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Efficient Distribution of Investment Capital

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Abstract. Kelly showed in his well-known paper that if a result of a bet or an investment is uncertain, it is not advisable to bet or invest the whole capital as this leads, with certainty, to bankruptcy. Instead of investing the whole capital Kelly proposed to invest a fraction of the capital. More on a proportional gambling, also known as Kelly gambling, was investigated by Cover and Thomas. Our paper uses principles of a log-optimal portfolio from both sources and an approximation of the main criteria is used instead. Doing this allows us effective statistical inference.
Usual procedure is to maximize expected value of the logarithm of the capital after an investment. The obtained solution is not comfortable for use in real-life situations; therefore, we propose alternative approach where the logarithm is substituted by the second-order Taylor polynomial. Same as in the case of a log-optimal portfolio we can obtain trivial solution, i.e. to invest all or nothing but usually a fraction of the capital will be invested. This fraction is based on simple characteristics that can be easily estimated from existing data: expected value and variance.

Keywords: Investment, optimal capital distribution, Kelly gambling, proportional gambling, log-optimal portfolio, approximation, Taylor polynomial.

JEL Classification: G110
AMS Classification: 91G10

1 Introduction

It is well known that to invest the whole capital is risky and therefore an investor wants to invest only some part of his capital. This paper answers the question how to find an optimal part of the capital that is invested. The findings obtained by Kelly [1] and Cover and Thomas [3] are used in the second section to find a log-optimal portfolio and an approximation of the main criteria is suggested to ensure that characteristics necessary to determine the optimal part of the capital can be estimated easily enough.

The theory is presented in the second section of this paper where all possible situations determining the optimal part of the invested capital are investigated. Simple rules that can be used to decide what is the optimal invested part of the capital are formulated in the third section of this paper. Two examples – test case and real-life example – are presented in the fourth section with aim to demonstrate usage of obtained theoretical results. The fifth section summarizes the paper and suggests the future work.

2 Optimizing Investment

Let us consider a recurring investment with an initial capital F_0 in an asset A. In the first round a part $s_1, s_1 \in [0,1]$, of the initial capital is invested in the asset A and the rest is left uninvested, i.e. without appreciation. $Z, Z > 0$ will denote a relative part of the original invested capital that is obtained after one round of investment ($Z > 1$ means that the initial invested capital is increased and $Z < 1$ means that the initial invested capital is decreased). Z is a random variable described by a probability density function $f_Z(x)$. Further, F_i will denote the capital after the ith round of an investment, $s_i, s_i \in [0,1]$ will denote a relative part of the capital invested in the ith round of an investment and Z_i will denote the relative part of the original invested capital that is obtained in the ith round of investment.

The capital after the first round of an investment is

$$F_1 = F_0 \cdot (1 - s_1) + F_0 \cdot s_1 Z_1 = F_0[(1 - s_1) + s_1 Z_1],\ \ 0 \le s_1 \le 1 \tag{1}$$

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the capital after the second round of an investment is

$$F_2 = F_1 \cdot (1 - s_2) + F_1 \cdot s_2 Z_2 = F_0[(1 - s_1) + s_1 Z_1] \cdot [(1 - s_2) + s_2 Z_2], \ \ 0 \le s_i \le 1 \tag{2}$$

and the capital after the nth round of an investment is

$$F_n = F_0 \prod_{i=1}^{n} [(1 - s_i) + s_i Z_i] = F_0 \prod_{i=1}^{n} [1 + s_i(Z_i - 1)], \ \ 0 \le s_i \le 1. \tag{3}$$

In the following parts the logarithm of (3) will be used. The reason for this modification is a computational complexity (it is more convenient to work with summation obtained by logarithm) and the fact that logarithm is a continuous strictly increasing function and therefore this change will have no effect on the position of the maxima. Logarithm of (3) is

$$\ln F_n = \ln F_0 + \sum_{i=1}^{n} \ln[1 + s_i(Z_i - 1)], \ \ 0 \le s_i \le 1. \tag{4}$$

Both F_n and $\ln F_n$ are random variables. The expected value can be used to convert (4) into a deterministic form given by

$$E(\ln F_n) = \ln F_0 + \sum_{i=1}^{n} E(\ln[1 + s_i(Z_i - 1)]), \ \ 0 \le s_i \le 1. \tag{5}$$

