September 10 - 12, 2007
Pilsen, Czech Republic

VARIANT OF THE LEVEL SET METHOD AND ITS APPLICATION TO IMAGE SEGMENTATION AND DENOISING

PROF. ING. LIBOR DĚDEK, CSC.

Abstract: *We show how the piecewise-constant Mumford-Shah segmentation problem can be solved using the level set method. The obtained algorithm can be simultaneously used to denoise, segment, detect edges, and perform active contours. The proposed model is also an extension to the case with more than two segments for piecewise-constant segmentation.*

Key words: *Image Segmentation, Active Contour Model, Level Set Functions, Denoising*

INTRODUCTION

The aim of this paper is to provide basic principles of the Leve Set Method (LSM). LSM has many applications in mathematics, physics, as well as in electrical and electronic engineering. In this paper, we present some results where the LSM is used for the image segmentation, debluring and denoising. Until now, the LSM is mostly developed by mathematicians, as it requires good knowledge of the calculus of variation and numerical solution of partial differential equations. Basic theory as well as many examples of application of LSM are given by professor Sethian in [1]. Practical examples from many branches of mathematics and physics are on his home page [8]. Books [2], [3], and [4] are sources of good information about the method. All three books are written by the excellent mathematicians and they are incomprehensible for technicians. Good source of the latest development about the LSM is on the home page of the Department of Mathematics, University of California (UCLA) [5], where can be found many scientific papers since year 2000.

The intention of the author of this paper is to attract the attention to LSM of those colleagues who are familiar with the above-mentioned mathematics methods. The application of LSM in the Electrical Engineering community is not yet familiar. Until now, the method has been proposed to solve such inverse problems as the Impedance Tomography, Ultrasound tomography and generally all processes governed by the Hamilton-Jacobi equation. However, most applications can be found in the area of image processing. We show in this paper possibility of LSM application to segmentation and denoising of images.

1 LEVEL SETS BASICS

Looking at an image it is usually very easy for a human to see what it represents. For a computer this is not so easy. Before computers "know" what is represented, objects must be first detected. Understanding images is very important in problems like stereo and motion estimation, part recognition or image indexing. The first step in image understanding is the image segmentation, denoising, debluring and registration. Segmentation is the problem of dividing an image into objects or distinguishing objects from the background.

In this paper, we utilize a multi-phase level set formulation and algorithm for the general Mumford-Shah minimization problem [3] in image processing, to compute piecewise-smooth optimal approximations of a noised image.

The accepted model is based on work of T. Chan and L. Vese [6], [7], which is known as Chan – Vese (CV) model.

1.1 Two-phase model

For simplicity, let us proceed with two-dimensional images. Let Γ be a closed curve in $\Omega \subset R^2$. Associated with Γ, we define a level set (LS) function as an initial signed distance function defined by:

$$\phi(x, t=0) = \text{distance}(x, \Gamma), \; x \in \text{interior of } \Gamma,$$
$$\phi(x, t=0) = -\text{distance}(x, \Gamma), \; x \in \text{exterior of } \Gamma. \qquad (1)$$

In this case, Γ is the zero level set (LS) of function ϕ. Once the LS function is defined, we can use it to represent general constant piecewise functions.

Assuming further that image $u(x)$ consists of two parts of different, but constant intensity, one equals to c_1 inside Γ and the second equals c_2 outside Γ, the image intensity u can be approximated as:

$$u = c_1 H(\phi) + c_2 (1 - H(\phi)), \qquad (2)$$

where the Heaviside function $H(\phi)$ is defined by

$$H(\phi) = 1, \qquad \phi > 0, \qquad (3)$$
$$H(\phi) = 0, \qquad \phi \le 0.$$

In order to approximate u, we just need to identify the level set function ϕ and the piecewise constant values c_1, c_2. An example of ϕ defined on the square region 200 x 100 pixels by *meshgrid( )* Matlab function is in Figure 1, LS value is given by (5).

