} f^{(j+1)} \cdot g^{(n-j-1)} = \binom{n}{j+1} f^{(j+2)} \cdot g^{(n-j)} + \binom{n}{j+1} f^{(j+1)} \cdot g^{(n-j-1+1)}.$$

It is easily seen, that the sum over all $j \in \mathbb{N}$ is

$$f \cdot g^{(n+1)} + \sum_{j=0}^{n-1} \left(\binom{n}{j+1} + \binom{n}{j} \right) f^{(j+1)} \cdot g^{(n-j)} + f^{(n+1)} \cdot g.$$

Thus we have (after reindexation)

$$(f \cdot g)^{(n+1)} = \sum_{j=0}^{n+1} \binom{n+1}{j} f^{(j)} \cdot g^{(n-j+1)}.$$

Step 3: The recursion stop when $n + 1 = m$.

□

Theorem 3.5 *Let* $p = 2i + 1$, $i \in \mathbb{N}$. *Then*

$$\sin_p(\cdot) \in C^p(-\frac{\pi_p}{2}, \frac{\pi_p}{2}) , \qquad \textit{but } \sin_p(\cdot) \notin C^{p+1}(-\frac{\pi_p}{2}, \frac{\pi_p}{2}) .$$

Proof: We use (29) with the substitution (30) again. By Lemma 3.2 the n-th derivative of $\sin_{2i+1}(x)$ can by discontinuous only if there is a $k_0 \in \mathbb{N}$, $k \leq 2^{n-2}$ for which $q_{k_0,n} \leq 0$. By (31) we know that q decreasing if we taking into derivative by $\sin^q_{2i+1}(x)$ and it decreasing by one per derivative. We begin with second derivative of $\sin_p(x)$

$$\begin{array}{rcll} \sin''_{2i+1} & = & -\sin^{2i}_{2i+1} {\sin'_{2i+1}}^{1-2i} & \text{for } x \in [0, \frac{\pi_{2i+1}}{2}) , \\ \sin''_{2i+1} & = & \sin^{2i}_{2i+1} {\sin'_{2i+1}}^{1-2i} & \text{for } x \in (-\frac{\pi_{2i+1}}{2}, 0) , \end{array}$$

which we taking into derivative. By Lemma 3.3 we show that the highest derivative of $\sin_{2i+1}(x)$ is at member $f(x)^{(t)} \cdot g$. For $f(x) = \sin^{2i}_{2i+1}(x)$, $g(x) = {\sin'_{2i+1}}^{1-2i}(x)$ and $t = 2i-1$ it is obvious that from chain rule we get for $x \in [0, \frac{\pi_{2i+1}}{2})$

$$\frac{d^{2i-1}}{dx^{2i-1}} \left(-\sin^{2i}_{2i+1}(x) \right) = (2i)(2i-1) \cdot \ldots \cdot 2 \cdot \sin^1_{2i+1}(x) \cdot {\sin'_{2i+1}}^0(x)$$

and for $x \in (-\frac{\pi_{2i+1}}{2}, 0)$

$$\frac{d^{2i-1}}{dx^{2i-1}} \sin^{2i}_{2i+1}(x) = (2i)(2i-1) \cdot \ldots \cdot 2 \cdot \sin^1_p(x) \cdot {\sin'_{2i+1}}^0(x) .$$

The member with the highest derivative of $\sin^q_{2i+1}(x)$, which is for the minimum of all $q_{k,n}$ has after $2i + 1$ derivatives following form for $x \in [0, \frac{\pi_{2i+1}}{2})$ and for $x \in (-\frac{\pi_{2i+1}}{2}, 0)$, respectively

$$\begin{array}{rcl} \sin^{(2i+1)}_{2i+1}(x) & = & -(2i)(2i-1) \cdot \ldots \cdot 2 \cdot \sin^1_{2i+1}(x) \cdot {\sin'_{2i+1}}^0(x) , \\ \sin^{(2i+1)}_{2i+1}(x) & = & (2i)(2i-1) \cdot \ldots \cdot 2 \cdot \sin^1_{2i+1}(x) \cdot {\sin'_{2i+1}}^0(x) , \end{array}$$