Logarithm in (5) provides a condition that $1 + s_i(Z_i - 1) > 0$, i.e. $Z_i > 0, i = 1,2, \dots, n$ but this is the assumption made in the previous paragraphs and therefore we can consider it as fulfilled. Now, it is possible to use the classical procedure to maximize (5), i.e. using partial derivatives and investigation of the interval endpoints. The first and second derivatives of (5) are

$$\frac{\partial}{\partial s_i} E(\ln F_n) = E\left(\frac{\partial}{\partial s_i} \ln[1 + s_i(Z_i - 1)]\right) = E\left(\frac{Z_i - 1}{1 + s_i(Z_i - 1)}\right), \tag{6}$$

$$\frac{\partial^2}{\partial s_i^2} E(ln\, F_n) = -E\left(\left(\frac{Z_i - 1}{1 + s_i(Z_i - 1)}\right)^2\right) \le 0. \tag{7}$$

It is clear that the decision made in the previous rounds of an investment will have no effect on the current round of investment and the current round will have no effect on the following rounds (from the point of choosing an optimal part of a capital to invest). Next, we are looking for the optimal fraction of a capital that will be invested and an amount of the current capital is therefore insignificant for the analysis. It can be seen from (5) that to maximize the sum we have to maximize each single term of the sum, i.e. to maximize outcome in each round of investment individually. Therefore, the following analysis will be restricted only on the outcome from one round of investment.

Solving (6) for s_i can be problematic in a practical task and therefore, in some parts, an approximation of the logarithm by the second-order Taylor polynomial at $x = 1$ will be used instead, i.e.

$$\ln(x) \approx g(x) = (x - 1) - \frac{1}{2}(x - 1)^2. \tag{8}$$

We can use following information to find a point s_i, $s_i \in [0,1]$, where (5) reaches the global maximum (we recall that $Z_i > 0$):

- The function given by (5) is continuous in $s_i, s_i \in [0,1]$;
- The first derivative given by (6) exists for each $s_i \in [0,1]$;
- The second derivative given by (7) is nonpositive, i.e. the $E(\ln F_n)$ as the function of s_i is concave.

Now, the situation will be examined in the following subsections for several possible cases that arise from the previous list.

2.1 Extreme in Endpoints

If the derivative in (6) is positive for each s_i in $[0,1]$, then $E(\ln F_n)$ as the function of s_i is decreasing. The maximum is therefore achieved in the point $s_i = 0$.

If we use the information that the function is continuous, the first derivative exists for each s_i and the function of s_i is concave then we are able to make the same decision from a simple situation where only the endpoints of interval for s_i are investigated. The situation described above corresponds to the situation where (6) is nonpositive for $s_i = 0$ and negative for $s_i = 1$. This gives us the following conditions which have to be fulfilled simultaneously.

$$\frac{\partial}{\partial s_i} E(\ln F_n)\bigg|_{s_i=0} = E(Z_i - 1) \leq 0 \tag{9}$$

$$\frac{\partial}{\partial s_i} E(\ln F_n)\bigg|_{s_i=1} = E\left(1 - \frac{1}{Z_i}\right) < 0 \tag{10}$$

In other words, the conditions (9) and (10) ensure that the function with given properties is decreasing in s_i and therefore the maximum is obtained for $s_i = 0$.

Similar procedure leads to the conclusion that the maximum of (5) is obtained for $s_i = 1$ when the following conditions are fulfilled simultaneously.

$$\frac{\partial}{\partial s_i} E(\ln F_n)\bigg|_{s_i=0} = E(Z_i - 1) > 0 \tag{11}$$

$$\frac{\partial}{\partial s_i} E(\ln F_n)\bigg|_{s_i=1} = E\left(1 - \frac{1}{Z_i}\right) \geq 0 \tag{12}$$

Here, the conditions (11) and (12) ensure that the function with given properties is increasing in s_i and therefore its maximum is obtained for $s_i = 1$.

In a special case the function is constant if both conditions, (11) and (12), are equal to zero. This also means that the derivative in (6) is zero on the whole interval. Using $s_i \in [0,1]$ in (6) we obtain that the only way how to fulfill (6) is that Z_i have to be deterministic and it is equal to 1 (however, this situation is almost impossible in a real-life case). Therefore, the capital does not change after the investment and this result is independent of s_i. Thus, any value of s_i can be used, e.g. $s_i = 0$. This value is reasonable because we know for sure that the result will be exactly the same as the invested capital and therefore we choose to safe our time and not to invest.