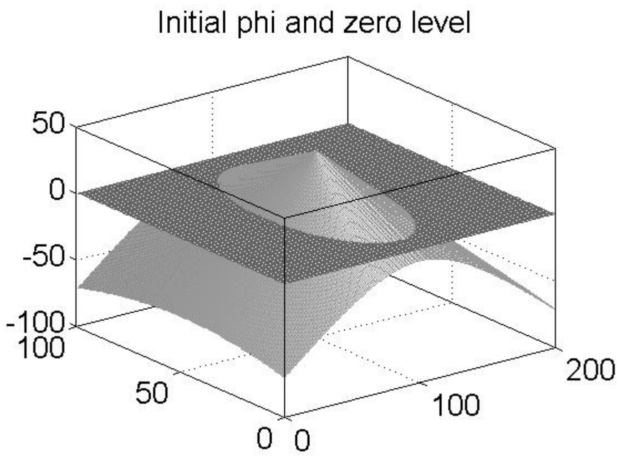

Fig.1: Level set ϕ and level zero

$$\phi(x, t=0) = -\sqrt{(x_1 - 100)^2 + (x_2 - 50)^2} + 30. \qquad (5)$$

Here ϕ represent a cone with vertex at (100, 50), circular zero level Γ is of radius 30. Corresponding Heaviside function $H(x)$ is a cylinder of unit height and of radius 30, but it is intentionally smoothed using, for example,

$$H_\varepsilon(\phi) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{\phi}{\varepsilon}. \qquad (6)$$

Differentiating H we obtain a smoothed Dirac function δ

$$\delta(\phi) = \frac{\varepsilon}{\pi(\phi^2 + \varepsilon^2)}. \qquad (7)$$

The corresponding shape of $H(\phi)$ evaluated from (4) is in Figure 2.

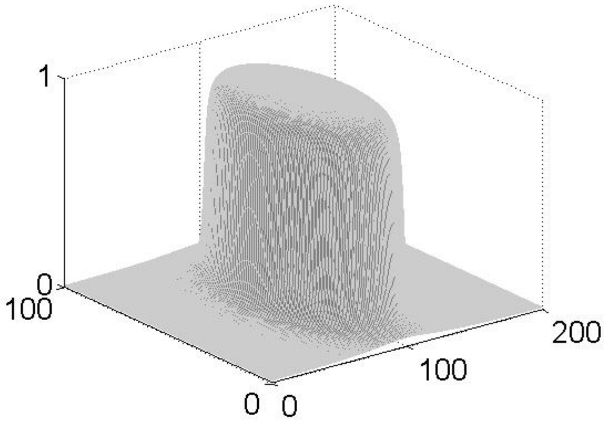

Fig. 2. Smoothed Heaviside function

More complicated level set initial function can be calculated using repeated concatenation of smaller initial H_i. Continuity of the resulting ϕ at least of C^0 must be satisfied to avoid discontinuity. An example of a small ϕ_i concatenated 4x in x_1 direction and then twice in x_2 direction is in Figure 3.

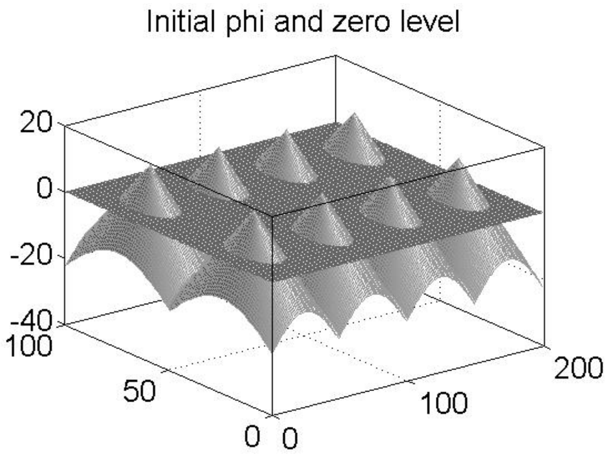

Fig.3 . An example of ϕ with more peaks

1.2 Four phase model

If $u(x)$ consists of more than two constant pieces, we need to use multiple level set functions. For example, two level set functions ϕ_1, ϕ_2 divide Ω into four parts:

$$\begin{aligned} \Omega_1 &= \{x \in \Omega, \quad \phi_1 > 0, \quad \phi_2 > 0\}, \\ \Omega_2 &= \{x \in \Omega, \quad \phi_1 > 0, \quad \phi_2 < 0\}, \\ \Omega_3 &= \{x \in \Omega, \quad \phi_1 < 0, \quad \phi_2 > 0\}, \\ \Omega_4 &= \{x \in \Omega, \quad \phi_1 < 0, \quad \phi_2 < 0\}. \end{aligned} \qquad (8)$$

Then we can express u with possibly up to four subregions of constant $u(x)$ value $c_{11}, c_{10}, c_{01}, c_{00}$.

$$u = c_{11}H(\phi_1)H(\phi_2) + c_{10}H(\phi_1)(1 - H(\phi_2)) + \\ c_{01}(1 - H(\phi_1))H(\phi_2) + c_{00}(1 - H(\phi_1))(1 - H(\phi_2)). \qquad (9)$$

Example of ϕ_1, ϕ_2 dividing Ω in four subregions is in Figure 4.