which can be rewritten ($\sin_p'(x) \in (0,1]$ on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$) as

$$\begin{array}{rclr} \sin_{2i+1}^{(2i+1)}(x) & = & -a_{2i+1} \cdot \sin_{2i+1}(x) & \text{for } x \in [0, \frac{\pi_{2i+1}}{2})\,, \\ \sin_{2i+1}^{(2i+1)}(x) & = & a_{2i+1} \cdot \sin_{2i+1}(x) & \text{for } x \in (-\frac{\pi_{2i+1}}{2}, 0)\,. \end{array} \tag{37}$$

By further taking (37) into derivative we obtain

$$\begin{array}{rclr} \sin_{2i+1}^{2i+2}(x) & = & -a_{2i+1} \cdot \sin_{2i+1}'(x) & \text{for } x \in [0, \frac{\pi_{2i+1}}{2})\,, \\ \sin_{2i+1}^{2i+2}(x) & = & a_{2i+1} \cdot \sin_{2i+1}'(x) & \text{for } x \in (-\frac{\pi_{2i+1}}{2}, 0)\,, \end{array}$$

It remains to evaluate limits

$$\begin{array}{rclcl} \lim_{x\to 0^+} \sin_{2i+1}^{2i+2}(x) & = & \lim_{x\to 0^+} -a_{2i+1} \cdot \sin_{2i+1}'(x) & = & -a_{2i+1}\,, \\ \lim_{x\to 0^-} \sin_{2i+1}^{2i+2}(x) & = & \lim_{x\to 0^-} a_{2i+1} \cdot \sin_{2i+1}'(x) & = & a_{2i+1}\,, \end{array}$$

which follows that $\sin_{2i+1}(\cdot) \notin C^{2i+2}(-\frac{\pi_{2i+1}}{2}, \frac{\pi_{2i+1}}{2})$. That $\sin_{2i+1}(\cdot) \in C^{2i}(-\frac{\pi_{2i+1}}{2}, \frac{\pi_{2i+1}}{2})$ follows from Lemma 3.2, because $q > 0$ and proof is complete.

□

Theorem 3.6 *Let $p \in \mathbb{R} \setminus \mathbb{N}$ such that $p > 1$. Then $\sin_p(\cdot)$ belongs $C^{[p]+1}(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$, but it does not belong $C^{[p]+2}(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$.*

Proof: We use (29) with the substitution (30) again. By Theorem 3.2 $\sin_p(\cdot) \in C^1(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. Considering (35) we get for all $n \leq [p] + 1$

$$q_{1,n} = p - n + 1 \geq q_{1,[p]+1} = p - [p] > 0\,,$$

and other $q_{k,n}$ for $k = 2\,, 3\,, 4\,, \ldots\,, 2^{n-2}$ are bigger than $q_{1,n}$ since $j \neq 0$ in these coefficients and we add $j(p-1) + j > 0$. Thus $\sin_p(\cdot)$ belongs $C^{[p]+1}(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ by Lemma 3.2 apply on (29). If we taking $\sin_p(x)$ into derivative $[p] + 1$-times, we get for $k = 1$

$$f(x) = a_{1,[p]+1} \sin_p^{p-[p]}(x) \cos_p^{1-p+[p]}(x)\,.$$

Taking $f(x)$ into derivative again, we get for $k = 1$

$$f'(x) = a_{1,[p]+2} \sin_p^{p-[p]-1}(x) \cos_p^{-p+[p]}(x)\,,$$

which is cleary undefined at $x = 0$. Since $\cos_p^{-p+[p]}(0) = 1$ and

$$\lim_{x\to 0^+} \sin_p^{p-[p]-1}(x) = +\infty$$

the function $\sin_p^{([p]+2)}(x)$ has the infinite discontinuity at $x = 0$, which implies that $\sin_p(\cdot) \notin C^{[p+2]}(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. Hence for $p \in \mathbb{R} \setminus \mathbb{N}$, $p > 1$

$$\sin_p(\cdot) \in C^{[p]+1}(-\frac{\pi_p}{2}, \frac{\pi_p}{2})\,,$$

and Theorem 3.6 is proved.

