FAKULTA APLIKOVANÝCH VĚD ZÁPADOČESKÉ UNIVERZITY V PLZNI
STUDENTSKÁ VĚDECKÁ KONFERENCE 2016

Biological reaction-diffusion models

Martin Fencl[1]

1 Introduction

We study a system of two partial differential equations describing diffusion and reaction of two chemical substances. We usually consider a linearized model

$$\begin{aligned} \frac{du}{dt} &= d_1 \Delta u + b_{1,1} u + b_{1,2} v, \\ \frac{dv}{dt} &= d_2 \Delta v + b_{2,1} u + b_{2,2} v, \end{aligned} \tag{1}$$

where d_1, d_2 are positive diffusion parameters and $b_{i,j}$ are constant elements of Jacobi matrix of certain mappings f, g, which describe the reaction of substances u, v. It was proposed by Turing (1952) that under some conditions the trivial stationary solution of the system (1) without diffusion ($d_1 = d_2 = 0$) is stable, but with diffusion it is unstable. Such a effect was later called ”Turing effect”. The loss of the stability of the trivial stationary solution give rise to the spatially non-homogeneous solutions. These solutions describe patterns, which have application as patterns on animal coat, for example.

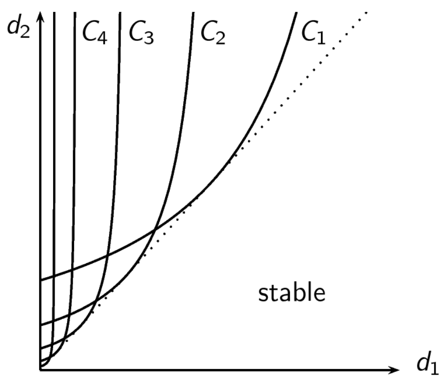


Figure 1: Hyperbolas in the plane $[d_1, d_2]$

The positive quadrant of parameters $[d_1, d_2] \in \mathbb{R}_+^2$ can be divided by curve C_E on two regions, i.e. region of stability and instability. The curve C_E is an envelope of certain hyperbolas $C_i, i \in \mathbb{N}$ illustrated on Figure 1.

2 Problem with a unilateral term

In most of the analytic part we focus on the stationary problem

$$\begin{aligned} d_1 \Delta u + b_{1,1} u + b_{1,2} v + \tau u^- &= 0, \\ d_2 \Delta v + b_{2,1} u + b_{2,2} v &= 0, \end{aligned} \quad \text{on } \Omega,$$
$$u = v = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N, \tag{2}$$

where $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial\Omega$. We distinguish two cases, $meas(\Gamma_D) > 0$ and $meas(\Gamma_D) = 0$, that is mixed and pure Neumann boundary conditions. We study an influence of a unilateral term τu^-

[1] student of the master study program Mathematics, field Mathematics, e-mail: fenclm37@students.zcu.cz

($\tau > 0$) on displacement of critical points of the system (2), i.e. points $[d_1, d_2]$ in the positive quadrant $\mathbb{R}^2_+$ such that there exists a non-trivial of the system (2).

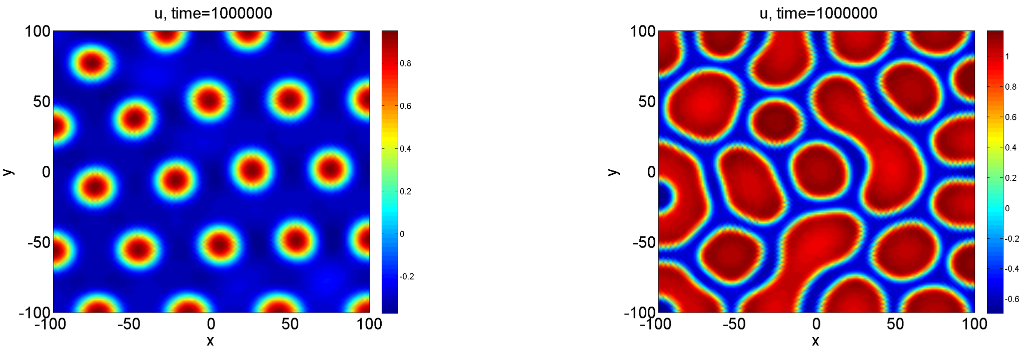
The main result is inspired by article about quasi-variational inequalities by Kučera (1997). In case of pure Neumann boundary conditions we proved that for every compact part of the envelope C_E there exist some ε-neighbourhood, where there are no critical points of the system (2). In case of mixed boundary conditions this statement holds for every compact part of $\overline{C_E \setminus C_1}$ (i.e. the envelope without the first hyperbola, see Figure 1).

3 Numerical experiments

We continue work of Vejchodský et al. (2015), who used model

$$\begin{aligned} \frac{du}{dt} &= D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2, \\ \frac{dv}{dt} &= \delta\Delta v - \alpha u + \beta v + r_2 uv + \alpha r_3 uv^2, \end{aligned} \tag{3}$$

for numerical experiments with unilateral terms of type τv^- (and its modifications) added to the second equation. The first goal is to study possible shapes of irregular patterns disturbed by unilateral terms. The second goal is to find maximal values of the parameter $D = \frac{d_1}{d_2}$, such that the model still generates patterns. We tested mostly the unilateral source terms with saturation. These terms lead to larger maximal value of the parameter D and some new shapes of the patterns. We also experimented with the unilateral term τu^- in the first equation. Examples of a regular pattern and an irregular pattern are illustrated on Figure 2. The pattern on Figure $2b$ is a product of the experiment with the unilateral term $\frac{\tau (v^-)^2}{1+\varepsilon (v^-)^2}$, the pattern on Figure $2a$ is a product of the experiment without any unilateral terms.



(a) Regular pattern **(b)** Irregular pattern

Figure 2: Comparison of the regular and the irregular pattern

References

Kučera M., 1997. Reaction-diffusion systems: Stabilizing effect of conditions described by quasivariational inequalities, *Czechoslovak Math. J.* 47 (122), 1997, p. 469-486.

Vejchodský T., Jaroš F., Kučera M., Rybář V. 2015. Unilateral regulation breaks regularity of Turing patterns, Preprint Math. Inst. ASCR No. 9-2015.

Turing A.M., 1952. Chemical basis of morphogenesis, *Philosophical Transactions of the Royal Society of London*. Series B, Biological Sciences, Vol.237, No. 641. (Aug. 14, 1952), pp. 37-72.