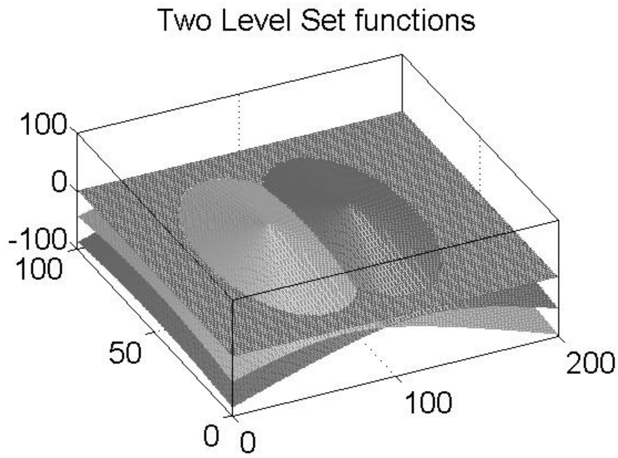

Fig. 4. Two Level Set functions with four subregions

2 IMAGE SEGMENTATION AND DENOISING TWO PHASE CASE

For some applications, we need to find the location of discontinuities for the intensity values of a given digital image and this process is often called the image segmentation.

2.1 Theory

Using one level set function, the minimization functional for two constant values of u introduced in Chan and Vese [6], [7] is

$$F_2(c_1, c_2, \phi) = \int_\Omega |u(x) - c_1|^2 H(\phi)dx + \\ \int_\Omega |u(x) - c_2|^2 (1 - H(\phi))dx + \nu \int_\Omega |\nabla H(\phi)| dx \quad . \qquad (10)$$

Minimizing the energy $F_2(c_1, c_2, \phi)$ with respect to c_1, c_2 we calculate two different levels of u as a mean value:

$$c_1 = \frac{\int_\Omega uH(\phi(x))dx}{\int_\Omega H(\phi(x))dx}, \qquad (11)$$

$$c_1 = \frac{\int_\Omega u(1 - H(\phi(x)))dx}{\int_\Omega (1 - H(\phi(x)))dx}. \qquad (12)$$

Minimizing the energy with respect to ϕ, we obtain the gradient descent flow equation for $\phi(x,t)$

$$\frac{\partial \phi}{\partial t} = \delta_\varepsilon(\phi)\left[\nu\nabla \cdot \frac{\nabla\phi}{|\nabla\phi|} - |u - c_1|^2 + |u - c_2|^2 \right] \qquad (13)$$

with initial condition $\phi(x, t = 0)$ such as in Figure 1 or Figure 3 and with zero Neumann boundary condition.

A program was written to solve (11) to (13) in the Matlab environment. Formula (13) was programmed using the semiimplicit finite differences.

2.2 Examples

Example 2.1

A synthetic noise image of two different levels u=100 and u = 50 is in Figure 5. The mesh size is 200 x 100 pixels. The initial LS function is such as in Figure 1. Segmentation and denosing results are in Figure 6 and 7.