□

Lemma 3.4 *Let $p \in \mathbb{R}$ such that $p > 1$ and let $k \in \mathbb{N}$. For all $k = 1\,,\,2\,,\,3\,,\,4\,,\ldots,\,k_0$, $r_k\,,\,q_k \in \mathbb{R}$*

$$\sum_{k=1}^{k_0} a_k \sin_p^{q_k}(x)\cos_p^{r_k}(x)$$

is continuous on $(\frac{\pi_p}{2},\pi_p)$ *.*

Proof: If the function $f(x)$ is continuous, then for all real constant α function $\alpha{\cdot}f(x)$ is also continuous. Thus we can consider simplifying assumption $a_k = 1$. We choose any $q\,,\,r \in \mathbb{R}$. By the continuity of $\sin_p(x)$ on $(\frac{\pi_p}{2},\pi_p)$, the fact that $\sin_p(x) \in (0,1)$ for $x \in (\frac{\pi_p}{2},\pi_p)$ and the continuity of $z \to z^q$ on $(0,1)$ we have the continuity of $\sin_p^q(x)$ on $(\frac{\pi_p}{2},\pi_p)$.
Similarly by the continuity of $\cos_p(x)$ on $(\frac{\pi_p}{2},\pi_p)$, the fact that $-\cos_p(x) \in (0,1)$ on $(\frac{\pi_p}{2},\pi_p)$ and the continuity of $z \to z^r$ on $(0,1)$ we find that $(-\cos_p(x))^r$ is continuous on $(\frac{\pi_p}{2})$. The proof is comleted by notation that sum or product of finite number of continuous functions is continuous function.

□

Lemma 3.5 *Let $p \in \mathbb{R}$ such that $p > 1$ and let $k_0 \in \mathbb{N}$, $k = 1\,,\,2\,,\,3\,,\,4\,,\,5\,,\ldots,k_0$, $q_k \in \mathbb{R}$, $r_k > 0$, a_k , $b_k \in \mathbb{R}$.*
For

$$f_1(x) = \sum_{k=1}^{k_0} a_k \sin_p^{q_k}(x)\cos_p^{r_k}(x)$$

defined on $(0,\frac{\pi_p}{2})$ *and*

$$f_2(x) = \sum_{k=1}^{k_0} b_k \sin_p^{q_k}(x)\,(-\cos_p(x))^{r_k}$$

defined on $(\frac{\pi_p}{2},\pi_p)$ *the function*

$$f(x) = \begin{cases} f_1(x) & x \in (0,\frac{\pi_p}{2})\,, \\ f_2(x) & x \in (\frac{\pi_p}{2},\pi_p) \end{cases}$$

is continuous on $(-\frac{\pi_p}{2},\frac{\pi_p}{2})$ *.*

Proof: By Lemma 3.1 the function $f_1(x)$ is continuous and by Lemma 3.4 the function $f_2(x)$ is continuous. Since $\cos_p^{r_k}(\frac{\pi_p}{2}) = 0$ for any $r > 0$ and $\sin_p^q(\frac{\pi_p}{2}) = 1$ for all $r \in \mathbb{R}$

$$0 = \lim_{x \to \frac{\pi_p}{2}^-} f_1(x) = \lim_{x \to \frac{\pi_p}{2}^+} f_2(x) = 0\,,$$

and function $f(x)$ is continuous, which conclude the proof.

