The Dynamic Behaviour of Wonderland Population–Development–Environment Model

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Abstract. The Wonderland Population-Development-Environment Model (PDE) allows to study the interactions between the economic, demographic and environment factors of an idealized world, thereby enabling them to obtain insights transferable to the real world. This model was first introduced by Sanderson in 1994 and now there are several modification of this model. From a mathematical perspective, the PDE model is a system of non-linear differential equations characterized by slow-fast dynamics. This means that some of the system variables vary much faster than others. The existence of speed of dynamical in variables in model implies problems with numerical solution of models. This article concentrates on the numerical solutions of model and on the visualization dynamical behaviour of a four dimensional continuous dynamical system, the Wonderland model. We analyse the behaviour of model for selected part of the parametric space and we showed that the system of four differential equations Wonderland model can generate behaviour typical for chaotic dynamic systems.

Keywords: Dynamical System, Population–Development–Environment Model, Slow–Fast Dynamics.

JEL classification: C63
AMS classification: 34C60, 91B55

1 Introduction

Many models focused on relations relationships between economic factors, demographic and environmental factors have been developed to address climate change policy and these models are used for analysing several scenarios of future development of population from a critical (nightmare) scenario to a dream scenario. Between these two extreme limitations is usually model of the sustainable development expected. Sustainable development, although a widely used phrase and idea, has many different meanings and therefore provokes many different responses. In broad terms, the concept of sustainable development is an attempt to combine growing concerns about a range of environmental issues with socio-economic issues.

Today's economic and civilization development does not imply, in the opinion of many experts, sustainable concept for future. The economic expansion leads to population growth. Population in the world is currently (2016) growing at a rate of around 1.13% per year. The current average population change is estimated at around 80 million per year. Annual growth rate reached its peak in the late 1960s, when it was at 2% and above. The rate of increase has therefore almost halved since its peak of 2.19 percent, which was reached in 1963. The annual growth rate is currently declining and is projected to continue to decline in the coming years. Currently, it is estimated that it will become less than 1% by 2020 and less than 0.5% by 2050. This means that world population will continue to grow in the 21st century, but at a slower rate compared to the recent past. World population has doubled (100% increase) in 40 years from 1959 (3 billion) to 1999 (6 billion). It is now estimated that it will take a further 39 years to increase by another 50%, to become 9 billion by 2038. The latest United Nations projections *http://esa.un.org/unpd/wpp/* indicate that world population will reach 10 billion persons in the year 2056 (six years earlier than previously estimated).

The endless population growth creates more pressure on economic development and the related increasing of economic production puts a strain on the ability of the natural environment to absorb the

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high level of pollutants that are created as a part of this economic growth. Therefore, solutions need to be found so that the economies of the world can continue to grow, but not at the expense of the public good. In the world of economics the amount of environmental quality must be considered as limited in supply and therefore is treated as a scarce resource. This is a resource to be protected and the only real efficient way to do it in a market economy is to look at the overall situation of pollution from a benefit-cost perspective.

This article focuses on the numerical solutions and the visualisation dynamical behaviour of the Wonderland model, which can be regarded as one of the simple dynamic models of geographic, economic and environmental interactions. The special feature of Wonderland model is the fact that not all system variables evolve with the same velocity. The velocity varies by two magnitudes and this slow fast dynamics makes solving difficult. The resulting mixture of slow and fast dynamics can lead to unpredictable, catastrophic transition even if all functions in model are deterministic, i.e. no stochastic force are introduced.

2 Sanderson Wonderland model

The first version of Wonderland model was published by Warren Sanderson as a part of IIASA study - The International Institute for Applied System Analysis in 1994 [8]. The original model was written in discrete time. Next, the model has been reformulated in continuous variables and this system has been used to investigate sustainability of economic growth [5],[2], [3] and as a benchmark problem for visualization techniques for higher dimensional dynamical systems [9].

In this parer the continuous version of Wonderland model of economic, demographic and environmental interaction is used. The dynamics in Wonderland is determined by four state variables:

- $x(t)$ - population, demographic variable;
- $y(t)$ - per capital output, economic performance;
- $p(t)$ - pollution per unit of output;
- $z(t)$ - quality of environment.

The state variables for population x and per capita output y can assume all non-negative real values $x, y \in [0, \infty)$, while the stock of natural capital z and the pollution per unit of output p are confined to the unit interval $z, p \in [0, 1]$. The variable z can be interpreted as a level of natural capital. If the natural capital is not polluted at all it takes on the value $z \approx 1$. On the other extreme, when the environmental is so polluted, that it produces the maximum possible damage to human health and to the economy, $z \approx 0$. The value of $p =\approx 1$ represent situation when there is maximal pollution per unit of output and on the other hand $p \approx 0$ implies no pollution per unit of output.