2.2 Extreme Inside Interval

The extreme is inside the considered interval if the following conditions are fulfilled simultaneously.

$$\frac{\partial}{\partial s_i} E(\ln F_n)\bigg|_{s_i=0} = E(Z_i - 1) > 0 \tag{13}$$

$$\frac{\partial}{\partial s_i} E(\ln F_n)\bigg|_{s_i=1} = E\left(1 - \frac{1}{Z_i}\right) < 0 \tag{14}$$

Remark 1. The rest situations that were not yet examined are not achievable because the function is convex, i.e. it is not possible to obtain these combinations of conditions:

- $E(Z_i - 1) \leq 0$ and $E\left(1 - \frac{1}{Z_i}\right) > 0$;
- $E(Z_i - 1) < 0$ and $E\left(1 - \frac{1}{Z_i}\right) = 0$.

For the situation where (13) and (14) are fulfilled the approximation of (5) with (8) is used. This results in

$$E(\ln F_n) \approx \ln F_0 + \sum_{i=1}^{n} E(g[1 + s_i(Z_i - 1)]), \quad 0 \leq s_i \leq 1. \tag{15}$$

The same approximation is used for (6) which is set to zero and solved

$$E\left(\frac{\partial}{\partial s_i} g[1 + s_i(Z_i - 1)]\right) = E\left(\frac{\partial}{\partial s_i}\left(s_i(Z_i - 1) - \frac{1}{2}s_i^2(Z_i - 1)^2\right)\right) = 0. \tag{16}$$

Using some algebra, it is obtained form (16)

$$E(Z_i - 1) - s_i E((Z_i - 1)^2) = 0 \tag{17}$$

and the solution is

$$s_i^{opt} = \frac{E(Z_i - 1)}{E((Z_i - 1)^2)}. \tag{18}$$

Next, the approximation of (7) gives

$$\frac{\partial^2}{\partial s_i^2} E(g[1 + s_i(Z_i - 1)]) = -E((Z_i - 1)^2) < 0. \tag{19}$$

We do not consider (19) to be equal to 0 as this would mean that $Z_i = 1$ with probability 1, i.e. it would mean that Z_i is not a random variable and after each round of investment we would have the same amount of the capital as before the investment. This result means that (19) guaranties that s_i^{opt} in (18) is the point where the maximum is realized. Using condition $s_i \in [0,1]$ and (18) we obtain

$$0 \leq \frac{E(Z_i - 1)}{E((Z_i - 1)^2)} \leq 1. \tag{20}$$

In the case where $E(Z_i - 1) > E((Z_i - 1)^2)$ we set $s_i^{opt} = 1$. Obviously, if $E(Z_i - 1) < 0$ then it is not reasonable to invest and we set $s_i^{opt} = 0$.

Obtaining a point estimate of $E(Z_i - 1)$ and $E((Z_i - 1)^2)$ in (18) is a standard statistical task whereas to solve (6) can be problematic.

3 Decision Summary

Using the theory described in the section 2 it is possible to formulate simple rules for the decision making. This decision is affected by random variable Z_i and it is necessary to know (or to estimate) its following characteristics:

- $E(Z_i - 1)$ which represents expected value of a net income in one round of an investment (before taxes);
- $E((Z_i - 1)^2)$ which represents variability of a net income in one round of an investment (before taxes);
- $E\left(1 - \frac{1}{Z_i}\right) = E\left(\frac{Z_i - 1}{Z_i}\right)$ which represents the expected value of a relative net income in one round of an investment (before taxes);

Decision is made by the rules in Table 1 (all rules were derived in the previous section).

Conditions	**Decision**
$E(Z_i - 1) \leq 0 \wedge E\left(1 - \frac{1}{Z_i}\right) < 0$	$s_i^{opt} = 0$, i.e. not to invest
$E(Z_i - 1) > 0 \wedge E\left(1 - \frac{1}{Z_i}\right) \geq 0$	$s_i^{opt} = 1$, i.e. invest the whole capital
$E(Z_i - 1) = 0 \wedge E\left(1 - \frac{1}{Z_i}\right) = 0$	$s_i^{opt} \in [0,1]$; as $Z_i = 1$ for all $s_i^{opt} \in [0,1]$; use $s_i^{opt} = 0$
$E(Z_i - 1) > 0 \wedge E\left(1 - \frac{1}{Z_i}\right) < 0$	use Table 2 for a decision, i.e. decision based on the approximation of the logarithm

Table 1 Decision making rules

Remark 2. We recall that the following situations are not achievable because the function is convex:

- $E(Z_i - 1) \leq 0$ and $E\left(1 - \frac{1}{Z_i}\right) > 0$;
- $E(Z_i - 1) < 0$ and $E\left(1 - \frac{1}{Z_i}\right) = 0$.