□

Theorem 3.7 *Let $p \in (1,2)$. If $p' \notin \mathbb{N}$, then $\sin_p(\cdot)$ belongs $C^{[p']}(0,\pi_p)$, but it does not belong $C^{[p']+1}(0,\pi_p)$.*

Proof: By Theorem 3.2 $\sin_p(\cdot)$ belongs $C^1(0,\pi_p) \subset C^1(\mathbb{R})$. Using Theorem 3.3 and (28) and considering the substitution (30)

$$\sin_p^{(n)}(x) = \begin{cases} \sum_{k=1}^{2^{n-2}} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x) & \text{for } x \in (0, \frac{\pi_p}{2}), \\ \sum_{k=1}^{2^{n-2}} b_{k,n} \sin_p^{q_{k,n}}(x) \left(-\cos_p(x)\right)^{1-q_{k,n}} & \text{for } x \in (\frac{\pi_p}{2}, \pi_p). \end{cases} \tag{38}$$

By Lemma 3.5 if $1 - q_{k,n} > 0$, then $\sin_p^{(n)}(x)$ is continuous. Modify (35) for n instead of $n+1$, i.e.

$$q_{k,n} = j(p-1) + (n-2-j)(-1) + q_{1,2} = (j+1)p - n + 1,$$

where $j \in \mathbb{N}$ denotes number of taking $\cos_p^q(x)$ into derivative to obtain k-th member of (38) from $\sin_p''(x)$. It is obvious that $j \le n-2$. Since $1 > q_{k,n}$

$$\begin{aligned} 1 &> (n-2+1)p + 1 - n > (j+1)p + 1 - n, \\ n &> (n-1)p, \\ n - np &> -p, \\ n(1-p) &> -p. \end{aligned}$$

Since $p \in (1,2)$

$$n < \frac{p}{p-1} = p'.$$

If $p' \notin \mathbb{N}$, then $\sin_p(\cdot) \in C^{[p']}(0,\pi_p)$ by Lemma 3.5. If we taking $\sin_p^{([p'])}(x)$ into derivative again, we get certainly for some k_0 $j = [p'] - 1$

$$q_{k_0,[p']+1} = ([p'])p - [p'] = [p'](p-1) > \frac{p}{p-1}(p-1) = p > 1,$$

which means $\cos_p^{1-q_{k_0,[p']+1}}(\frac{\pi_p}{2})$ is not defined and since $\sin_p^{q_{k_0,[p']+1}}(\frac{\pi_p}{2}) = 1$

$$\lim_{x \to \frac{\pi_p}{2}+} \sin_p^{q_{k_0,[p']+1}}(x) \cos_p^{1-q_{k_0,[p']+1}}(x) = +\infty.$$

Hence $\sin_p^{([p']+1)}(x)$ can not be defined at $x = \frac{\pi_p}{2}$ and proof is complete.

□

p	**x in**	$(0,\frac{\pi_p}{2})$	$(-\frac{\pi_p}{2},\frac{\pi_p}{2})$	$\mathbb{R}$	$(0,\pi_p)$
$p=2$		C^∞	C^∞	C^∞	C^∞
$p=2k$	$k\in\mathbb{N}\setminus\{1\}$	C^∞	C^∞	C^1	C^1
$p=2k+1$	$k\in\mathbb{N}$	C^∞	C^p	C^1	C^1
$p\in(1,2)$	$p'\notin\mathbb{N}$	C^∞	C^2	C^2	$C^{[p']}$
$p\in\mathbb{R}\setminus\mathbb{N}$	$p>2$	C^∞	$C^{[p]+1}$	C^1	C^1