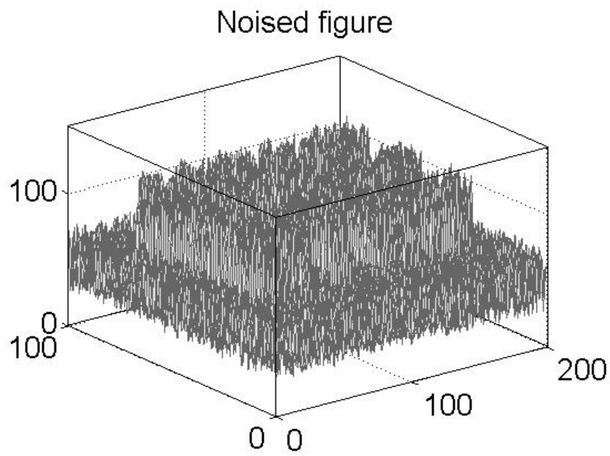

Fig. 5. Noise image, SNR=3 dB

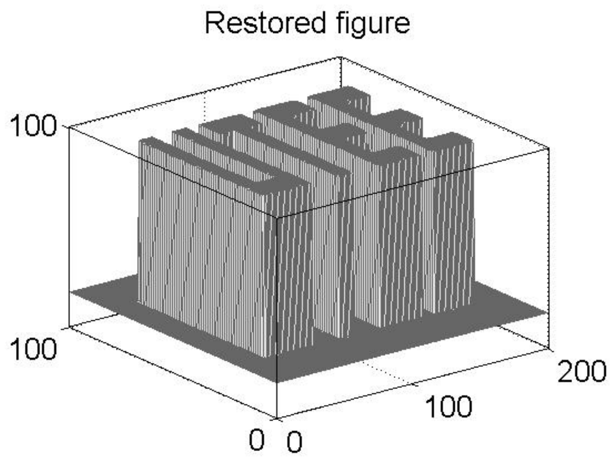

Fig. 6. Restored image u after 1000 iterations

Recovered value of u is evaluated substituting the steady state value of $H(\phi)$ after 1000 iterations in (2). This u is in Figure 6. The Heaviside function shape starts from that in Figure 2 and its final shape is of the same shape like u in Figure 6, but with limits between 0 and 1. Evolution of the Heaviside function edge Γ during the iteration process is in Figure 7. The shape of boundary is visualized in Matlab using function *isosuface*( ) and *isocaps*( ).

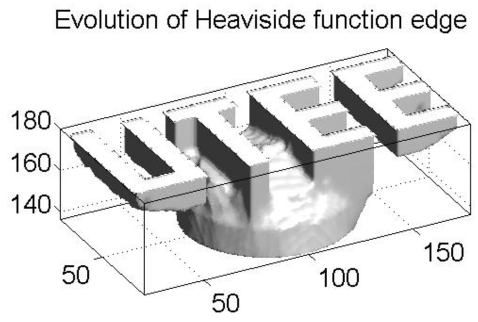

Fig. 7. Level H=0.5 evolution during iterations

Example 2.2

Denoising and segmentation of the noise image in Example 2.1 started from initial ϕ with 8 peaks, such as in Figure 3. To obtain the acceptable solution, the sufficient number of iterations decreased to 500. Evolution of the Heaviside function during the iteration process is in Figure 8.

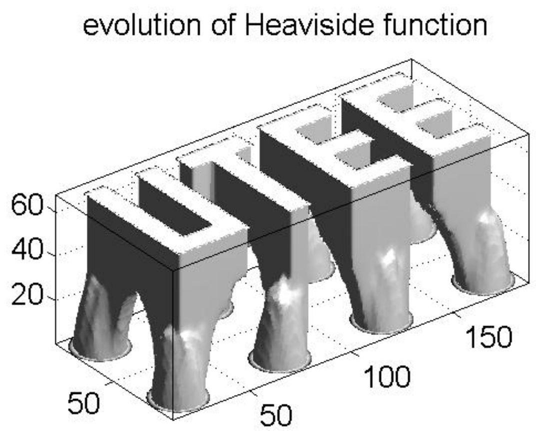

Fig. 8. Evolution of H=.5 level during the iteration process

3 IMAGE SEGMENTATION AND DENOISING FOUR PHASE CASE

3.1 Theory

For two different level set functions ϕ_1, ϕ_2, such as in Figure 4, the minimization functional is

$$\begin{aligned}
F(c_{11}, c_{10}, c_{01}, c_{00}, \phi_1, \phi_2) = \\
&\int_\Omega |u(x) - c_{11}|^2 H(\phi_1) H(\phi_2) dx + \\
&\int_\Omega |u(x) - c_{10}|^2 H(\phi_1)(1 - H(\phi_2)) dx + \\
&\int_\Omega |u(x) - c_{01}|^2 H(\phi_2)(1 - H(\phi_1)) dx + \\
&\int_\Omega |u(x) - c_{00}|^2 (1 - H(\phi_1))(1 - H(\phi_2)) dx + \\
&\nu \int_\Omega |\nabla H(\phi_1)| dx + \nu \int_\Omega |\nabla H(\phi_2)| dx .
\end{aligned} \tag{14}$$

Minimizing energy $F(c, \phi_1, \phi_2)$ with respect to $c_{11}, c_{10}, c_{01}, c_{00}$ we obtain corresponding mean values:

$$\begin{aligned}
c_{11} &= \frac{\int_\Omega uH\ (\phi_1(x))H\ (\phi_2(x))dx}{\int_\Omega H\ (\phi_1(x))H\ (\phi_2(x))dx}, \\
c_{12} &= \frac{\int_\Omega uH\ (\phi_1(x))[1 - H\ (\phi_2(x))]dx}{\int_\Omega H\ (\phi_1(x))[1 - H\ (\phi_2(x))]dx}, \\
c_{11} &= \frac{\int_\Omega u[1 - H\ (\phi_1(x))]H\ (\phi_2(x))dx}{\int_\Omega [1 - H\ (\phi_1(x))]H\ (\phi_2(x))dx}, \\
c_{11} &= \frac{\int_\Omega u[1 - H\ (\phi_1(x))][1 - H\ (\phi_2(x))]dx}{\int_\Omega [1 - H\ (\phi_1(x))][1 - H\ (\phi_2(x))]dx}.
\end{aligned} \tag{15}$$

