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Parciální dynamické rovnice na diskrétních prostorových oblastech

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disertační práce
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v oboru **Aplikovaná matematika**

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Partial dynamic equations on discrete spatial domains

Jonáš Volek

dissertation thesis
for taking academic degree **Doctor of Philosophy (Ph.D.)**
in specialization **Applied Mathematics**

Supervisors: Petr Stehlík
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Department of Mathematics

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Deklarace, Declaration

Prohlašuji, že jsem tuto práci zpracoval samostatně. Je založena na vědeckých výsledcích, jejichž jsem autorem nebo spoluatorem, a na zdrojích, které jsou citovány a uvedeny v seznamu literatury.

Plzeň, Obora, 19. prosinec 2016

I declare that the presented thesis is my original work. It is based on the scientific results which I have authored or co-authored and on cited sources which are contained in the list of references.

Pilsen, Obora, December 19, 2016

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Jonáš Volek

Abstrakt

Tato disertace prezentuje naše výsledky týkající se parciálních dynamických rovnic na oblastech s diskrétní prostorovou proměnnou (tzv. mřížkách). Tyto problémy slouží k modelování procesů probíhajících v prostorově strukturovaném prostředí (např. buňkách, krystalových mřížkách, elektronických obrázcích). Proto studujeme parciální rovnice s diskrétní prostorovou proměnnou a časovou proměnnou uvažujeme buď spojitou, diskrétní nebo z obecné časové škály.

Práce je rozdělena do dvou částí. První část disertace lze vnímat jako komentář s historickými souvislostmi. Popisujeme zde nejen historii diferenčních a diferenciálních rovnic, ale i modelování procesů, které lze popsat parciálními dynamickými rovnicemi na oblastech s diskrétní prostorovou proměnnou. Odvozujeme zákony zachování a následně vysvětlujeme, jak z nich vznikají transportní a difúzní rovnice. Poté představujeme teoretické výsledky, které se zabývají transportní rovnicí. V lineárním případě odvozujeme explicitní řešení a pro nelineární rovnici dokazujeme rozličné vlastnosti (např. principy maxima, globální existenční věty). Poté se zabýváme reakčně-difúzní rovnicí na mřížkách, pro kterou studujeme podobné teoretické vlastnosti. Diskutujeme také otázky týkající se implicitní diskretizace reakčně-difúzní rovnice. Na závěr, abychom byli schopni studovat stacionární řešení, dokazujeme existenci pro diskrétní okrajové úlohy.

Druhá část práce se skládá ze šesti publikací. Tyto články jsou přiloženy v originální podobě a obsahují všechny technické detaily pro zainteresovaného čtenáře.

Klíčová slova

parciální dynamické rovnice, mřížky, transportní rovnice, reakčně-difúzní rovnice, existence, jednoznačnost, principy maxima, variační metody

Abstract

The dissertation thesis presents our recent results about the partial dynamic equations on discrete spatial domains (so-called lattices). These problems arise from the modeling of processes on spatially structured environment (e.g., cells, crystal lattices, electronic images). Consequently, we study partial equations with discrete spatial variable and suppose time variable being either continuous, discrete or, generally, from a time scale.

The thesis is divided into two parts. The first part is a commented overview with historical context. We summarize the history of difference and differential equations and introduce the modeling of processes described via partial dynamic equations on discrete spatial domains. We derive conservation laws and then explain how transport and diffusion equations arise. Next, we present theoretical results for transport equation. We derive the explicit formula for the solution in the linear case and prove various properties for the nonlinear equation. Further, we study the reaction-diffusion equations and show similar properties. We also discuss implicit discretization of the reaction-diffusion equation. Finally, we prove the existence for discrete boundary value problems for the analysis of stationary solutions.

The second part of the dissertation thesis is composed of six appendices containing the original publications. There are all technical details for interested readers.

Keywords

partial dynamic equation, lattices, transport equation, reaction-diffusion equation, existence, uniqueness, maximum principles, variational methods

Zusammenfassung

Diese Dissertation präsentiert unsere Ergebnisse über partielle dynamische Gleichungen auf Gebieten mit diskreten Raumvariablen (sog. Gittern). Solche Probleme dienen der Modellierung der Vorgänge, die auf strukturierten Raumbereiche verlaufen (z.B. Zellen, Kristallgittern oder elektronische Bilddateien). Deshalb studieren wir die partiellen Gleichungen mit der diskreten Raumvariable und betrachten sowohl stetige, diskrete als auch zeitskalige Zeitvariablen.