Table 1: The continuity of n-th derivative of $\sin_p(x)$

4 Explicit expression of $\sin_p(x)$

This section is devoted to a few ways of expression $\sin_p(x)$ in the term of power series. We use inverting of power series of $\operatorname{arcsin}_p(x)$, i.e. (11) for this purpose. Since this series is defined only on $(0,1)$, function $\sin_p(x)$ is defined only on $(0,\frac{\pi_p}{2})$. In whole section we will consider $p\in\mathbb{N}$ such that $p>1$.
Using (9) to invert power series (11) for $p\in\mathbb{N}$ such that $p>1$ and for $x\in(0,1)$

$$b_{np+1}=\frac{1}{(np+1)\cdot a_1^{np+1}}\sum_{s,t,u,\dots}(-1)^\sigma\cdot\frac{(np+1)(np+2)\cdot\ldots\cdot(np+\sigma)}{s!t!u!\dots}\left(\frac{a_2}{a_1}\right)^s\left(\frac{a_3}{a_1}\right)^t\dots, \tag{39}$$

where $\sigma=s+t+u+\dots$. For $n\in\mathbb{N}$ if $k\neq np+1$ then $a_k=0$ and if $k=np+1$ then

$$a_{np+1}=\frac{\Gamma\left(n+\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)(n\cdot p+1)\cdot n!}. \tag{40}$$

Since $\Gamma(z+1)=z\cdot\Gamma(z)$ for $n\in\mathbb{N}$

$$a_{np+1}=\frac{\left(n-1+\frac{1}{p}\right)\left(n-2+\frac{1}{p}\right)\dots\left(1+\frac{1}{p}\right)\frac{1}{p}\cdot\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)(n\cdot p+1)\cdot n!}=\frac{(p(n-1)+1)(p(n-2)+1)\dots 1}{(p\cdot n+1)p^n\cdot n!}. \tag{41}$$

Theorem 4.1 *Let $a_{n_0}=0$ is n_0-th coefficient of (11), then $b_{n_0}=0$ in the power series for* $\sin_p(x)$.

Proof: Let us assume that for any $n_0\in\mathbb{N}$ is $a_{n_0}=0$. From (11) if for all $i\in\mathbb{N}$ the index $l\neq i\cdot p+1$, then $a_l=0$. Now we use a condition from (9), i.e.

$$1\cdot s+2\cdot t+3\cdot u+\ldots=n-1, \tag{42}$$

where we denote $m_1=s$, $m_2=t$, $m_3=u,\dots$. If $a_{n_1}=0$, then the corresponding member of (9) is equal to zero since we multiply by a_{n_1}. It follows that we must not consider such

m_l for which $a_{l+1} = 0$. Thus we can rewrite (42) as

$$p \cdot m_p + 2p \cdot m_{2p} + 3p \cdot m_{3p} + \ldots = n_0 - 1\,.$$

Divide this equation by $p > 1$

$$m_p + 2 \cdot m_{2p} + 3 \cdot m_{3p} = \frac{n-1}{p}\,. \tag{43}$$

Since $a_{n_0} = 0$ for all $i \in \mathbb{N}$, $n_0 \neq i \cdot p + 1$ and

$$\frac{n_0 - 1}{p} \notin \mathbb{Z}\,.$$

But for all $i \in \mathbb{N}$, $m_{ip} \in \mathbb{N}$, which implies that there is no acceptable member combination for condition of (9). Hence $b_{n_0} = 0$ and proof is complete.

□

Let us note that since b_n are Taylor coefficients, it correspond to the fact, that if for all $k \in \mathbb{N}$, $k < 2^{n-2}$ is $q_{k,n} > 0$, then

$$\sin_p^{(n)}(0) \;=\; \sum_{k=1}^{2^{n-2}} a_{k,n} \sin_p^{q_{k,n}}(0) \cos_p^{1-q_{k,n}}(0) \;=\; 0\,,$$