Minimization of (14) with respect to ϕ_1, ϕ_2 results in a mutually solved couple of equations

$$\begin{aligned}
&\frac{\partial \phi_1}{\partial t} = \delta_\varepsilon(\phi_1) \nu \nabla \cdot \frac{\nabla \phi_1}{|\nabla \phi_1|} \\
&- \delta_\varepsilon(\phi_1)\left(|u - c_{11}|^2 - |u - c_{01}|^2 H(\phi_2)\right) \\
&+ \delta_\varepsilon(\phi_1)\left(|u - c_{10}|^2 - |u - c_{00}|^2\right)(1 - H(\phi_2)),
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\frac{\partial \phi_2}{\partial t} = \delta_\varepsilon(\phi_2) \nu \nabla \cdot \frac{\nabla \phi_2}{|\nabla \phi_2|} \\
&- \delta_\varepsilon(\phi_2)\left(|u - c_{11}|^2 - |u - c_{01}|^2 H(\phi_1)\right) \\
&+ \delta_\varepsilon(\phi_2)\left(|u - c_{10}|^2 - |u - c_{00}|^2\right)(1 - H(\phi_1)).
\end{aligned} \tag{17}$$

A program was written in Matlab to solve system (15) to (17) with appropriate initial conditions of ϕ_1, ϕ_2 and zero Neumann boundary conditions. Some results giving insight in the evaluation are further

3.2 Examples

Example 3.1

Original image consists of a triangular prism with $u = 50$, a block and a tube of $u = 100$ and on the bottom is $u = 10$, as in Figure 9. To this image is add noise so that the initial image in Figure 10 has SNR = 1.16 dB. Initial LS functions are in Figure 11. After 1200 iterations the image was restored with the trace of noise observed in Figure 12.