Die Arbeit ist in zwei Teile eingeteilt. Der erste Teil der Doktorarbeit kann man als ein Kommentar mit den historischen Zusammenhänge betrachten. Hier schildert man nicht nur die Geschichte der Differenz- und Differentialgleichungen aber erklärt auch die Modellierung der Vorgänge, die man mit partiellen dynamischen Gleichungen auf Gebieten mit diskreten Raumvariablen beschreiben kann. Man leitet verschiedene Erhaltungsätze ab und zeigt, wie die lineare und nichtlineare Transport- oder Diffusionsgleichung entsteht. Dann stellen wir theoretische Ergebnisse vor, die sich mit der Transportgleichung beschäftigen. Wir berechnen die expliziten Lösungen in dem linearen Fall und beweisen verschiede Eigenschaften in dem nichtlinearen Fall (z.B. Maximumprinzipien, globale Existenzaussagen und stetige Abhängigkeit). Danach beschäftigen wir uns mit der Reaktionsdiffusionsgleichung auf Gittern. Wir untersuchen ähnliche theoretische Eigenschaften. Wir diskutieren auch die Fragen der impliziten Diskretisierung der Reaktionsdiffusionsgleichung. Um die stationären Lösungen studieren zu können, beweisen wir auch einige Existenzsätze für diskrete Randwertaufgaben.

Der zweite Teil der Arbeit besteht aus sechs Veröffentlichungen. Diese Artikel werden in der Originalform angehängt und enthalten alle technischen Details für interessierte Leser.

Schlüsselwörter

partielle dynamische Gleichungen, Gittern, Transportgleichung, Reaktionsdiffusionsgleichung, Existenz, Eindeutigkeit, Maximumprinzipien, Variationsmethoden

Preface and acknowledgments

This work is an overview of results about partial dynamic equations with a spatially structured environment which I have authored or co-authored. Fundamentally, it is based on the following six papers which have been published or submitted for publication to several mathematical journals during the study on the Department of Mathematics of the Faculty of Applied Science, University of West Bohemia in Pilsen, Czech Republic:

[79] P. Stehlík, J. Volek, *Transport equation on semidiscrete domains and Poisson-Bernoulli processes*, Journal of Difference Equations and Applications 19(3) (2013), 439–456.[1]

[85] J. Volek, *Maximum and minimum principles for nonlinear transport equations on discrete-space domains*, Electronic Journal of Differential Equations 2014 (2014), no. 78, 1–13.

[80] P. Stehlík, J. Volek, *Maximum principles for discrete and semidiscrete reaction-difussion equation*, Discrete Dynamics in Nature and Society 2015 (2015), Article ID 791304, 1–13.

[77] A. Slavík, P. Stehlík, J. Volek, *Well-posedness and maximum principles for lattice reaction-diffusion equations*, submitted (2016).

[81] P. Stehlík, J. Volek, *Implicit discrete Nagumo equation and variational methods*, Journal of Mathematical Analysis and Applications 438 (2016), 643–656.

[84] J. Volek, *Landesman-Lazer conditions for difference equations involving sublinear perturbations*, Journal of Difference Equations and Applications, online (2016), DOI 10.1080/10236198.2016.1234617.

The text follows from the preliminary work submitted for the state doctoral exam in 2015. It has been later completely revised and amended by new scientific results from the last two years of the doctoral study.

At this stage I would like to emphasize the role of several people which have influenced not only my doctoral study and scientific thinking but also my way of thinking about the whole real world as well as my life. I would like to pay my deepest thanks them.

I want to thank very much my supervisor Petr Stehlík and whole his family. Although he has great and wide family, he has given me a lot of his time during not only last four years of my doctoral study but

[1]The paper P. Stehlík, J. Volek [79] has been published before the doctoral study. We include it into the thesis for the sake of compactness.

for seven years of our cooperation. I appreciate him for his strong values and opinions on the science as well as on human life. I am happy that he is not already only my supervisor and colleague, but during the years he became my friend. I thank for our discussions. They have influenced my view on mathematics and also on the ethics in the science very strongly. I thank Petr for the cooperation and hope that it will continue for a long time.