These four state variables evolve according to following set of non-linear, difference equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \cdot n(y, z) \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y \cdot \left[\gamma - (\gamma + \eta)(1 - z)^{\lambda}\right] \tag{2}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \nu z(1 - z) \left[\mathrm{e}^{\omega(g(z) - f(x,y,z,p)) - 1}\right] \tag{3}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\chi p \tag{4}$$

where

$c(y,z) = \varphi(1-z)^{\mu} y$	is pollution control,
$\overline{y}(y,z) = y - c(y,z)$	is net per capital output,
$f(x,y,z,p) = pxy - \kappa \left[\frac{e^{\sigma c(y,z)x}}{1+e^{\sigma c(y,z)x}} - \frac{1}{2} \right]$	is pollution flow,
$n(y,z) = b(y,z) - d(y,z)$	is population growth rate, that is calculated as difference between crude birth rate $b(y,z)$ and crude death rate $b(y,z)$.

The complex dynamical model includes 20 parameters, which can be grouped as follows

demographic:	β_1, β_2, β, δ_1, δ_2, δ_3, α, ϑ,
economy:	γ, η, λ,
environmental:	κ, σ, δ, ρ, ω, ν,
environmental policy:	φ, μ, χ.

2.1 Population $\frac{dx}{dt} = x \cdot n(y,z)$

The first differential equation describes population growth which depends endogenously only on per capital output y and the level of natural capital z. In eq. (1), the population growth rate, $n(y,z)$, can be written as the difference between the crude birth rate, $b(y,z)$ (number of births per 1000 population per time at a given time step) and the crude death rate, $d(y,z)$ (number of deaths per 1000 population per time at a given time step):

$$b(y,z) = b(\overline{y}) = \beta_1 \left[\beta_2 - \frac{e^{\beta \overline{y}}}{1+e^{\beta \overline{y}}} \right] \tag{5}$$

$$d(y,z) = d(\overline{y},z) = \delta_1 \left[\delta_2 - \frac{e^{\alpha \overline{y}}}{1+e^{\alpha \overline{y}}} \right] \cdot \left[1 + \delta_3 (1-z)^{\vartheta} \right] \tag{6}$$

The parameters β_1, β_2, β govern the birth rate, while the parameters δ_1, δ_2, δ_3, α and ϑ govern the death rate. At eq. (5) and (6) it can be seen how both the birth rate and death rate decrease with increases in the per capita output, y. Furthermore, in eq. (6), the death rate is seen to increase when the environment deteriorates, i.e., when z decreases. These effects are in line with recent studies relating population growth with industrial output [1]. The maximum crude birth rate is realised for zero net per capital output $b(\overline{y} = 0) = \beta_1(\beta_2 - \frac{1}{2})$ and minimal crude birth rate is the limit situation for unlimited grow of output: $\lim_{\overline{y} \to \infty} b(\overline{y}) = \beta_1(\beta_2 - 1)$.

In the literature [9] there exists also other type of function, that describes the population's growth in form $b_2(\overline{y},z) = \beta_1 \left[\beta_2 - \frac{1}{2} \left(1 + \frac{\beta \overline{y}}{1+\beta \overline{y}} \right) \right]$. The limit behaviour of this alternative function is the same as for the function previous formulated $b(\overline{y})$.

2.2 Product $\frac{dy}{dt} = y \cdot \left[\gamma - (\gamma + \eta)(1-z)^{\lambda} \right]$

Net per capital output $\overline{y}(y,z)$ is defined as per capital output y minus expenditures on pollution control $c(y,z)$. Pollution control expenditures $c(y,z) = \varphi(1-z)^{\mu} y$ depend on how polluted environmental is and not on the current flow emissions. The availability of natural capital also influence the growth rate of economy as indicated by equation (2). When environmental is totally polluted, i.e. $z \approx 0$, per capital output shrink at the rate $-\eta$, while if environmental is not polluted at all, i.e $z \approx 1$ per capital output increases at rate γ.

2.3 Quality of environmental (natural capital) $\frac{dz}{dt} = \nu z(1-z) \left[e^{\omega(g(z) - f(x,y,z,p)) - 1} \right]$

The growth of quality of environmental is depended on all state variables. The first part of equation (3) assumes the logistic model for quality of environmental. The speed at which natural capital regenerates is indicated by term $\nu \left[e^{\omega(g(z) - f(x,y,z,p)) - 1} \right]$.

This term depends positively on the self-cleaning ability of nature, which is described by function $g(z) = \frac{\delta}{\omega} z^{\rho}$, and this term depends negatively on the amount of pollution, which is modelled function

$$f(x, y, z, p) \quad = \quad pxy - \kappa \left[\frac{e^{\sigma c(y,z)x}}{1 + e^{\sigma c(y,z)x}} \right] \tag{7}$$

The difference between two flows $g(z) - f(x, y, z, p)$ is the net effect of natural and human forces on environmental. As we can see the model of dynamic behaviour quality of environmental is based on the assumption that the interaction of net effect on environmental is non-linear.