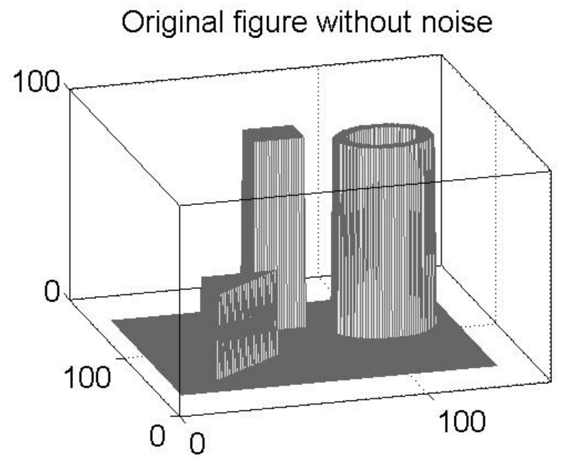

Fig. 9. Initial image level without noise

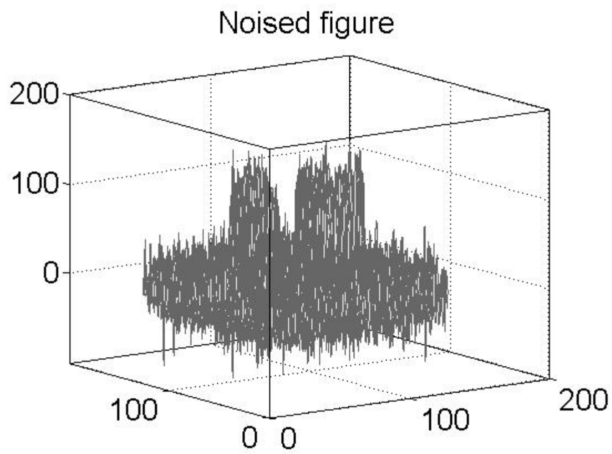

Fig. 10. Strongly noise initial image, SNR = 1.16 dB

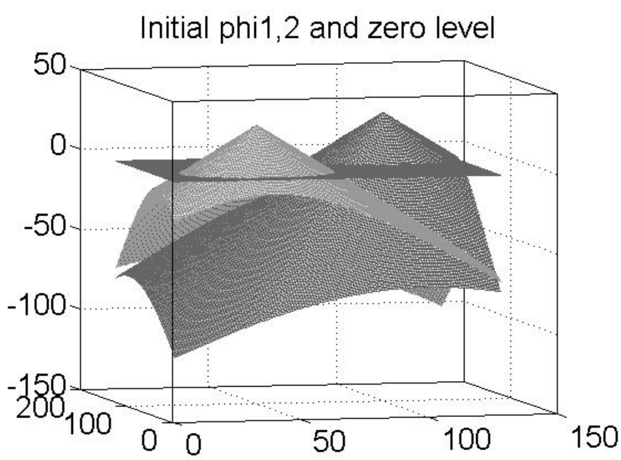

Fig. 11. Initial LS ϕ_1, ϕ_2 with zero plane

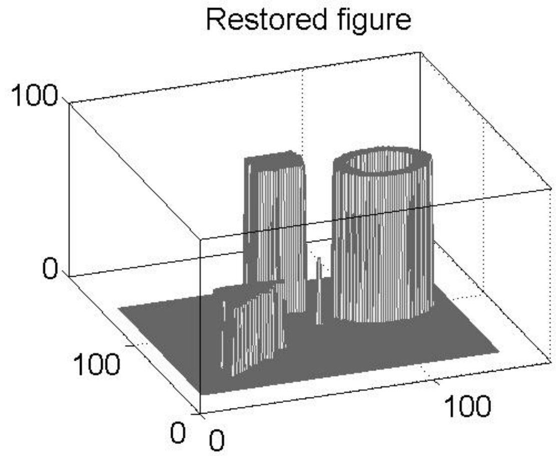

Fig. 12. Restored figure

Example 3.2

Original image is of the shape identical with that in Figure 6, but with three different values of u = 20, 50, and 100. A noise is added with the resulting initial u in Figure 13. Using two initial LS we obtained the restored image such as in Figure 14.

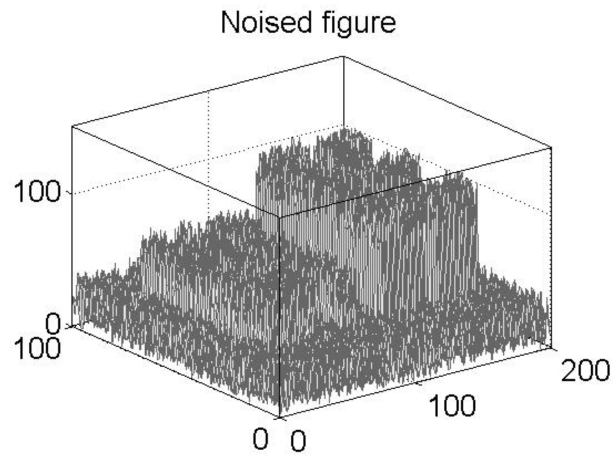

Fig. 13. Noise image

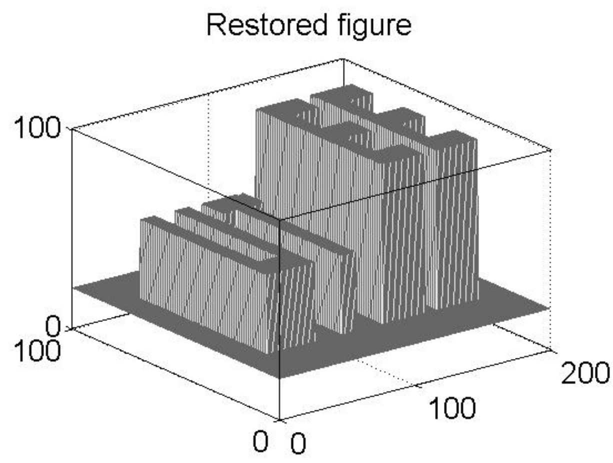

Fig. 14. Restored image

Final value of the Heavidside function products in (9), which determine final u is in Figure 15 to 18.