I would like to thank also my former supervisor Pavel Drábek. He has had always time to give me his opinion on my problems and to encourage me, although he has been a very busy man. He is a very respected mathematician, however, he has never forgotten to provide inspiration for his students.

Thirdly, I would like to thank my former opponent and nowadays my co-author Antonín Slavík. It has been a great pleasure to meet his way of thinking – deep and wide at the same time, his rigorous style of scientific work and his modesty.

From the scientific point of view, I want to learn more from Petr Stehlík's searching of new unconventional problems and width of his interests, Pavel Drábek's enthusiasm for nonlinear analysis and his insight, and Antonín Slavík's deep and modest way of thinking.

Certainly, I also thank all my colleagues and friends.

At the end, most importantly, I would like to highlight the role of my family and express my appreciation to all of them. My best thanks belong to my wife Radka. She has been as supportive as she could and has been patient with me when my mind has not been at home. She has always stood by me, encouraged me and made a warm spirit of our home. Furthermore, I would like to thank my little daughter Viktorie. Although she is so small, she has brought new sunny light into my days and gave the sense to many steps of my life.

Pilsen, Obora, December 19, 2016 ***Jonáš Volek***

Contents

CHAPTER 1

Introduction

Differential equations play one of the most important roles in the theory of mathematical analysis already from the invention of differential and integral calculus in the end of 17th century by I. Newton and G. W. Leibnitz. The reason is straightforward, in that times the endeavor to understand and mathematically describe the physical laws in the real world culminated. The infinitesimal calculus showed new ways how to describe physical processes. Firstly, considering time-dependent magnitudes, the notion of derivative gave a method how to mathematically establish their instantaneous changes. Secondly, differential and integral calculus gave a possibility how to introduce balances, namely, conservation laws for continua. Consequently, many physical laws have been naturally expressed in the form of differential equations from that times.

Later, not only physical, but also economic, biological, chemical and social phenomena have been described via differential models. Therefore, the detailed qualitative analysis of differential equations and establishing methods for solving them have taken great attention of thousands of mathematicians over the world. Generally, we can say that differential equations serve for modeling and better understanding of real world around us. These models could help with hard decisions and prevent some undesirable effects and damages.

Furthermore, differential equations are important and interesting even from purely mathematical reasons. They serve as a useful tool in other areas of mathematics, e.g., in algebraic and differential geometry, manifold theory, differential topology, etc. On the other hand, this cooperation among seemingly distinct fields of mathematics works naturally in both ways and the theory of differential equations often uses deep results from algebra, topology, functional analysis, etc.

1.1 Difference equations

The history of difference equations is much older than the history of differential equations. The main reason is straightforward, one does not need any special mathematical instrument to formulate problems in the form of difference equations. Contrary to the differential equations which had to wait until 17th century when I. Newton and G. W. Leibnitz came with the idea of infinitesimal calculus, difference equations appeared in the form of recurrence relations already in the ancient history.

One of the most known examples of recurrence relations is from the book Liber Abaci by Leonardo of Pisa (better known as Fibonacci) from the beginning of 13th century. His example for the reproduction of rabbits reads as follows. At the beginning one has a pair of rabbits (male and female). When the pair matures, they have a new pair of immature rabbits. After one reproductive season, the new pair mature and the old pair has another pair of immature rabbits, etc. The question is how many pairs of rabbits are there after t-th reproductive season, $t = 0, 1, 2, \dots$ This can be formulated in the recurrence form

$$N_{t+1} = N_t + N_{t-1} \quad \text{for} \quad t = 2, 3, \dots, \qquad N_0 = 1, \quad N_1 = 1,$$

where N_t denotes the number of pairs after t-th reproductive season. This recurrence scheme produces the so-called Fibonacci sequence[1]

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \dots\}.$$

Let us emphasize that Fibonacci was not the founder of this interesting sequence. This has been known since ancient history thanks to Indian mathematicians (see, e.g., P. Singh [71]).