Table 2 contains decision for the last situation given in Table 1 ($E(Z_i - 1) > 0 \wedge E\left(1 - \frac{1}{Z_i}\right) < 0$).

Conditions	Decision
$0 \leq \frac{E(Z_i - 1)}{E((Z_i - 1)^2)} \leq 1$	$s_i^{opt} = \frac{E(Z_i-1)}{E((Z_i-1)^2)}$, condition guaranties that $s_i^{opt} \in [0,1]$
$E(Z_i - 1) > E((Z_i - 1)^2)$	$s_i^{opt} = 1$, i.e. invest the whole capital
$E(Z_i - 1) < 0$	$s_i^{opt} = 0$, i.e. not to invest

Table 2 Decision making rules in the case of an approximation

Standard statistical procedures can be used to test the rules given in Table 1 and Table 2 when a large enough random sample is available. This is usually not a problem when we are dealing with a financial data set that tends to be large.

We recall that strategy used in a given round of an investment is independent of strategies in the previous rounds of an investment. Therefore, it is also possible to use information on changing distribution of $f_Z(x)$ (by change it is meant change in time). This means that an asset used for the investment can be completely different from an asset used in the previous rounds. The presented procedure can be easily used and generalized for an investment of a money or a situation with more assets. Using more assets will require usage of nonlinear programing.

A demonstration that the objective function given by (5) and its approximation introduced in (15) are similar and flat around the point of maximum will be shown in the following section. The fact that functions are flat around the point of maximum means that the obtained solution is robust and it can be used in a real-life investment where we have to deal with problems which are extremely hard to describe analytically, i.e. a delay between time of decision and time of investment (the price can differ from the price used in analysis); different time between listing days; a difference between function (5) and its approximation; a transaction costs and taxes.

4 Results Presentation

Two examples are shown in this section, the first one where generated values are used and the second one where real data set is used.

4.1 Test Case

We will use model situation to present course of the criterion (5) for one round of investment where the initial capital is set to one, i.e. $F_0 = 1$, this results in

$$E(\ln F_1) = E(\ln[1 + s_1(Z_1 - 1)]), \;\; 0 \leq s_1 \leq 1 \tag{21}$$

and the approximation of the criterion based on (8) results in

$$E(\ln F_1) \approx E\left(s_1(Z_1 - 1) - \frac{1}{2}s_1^2(Z_1 - 1)^2\right), \;\; 0 \leq s_1 \leq 1. \tag{22}$$