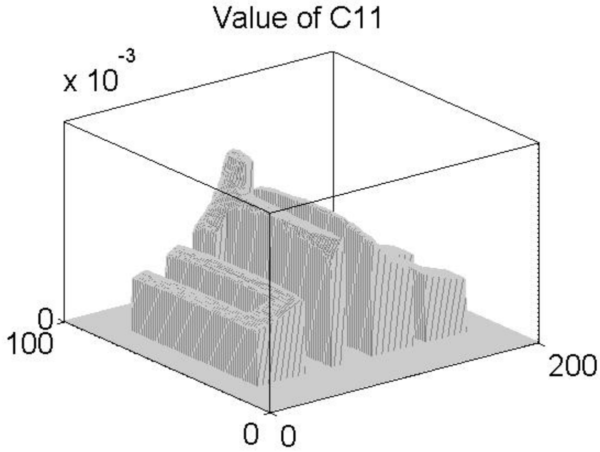

Fig. 15. Small nonzero value of final product $H_1 H_2$

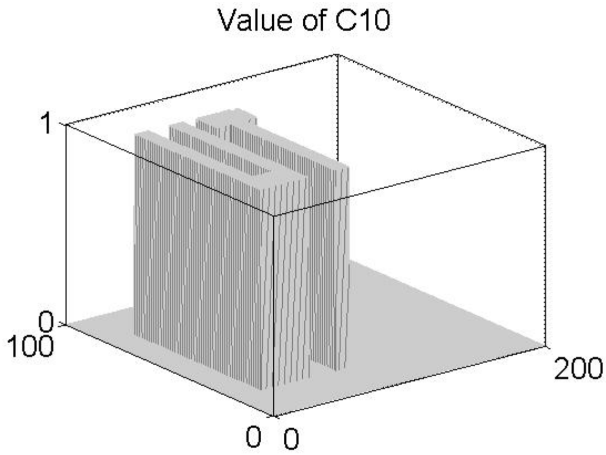

Fig. 16. Final value of $H_1(1-H_2)$

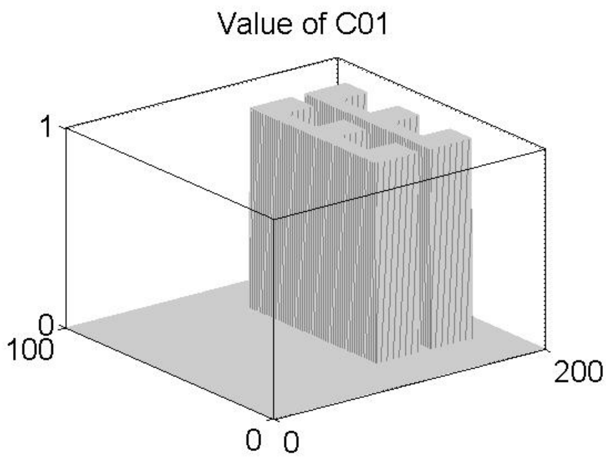

Fig. 17. Final value of $(1-H_1)H_2$

The change of the level set function during the iteration process can be observed from Figure 19, 20, and 21.