Due to Theorem 4.1 we can compute only coefficients b_{ip+1} for $i \in \mathbb{N}$. Let us express few coefficients. From (39), (40) and (41) it is easily seen that

$$\begin{aligned}
b_1 &= \tfrac{1}{a_1} && = 1\,,\\
b_{p+1} &= \tfrac{1}{(p+1)\cdot a_1^{p+1}} \left(-\tfrac{p+1}{1!} \left(\tfrac{a_{p+1}}{a_1} \right)^1 \right) && = -\tfrac{1}{p(p+1)}\,,\\
b_{2p+1} &= \tfrac{1}{(2p+1)\cdot a_1^{2p+1}} \left(\tfrac{(2p+1)(2p+2)}{2!} \left(\tfrac{a_{p+1}}{a_1} \right)^2 - \tfrac{(2p+1)}{1!} \left(\tfrac{a_{2p+1}}{a_1} \right)^1 \right) && = \tfrac{1}{p^2(p+1)} - \tfrac{p+1}{2!(2p+1)p^2}\\
&= -\tfrac{p^2-2p-1}{4p^4+6p^3+2p^2}\,,
\end{aligned}$$

which are also given in [15]. Further two coefficients are

$$\begin{aligned}
b_{3p+1} &= \tfrac{1}{(3p+1)a_1^{3p+1}} \cdot \left(-\tfrac{(3p+1)(3p+2)(3p+3)}{3!} \left(\tfrac{a_{p+1}}{a_1} \right)^3 + \tfrac{(3p+1)(3p+2)}{1!1!} \left(\tfrac{a_{p+1}}{a_1} \right) \left(\tfrac{a_{2p+1}}{a_1} \right) - \tfrac{3p+1}{1!} \left(\tfrac{a_{3p+1}}{a_1} \right) \right)\\
&= -\tfrac{(3p+2)(p+1)}{2!} \cdot \tfrac{1}{p^3(p+1)^3} + (3p+2) \cdot \tfrac{1}{p(p+1)} \tfrac{p+1}{2!(2p+1)p^2} - \tfrac{(2p+1)(p+1)}{3!(3p+1)p^3}\\
&= -\tfrac{4p^5+11p^4+2p^3-13p^2-7p-1}{36p^7+102p^6+102p^5+42p^4+6p^3}
\end{aligned}$$

and

$$\begin{aligned}
b_{4p+1} &= \tfrac{1}{(4p+1)a_1^{4p+1}} \cdot \left(\tfrac{(4p+1)(4p+2)(4p+3)(4p+4)}{4!} \left(\tfrac{a_{p+1}}{a_1} \right)^4 - \tfrac{(4p+1)(4p+2)(4p+3)}{2!1!} \left(\tfrac{a_{p+1}}{a_1} \right)^2 \left(\tfrac{a_{2p+1}}{a_1} \right) \right) +\\
&+ \tfrac{1}{(4p+1)a_1^{4p+1}} \cdot \left(\tfrac{(4p+1)(4p+2)}{2!} \left(\tfrac{a_{2p+1}}{a_1} \right)^2 + \tfrac{(4p+1)(4p+2)}{1!1!} \left(\tfrac{a_{p+1}}{a_1} \right) \left(\tfrac{a_{3p+1}}{a_1} \right) - \tfrac{4p+1}{1!} \left(\tfrac{a_{4p+1}}{a_1} \right) \right)\\
&= \tfrac{(2p+1)(4p+3)(p+1)}{3} \cdot \tfrac{1}{p^4(p+1)^4} - (2p+1)(4p+3) \cdot \tfrac{1}{p^2(p+1)^2} \cdot \tfrac{p+1}{2!(2p+1)p^2} +\\
&+ (2p+1) \cdot \tfrac{(p+1)^2}{(2p+1)^2p^4(2!)^2} + (4p+2) \cdot \tfrac{1}{p(p+1)} \cdot \tfrac{(2p+1)(p+1)}{(3p+1)p^3(3!)} - \tfrac{(3p+1)(2p+1)(p+1)}{(4p+1)p^4(4!)}\\
&= -\tfrac{36p^8-124p^7+27p^6+206p^5-15p^4-140p^3-71p^2-14p-1}{576p^{10}+2352p^9+3816p^8+3120p^7+1344p^6+288p^5+24p^4}
\end{aligned}$$

Computing of these coefficients is quite difficult. In section 2 we introduce Bell polynomials, which we use to express coefficients b_{np+1}. Comparing (10) and (9) we get following theorem