2.4 The pollution flow $\frac{\mathrm{d}p}{\mathrm{d}t} = -\chi p$

Pollution per units of output p exponential falls towards to zero as technology improves and environmental consciousness growth with rate χ.

3 Numerical experiment

The numerical experiments were inspirited by the origin Sanderson's work [8] and special attention was focused on analysis the behaviour the solving of differential system equations for different values of parameters χ, φ and κ. These parameters involve to the governing the expenditures for pollution abatement. The parameter φ from function $c(y, z)$ influences how expenditure increase.

3.1 Slow–fast dynamics

The numerical calculation have been performed using the Matlab R2013b. For numerical solving a stiff solver was used, that is based on an implicit Runge-Kutta formula for solving stiff differential equations with a trapezoidal rule step as its first stage and a backward differentiation formula of order two as its second stage.

The Wonderland model is an example of set differential equations exhibits slow-fast behaviour. One system variable (in this model z) varies much faster then the others (x, y and p). The analyse this phenomenon, we can rewrite the system using a scaling factor ε with $0 < \varepsilon \ll 1$. The scaling factor is called the perturbation parameter in singular perturbation theory [4]. The application of methods of perturbation theory enables us to separate slow and fast components of the system and to analyse both components separately. The reduced system captures the slow dynamics reformulate the equation (3) to form $0 = \nu z(1 - z) \left[e^{\omega(g(z) - f(x,y,z,p)) - 1} \right]$. The layer system captures the fast dynamics, where right sides in equations (1),(2) and (4) are zero and dynamic of system reduced only on the dynamic equation (3) for pollution per unit of output.

3.2 Visualization results and dynamics

Visualisation the behaviour of dynamical systems can provide us a deeper understanding of underlying dynamics. For analysing the model we used the following value of parameters: **demographics factors** $\beta_1 = 0.04$, $\beta_2 = 1.375$, $\beta = 0.16$, $\delta_1 = 0.01$, $\delta_2 = 2.5$, $\delta_3 = 4$, $\alpha = 0.18$, $\vartheta = 15$, **economy factors** $\gamma = 0.02$, $\eta = 0.1$, $\lambda = 2$, **environmental** $\kappa \in (1, 100)$, $\sigma = 0.02$, $\delta = 1$, $\rho = 2$, $\omega = 0.1$, $\nu = 1$ and **environmental police** $\varphi \in (0, 1)$, $\mu = 2$, $\chi \in (0.01, 0.02)$.

Our analysis has concentrated on changing three parameters χ, φ and κ while holding all the other parameters fixed. The parameter χ is one of the most responsible parameter for system, this one governs the economic decoupling rate, i.e. the rate at which technological innovation reduce the pollution flow per unit of output. Parameters φ and κ describe policy planning, the parameter φ can be interpreted as a rate at which expenditures increase, so the net per capital output is calculated as $\overline{y}(y, z) = y - \varphi(1 - z)^{\mu} y$. Since we require $c < y$, this limit $\varphi < 1$. The parameter κ is included in pollution function $f(x, y, z)$ (7) and determines the effectiveness of expenditures.

The future of Wonderland economy modelled by our system of differential equations is demonstrated for three sets of parameters that describe the three basic scenarios:

(A) Economists' dream scenario $\chi = 0.02$, $\kappa = 20$ and $\varphi = 0.5$. In this case, the per capital output increases over time, population converges toward stationary level of zero growth rate and the pollution flows steadily decreases.

(B) Environmentalists' nightmare scenario $\chi = 0.01$, $\kappa = 20$ and $\varphi = 0.5$. In this case, when parameter χ is reduced from 0.02 to 0.01, the system don't obeys the criteria of sustainability.

(C) Sustainable development scenario $\chi = 0.01$, $\kappa = 70$ and $\varphi = 0.5$ describes the situation when the progress of technology can help to sustainable economic increase.