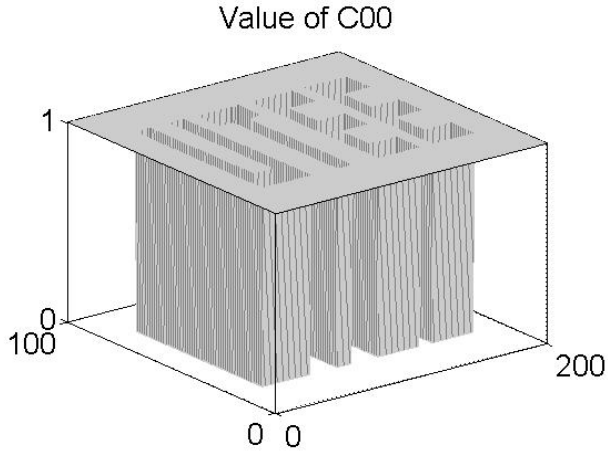

Fig. 18. Final value of $(1-H_1)(1-H_2)$

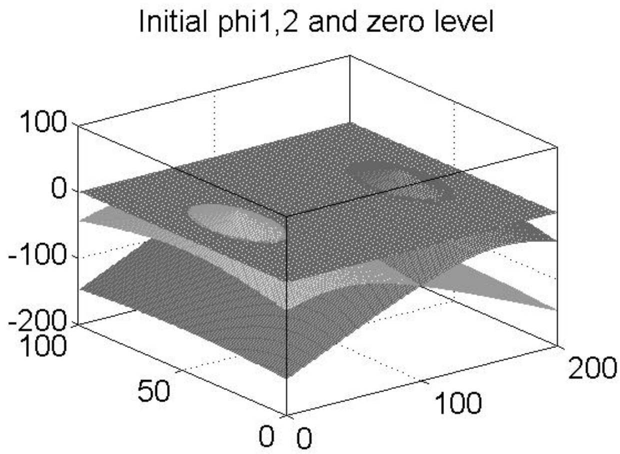

Fig. 19. Initial value of ϕ_1 *and* ϕ_2

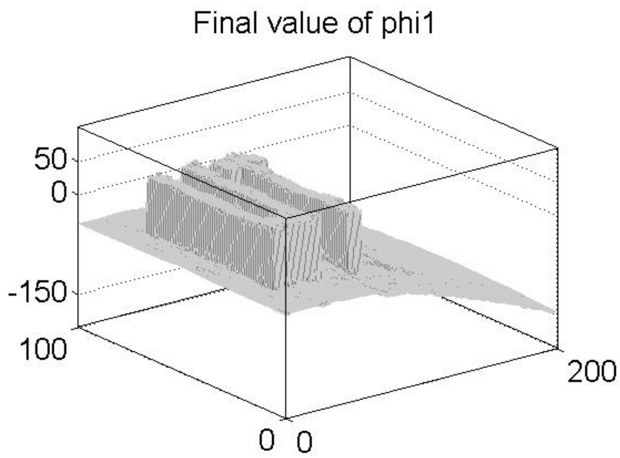

Fig. 20. Final value of ϕ_1

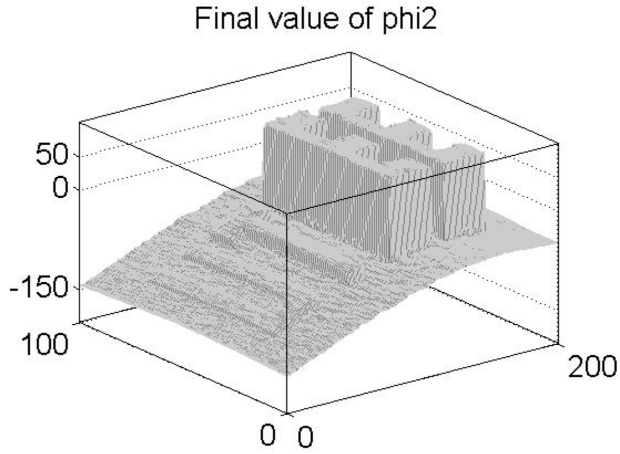

Fig. 21. Final value of ϕ_2

The decrease of Energy function $F4$ in (14) during the solution is in Figure 22.

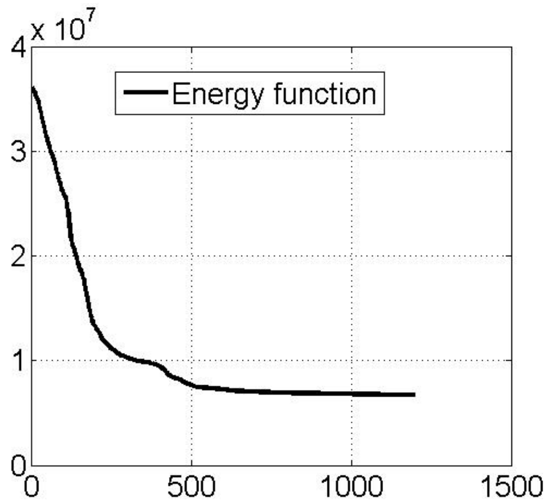

Fig. 22. Decrease of energy F4 on iterations

4 CONCLUSION

The numerical experiments proved some specific problems in the area, which is known as experimental mathematics. The solution was found to be significantly sensitive to the choice of the initial position and to the shape of LS functions, which represent initial conditions of PDE.

The next research of the LSM will be directed to the application on the inverse problems connected with different kinds of tomography, namely on the segmentation of MR images, to the impedance tomography and to the ultrasound tomography. The research will continue in the area of general image processing.

5 REFERENCES

[1] Sethian J. A.:Level set method and fast marching methods. Cambridge University Press, 1999.
[2] Yoo T. S. (editor): Insight into images. A. K. Peters, Wellesley, Massachusetts, 2004.
[3] Osher S., Fedkiw R.: Level set methods and dynamic implicit surfaces. Springer, 2003.
[4] Aubert G., Kornprobst P.: Mathematical problems in image processing. Springer, 2006.
[5] http://www.math.ucla.edu/applied/cam/index.html
[6] T. Chan and L. Vese. Image segmentation using level sets and the piecewise-constant Mumford-Shah model. *UCLA CAM Report 00-14, submitted to IJCV*, 2000.
[7] Chan, T., Vese, L.: Active Contours Without Edges. *IEEE Transactions on Image Processing*, 10(2):266–277, 2001.
[8] http://math.berkeley.edu/~sethian/2006/ Applications/Menu_Expanded_Applications.html
.

Libor Dědek, Professor,
VUT FEKT UTEE
Kolejní 4
612 00 Brno
Czech Republic
ldedek@nbox.cz