Theorem 4.2 *Let $p \in \mathbb{N}$ such that $p > 1$, $n \in \mathbb{N}$ and b_{np+1} is coefficient of series expansion of $\sin_p(x)$. Then*

$$b_{np+1} = \sum_{k=1}^{n} \frac{(np+k)!}{(np+1)!} \cdot \frac{(-1)^k}{n!} B_{n,k}\left(\frac{a_{p+1}}{1!}, \frac{a_{2p+1}}{2!}, \ldots, \frac{a_{np+1}}{n!}\right).$$

Proof: We use formula (9) and we get considering (43) with substitution $\sigma = m_p + m_{2p} + m_{3p} + \ldots$

$$b_{np+1} = \frac{1}{(np+1)a_1^{np+1}} \sum_{m_p, m_{2p}, \ldots} (-1)^{\sigma} \frac{(np+1)(np+2)\cdot \ldots \cdot (np+\sigma)}{m_p! m_{2p}! \ldots} \left(\frac{a_{p+1}}{a_1}\right)^{m_p} \left(\frac{a_{2p+1}}{a_1}\right)^{m_{2p}} \ldots .$$

Since $a_1 = 1$ for all $p \in \mathbb{N}$ such that $p > 1$

$$b_{np+1} = \frac{1}{np+1} \sum_{m_p, m_{2p}, \ldots} (-1)^{\sigma} \frac{(np+1)(np+2)\cdot \ldots \cdot (np+\sigma)}{m_p! m_{2p}! \ldots} (a_{p+1})^{m_p} (a_{2p+1})^{m_{2p}} \ldots .$$

For any given $\sigma = \sigma_0$ we denote

$$L(m_p, m_{2p}, \ldots, m_{np}) = \text{ Boolean } [m_p + 2m_{2p} + \ldots + n \cdot m_{np} = n \wedge m_p + m_{2p} + \ldots + m_{np} = \sigma_0],$$

which ensures compliance with (42) and we obtain

$$\begin{aligned} & \frac{1}{np+1} \sum_{m_p, m_{2p}, \ldots} (-1)^{\sigma_0} \frac{(np+1)(np+2)\cdot \ldots \cdot (np+\sigma_0)}{m_p! m_{2p}! \ldots} \prod_{s=1}^{n} (a_{sp+1})^{m_{sp}} \\ = & \sum_{m_p=0}^{n} \sum_{m_{2p}=0}^{n} \ldots \sum_{m_{np}=0}^{n} L(m_p, \ldots, m_{np}) \frac{(np+\sigma_0)!}{(np+1)!} \cdot \frac{(-1)^{\sigma_0}}{n!} \cdot \frac{n!}{m_p! m_{2p}! \ldots} \prod_{s=1}^{n} (a_{sp+1})^{m_{sp}} \\ = & \frac{(np+\sigma_0)!}{(np+1)!} \cdot \frac{(-1)^{\sigma_0}}{n!} \cdot B_{n,\sigma_0}\left(\frac{a_{p+1}}{1!}, \frac{a_{2p+1}}{2!} \ldots \frac{a_{np+1}}{n!}\right). \end{aligned}$$

Sum over all $\sigma \in \mathbb{N}$ such that $\sigma \leq n$ gives us desired formula, e.g.

$$b_{np+1} = \sum_{\sigma=1}^{n} \frac{(np+\sigma)!}{(np+1)!} \cdot \frac{(-1)^{\sigma}}{n!} \cdot B_{n,\sigma}\left(\frac{a_{p+1}}{1!}, \frac{a_{2p+1}}{2!} \ldots \frac{a_{np+1}}{n!}\right),$$

which complete the proof.