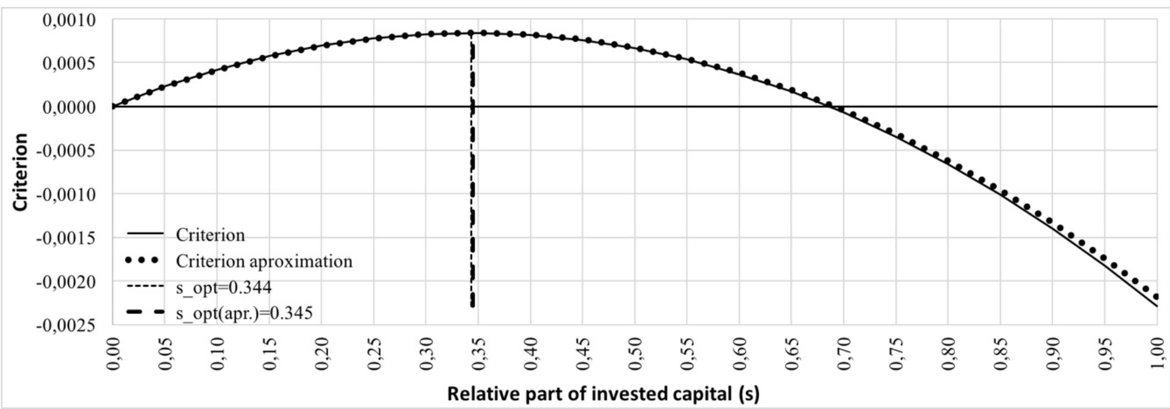

Figure 1 Demonstration of results – test case

Figure 1 shows comparison of (21) and (22). Optimal relative part of invested capital is 0.344 when the original criterion is used and 0.345 when the approximation is used (difference 0.409%). The value of the original criterion function for the optimal relative part of invested capital is $8.3957 \cdot 10^{-4}$ and the value of the approximation of the criterion function for the optimal relative part of invested capital is $8.3958 \cdot 10^{-4}$ (difference -0.002%). The results suggest that the method is robust and provides similar results even when the approximation is used.

4.2 Real Data

In this part, the application of the theory from section 3 is used for an investment to Amundi Funds Equity US Relative Value (CZK), ISIN LU0568606221. Historical prices of this fund are available at [2]. We test the time frame from 2. 1. 2014 to 19. 4. 2016. The statistical inference is made each day using daily prices in the last year (first estimation is made from historical data in 2013 and investing starts on 2nd January 2014). This produces estimation of rules used in Table 1 and Table 2. According to the obtained value of s^{opt} (relative part of the invested capital) we adjust our position in this fund each day. Comparison of this strategy with *strategy buy-and-hold*, i.e. where the whole capital is invested to the fund with no change during the whole interval, is shown in Figure 2.

Remark 3. We do not use any limitation in this example, i.e. we assume that the fund is frictionless and that there are no taxes and transaction costs. On the other hand, we use a delay between time of decision and time of investment, so the fund is not bought for the price that were used for the decision but it is bought for the price that is available at the next day. We also assume that money left aside are not subject to interest.

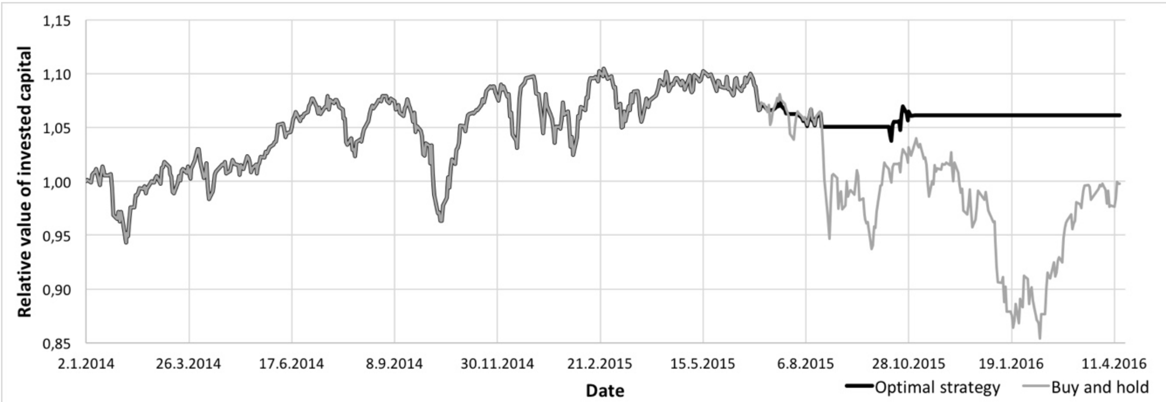

Figure 2 Demonstration of results – real example

As can be seen in Figure 2, the strategy started with $s^{opt} = 1$ and maintained this value for more than one and half year, i.e. the same results as the *strategy buy-and-hold*. Courses of both strategies went to different directions in mid-2015 and the optimal strategy produced better results.

5 Conclusion

This paper showed how to allocate investor's capital between money and a risky asset so that the log-optimal strategy is obtained. To achieve easy enough procedure an approximation of the main criteria was used. This approximation has a little impact on the solution as was presented in the fourth section of this paper and it is based on easily estimated characteristics.

In the fourth section, a test case and a real-life example were presented. In the test case, it was shown that the main criteria and its approximation has small difference. The real-life example demonstrated that the presented procedure using approximation can produce better results than *buy-and-hold strategy*.

Future works will be focused on investigation of a delay between time of a decision and time of an investment; developing simple trading strategies that will allow to avoid too frequent trading with high transaction costs. More work has to be done for a testing real-life data, i.e. more assets has to be evaluated to strongly support presented procedure.

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