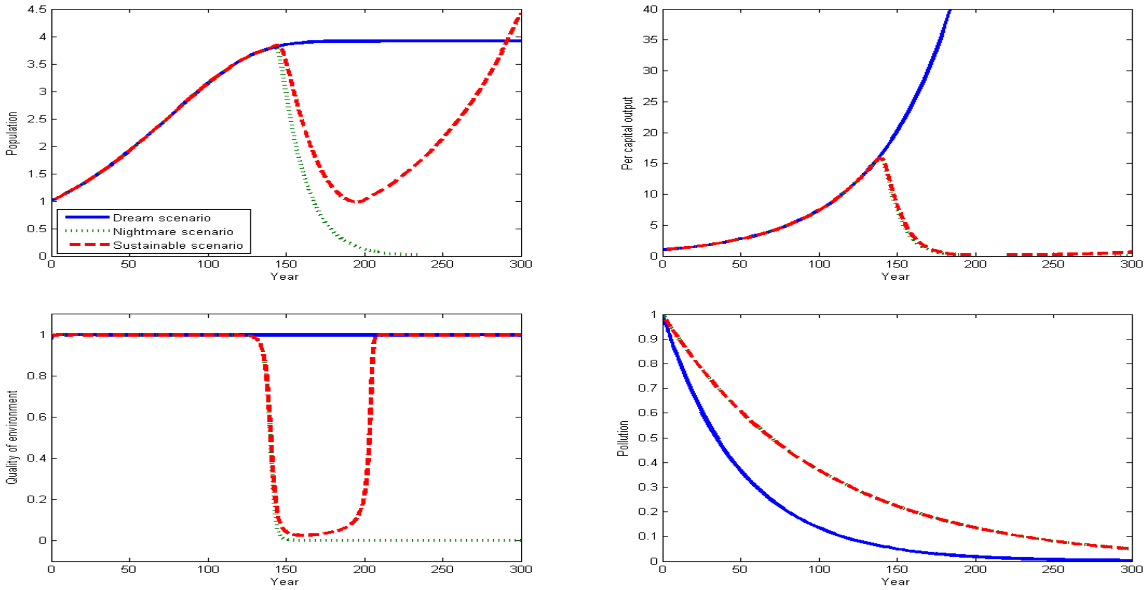

Figure 1 The population $x(t)$, per capital output $y(t)$, quality of environment $z(t)$ and pollution $p(t)$ for scenarios **(A)**, **(B)** and **(C)**

The other numerical experiments has been focused on the stability of the solution. We analyse the behaviour of the solution system for parameter κ from interval 40 to 100 and parameter φ from interval $(0.2, 0.5)$. The modelled value of per capital output at the time $t = 300$ is in the pictures bellow as a scaled coloured graph. There is the region of nightmare scenarios, where per capital output tends for $t \to 0$ to zero (black color) and the region of dream scenarios where per capital output tends to ∞ (white color). As we can see in detail, the border between this two scenarios is not exactly a clear line. This type of behaviour is typical of dynamic system operating in chaotic regime.

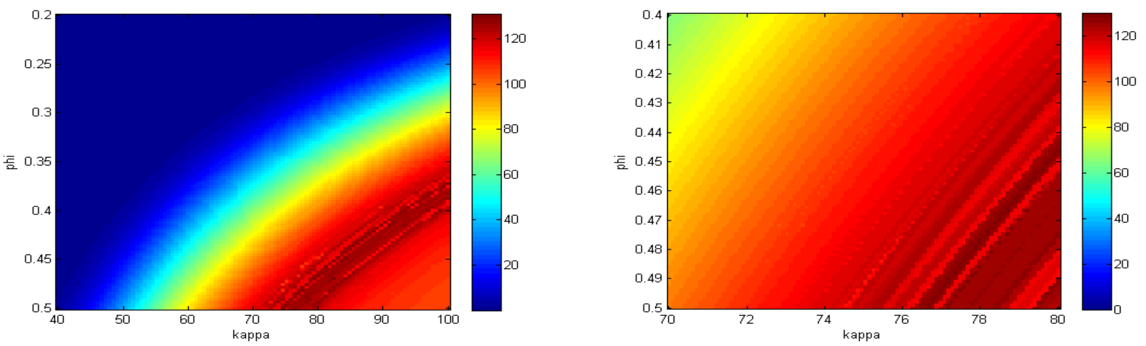

Figure 2 The scaled color graph of per capital output at time $t = 300$ for parametric region $\kappa \in (40, 100) \times \varphi \in (0.2, 0.5)$ (left picture) and detail for parameters $\kappa \in (70, 80) \times \varphi \in (0.4, 0.5)$ (right picture)

4 Conclusion

The behaviour of the four dimensional dynamical system which describe Wonderland Population-Development-Environment Model was presented. The future of the model for three basic scenarios – dream scenario, nightmare scenario and sustainable development scenario – was demonstrated. The behaviour of state variables for several part of parametric space (parameters χ, φ and κ) have been detailed analysed. The detailed comparison of parameters in the Wonderland model with estimation of this parameters based on real data sets are great challenge for our future works.

Of course, Wonderland is a toy model which captures the global features of complete human–environment interaction, but not its a detail. On the other hand though there are no stochastic, exogenous shocks, the future of state variables seems quite unpredictable in some part of parametric space that is typical for chaotic dynamic systems. The next studying of a chaotic behaviour of Wonderland model is the second motivation for our continuing interest in this model.

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