□

There is also other way to invert power series by using Bell polynomias. Directly by Theorem 2.1 we get for $k = 1$

$$g_n^{(-1)} = \left(\begin{array}{c} n-1 \\ 0 \end{array} \right) \sum_{i=0}^{n-1} (-1)^i g_1^{-n-i} B_{n-1+i,i}(0, g_1, g_2, g_3, g_4 \ldots) .$$

Hence

$$\begin{aligned}
g_1^{(-1)} &= \textstyle\sum_{i=0}^{0} (-1)^i g_1^{-i-1} B_{i,i}(0, g_2, g_3, \ldots) = \frac{1}{g_1} , \\
g_2^{(-1)} &= \textstyle\sum_{i=0}^{1} (-1)^i g_1^{-2-i} B_{1+i,i}(0, g_2, g_3, \ldots) = g_1^{-2} \cdot 0 + (-1) g_1^{-3} B_{2,1}(0, g_2, g_3, \ldots) \\
&= -\frac{1}{g_1^3} \cdot \frac{2!}{1!} \cdot \frac{g_2}{2!} = -\frac{g_2}{g_1^3} , \\
g_3^{(-1)} &= \textstyle\sum_{i=0}^{2} (-1)^i g_1^{-3-i} B_{2+i,i}(0, g_2, g_3, \ldots) \\
&= g_1^{-3} \cdot 0 + (-1) g_1^{-4} B_{3,1}(0, g_2, g_3, \ldots) + (-1)^2 g_1^{-5} B_{4,2}(0, g_2, g_3, \ldots) \\
&= -\frac{1}{g_1^4} \cdot \frac{3!}{1!} \cdot \frac{g_3}{3!} + \frac{1}{g_1^5} \left(\frac{4!}{2!} \cdot \left(\frac{g_2}{2!}\right)^2 + \frac{4!}{1!1!} \cdot \frac{0}{1!} \cdot \frac{g_3}{3!} \right) = -\frac{g_3}{g_1^4} + 2\frac{g_2^2}{g_1^5} .
\end{aligned}$$

For inversion of $\arcsin_p(x)$ we have for $g_1 = a_1$, $g_2 = a_2$, $g_3 = a_3$,

$$\begin{aligned}
b_1 &= \frac{1}{a_1} &&= 1 , \\
b_2 &= -\frac{a_2}{a_1^3} &&= -a_2 = 0 , \\
b_3 &= -\frac{a_3}{a_1^4} + 2\frac{a_2^2}{a_1^5} &&= -a_3 .
\end{aligned}$$

In the same way we can compute other coefficient. Let us note, that Theorem 4.1 hold here too, but theire aplication is more complicated because of properties of Bell polynomials.

5 Conclusion

The main motivation for this work is to show that power series expansion of function $\sin_p(x)$ is convergent on some neighborhood of $x_0 = 0$. Section 4 deals with a number of ways we can express $\sin_p(x)$ in the term of power series without considering the domain of convergence of these series. The most natural way is using Taylor series expansion, but we need $\sin_p(\cdot) \in C^\infty$. In section 3 is given Theorem 3.4, which allows us to realize the expansion at $x = 0$ for even p, such that $p \neq 2$. It does not solve problem of convergence too, but let us mention a hypothesis.

Hypothesis 5.1 *Let* $p = 2(i+1)$, $i \in \mathbb{N}$. *Then we can expand* $\sin_p(x)$ *into Taylor series at* $x = 0$, *i.e.*

$$f(x) = \sin_p(0) + \frac{\sin_p'(0)}{1!}(x - x_0) + \frac{\sin_p''(0)}{2!}(x - x_0)^2 + \ldots + \frac{\sin_p^{(n)}(0)}{n!}(x - x_0)^n \,,$$

which converges for $n \to +\infty$ *to* $\sin_p(x)$ *at least on* $(-1, 1)$ *and at most on* $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$.

The maximal interval $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ follows from the fact that Theorem 3.4 holds only on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. The minimal interval is given by the fact that by Theorem 3.4 $\sin_p^{(n)}(0)$ can be express in form $\frac{P(p^\alpha)}{n!}$, where $\alpha \leq n - 2$ and $P(p^\alpha)$ denote a polynomial in p, which have degree α.

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