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Mathematical model of a suspension bridge in 1D: Revision of uniqueness results

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1 Introduction

The collapse of Tacoma Narrows Bridge in 1940 was the cause of major interest in modelling the behaviour of suspension bridges. In 1980s and 1990s, many authors presented new models and first theoretical results. Our work is based on a one-dimensional nonlinear model originally presented by Lazer and McKenna (1990) and on the previous work of Tajčová (1997).

2 Problem setting

In view of Lazer and McKenna's model, we consider the following boundary value problem

$$\begin{aligned} &u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ = h(x,t),\\ &u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0,\\ &u(x,t+2\pi) = u(x,t),\ -\infty < t < +\infty,\ x \in (0,\pi). \end{aligned} \tag{1}$$

By setting $\Omega = (0,\pi) \times (0,2\pi)$ and searching for a weak solution in the space $L^2(\Omega)$ for an arbitrary right-hand side $h \in L^2(\Omega)$, we are able to interpret (1) as the operator equation

$$Lu = -ku^+ + h \tag{2}$$

and consequently as the fixed point formulation

$$u = L^{-1}(-ku^+ + h). \tag{3}$$

With the help of Banach Contraction Theorem (also used by Tajčová (1997)), we can obtain a condition for the stiffness parameter k that implies the existence and uniqueness of the weak solution. In order to determine for which setting of parameters α, β and k is the operator $L^{-1}(-k(\cdot)^+ + h)$ contractive, we have to estimate the norm of L^{-1}. The eigenvalues of L, which are essential for this estimate, are complex numbers

$$\lambda_{mn} = \alpha^2 m^4 - n^2 + \mathrm{i}\beta n, \quad m \in \mathbb{N},\ n \in \mathbb{Z}. \tag{4}$$

It is an important fact that these eigenvalues can be interpreted as intersection of parabolas

$$p_m = \left\{(x,y) :\ x = \alpha^2 m^4 - \frac{y^2}{\beta^2}\right\}, \quad m \in \mathbb{N},$$

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with horizontal lines

$$l_n = \{(x,y) :\ y = \beta n\}, \quad n \in \mathbb{Z}.$$

For illustration, see Fig. 1. This geometric interpretation allows a rather straightforward improvement of previous results.

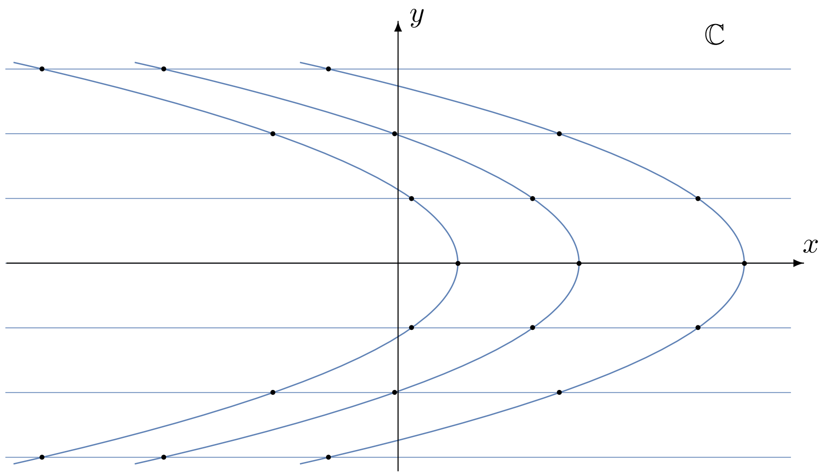

Figure 1: Eigenvalues of L in the complex plane.

3 Results

By employing Banach Contraction Theorem and the new geometric arguments, we are able to provide a refining of the previous uniqueness result in the form of a sufficient condition. Moreover, considering an equivalent ε-shifted problem

$$(L - \varepsilon I)u = -(k+\varepsilon)u^+ + \varepsilon u^- + h, \tag{5}$$

its fixed point reformulation via the resolvent operator, i.e.,

$$u = (L - \varepsilon I)^{-1}\left(-(k+\varepsilon)u^+ + \varepsilon u^- + h\right) \tag{6}$$

and, again, applying Banach Contraction Theorem and some more geometric arguments, we provide a significant extension of the "uniqueness interval" for the stiffness parameter k.

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