Surprisingly, this sequence occurs in many places in the nature (e.g., the seeds in the floret of sunflower are placed in spirals which consist of a Fibonacci number of seeds). It is also connected with the famous golden ratio which has been intensively studied since the times of Euclid in Ancient Greece (and often appearing in nature, culture, etc.). The limit of ratio of succeeding numbers of Fibonacci sequence converges to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

Next interesting problem involving recurrence relations which goes to the old history is bearing interests in money lending. The simplest case is well known. If you lend somebody N_0 amount of money and you make a settlement that after each month the debt increases by r %, then the debt after t months, $t = 0, 1, 2, \dots$, is given by the following recurrence relation

$$N_t = \left(1 + \frac{r}{100}\right) N_{t-1} \quad \text{for} \quad t = 1, 2, 3, \dots,$$

which has the explicit solution $N_t = \left(1 + \frac{r}{100}\right)^t N_0$.

From the numerical mathematics' point of view, the difference equations have started to play an important role after the development of infinitesimal calculus and differential equations. Since the mathematicians could not solve many differential equations analytically, they wanted to introduce a method how to find solutions at least approximately. The pioneering work in this field was done by L. Euler in 18th century who established a method for approximation of solution to initial value problem for ordinary differential equation (ODE)

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \geq 0, \quad f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}, \\ u(0) = u_0 \in \mathbb{R}. \end{cases}$$

The Euler method is based on discretization of the interval $[0, +\infty)$ with the discretization step $h > 0$ into the set of points $\{0, h, 2h, \dots\}$ and on the recurrence scheme when the value of the exact solution u at discretization points are approximated by the values

$$v(t + h) = v(t) + hf(t, v(t)), \quad t = h, 2h, \dots, \quad v(0) = u_0. \tag{1.1}$$

Between the discretization points it is approximated by the linear function characterized by the end points. The problem to establish the values $v(t)$ at the discretizations points (1.1) is in fact an initial value problem for (generally nonlinear) difference equation of the first order. See, e.g., E. Hairer, S. P. Nørsett,

[1] The closed formula for the solution of the Fibonacci recurrence relation is

$$N_t = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^{t+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{t+1} \right),$$

see, e.g., J. D. Murray [60].

G. Wanner [38] for more information and deep investigation of mathematical properties of the Euler method.

The detailed analysis of difference equations is very important, since the discrete analogues of continuous problems could exhibit qualitatively different behavior. To illustrate this, consider for example the initial value problem for linear ODE

$$\begin{cases} u'(t) = -u(t), \quad t \geq 0, \\ u(0) = 1. \end{cases}$$

The unique solution is obtained simply as $u(t) = \mathrm{e}^{-t}$ which is everywhere positive and satisfies $u(t) \to 0+$ for $t \to +\infty$. The discrete analogue for this problem obtained via the Euler method is

$$v(t+h) = (1-h)v(t), \quad t = h, 2h, \dots, \quad v(0) = 1,$$

which has the unique solution $v(t) = (1-h)^{\frac{t}{h}}$. Let us discuss and compare the solutions of continuous and discrete problem. Firstly, observe that for discretization steps $h > 1$ the solution of difference equation changes sign. Moreover,

- for $1 < h < 2$, there is $v(t) \to 0$ for $t \to +\infty$,
- for $h = 2$, there is $v(t) = (-1)^{\frac{t}{h}}$,
- for $h > 2$, there is $\liminf_{t \to +\infty} v(t) = -\infty$, $\limsup_{t \to +\infty} v(t) = +\infty$.

If $h = 1$, there is $v(t) = 0$ for all $t = h, 2h, \dots$ On the contrary, if $h < 1$ and $t > 0$ is fixed, we obtain

$$v(t) = (1-h)^{\frac{t}{h}} \to e^{-t} = u(t) \quad \text{for} \quad h \to 0+,$$

i.e., we obtain better approximations of the original continuous problem for smaller discretization steps. Secondly, the solution of the initial value problem for ODE exists for all $t \in \mathbb{R}$ and is given by $u(t) = \mathrm{e}^{-t}$, i.e., the backward solution also exists. However, if $h = 1$ there is no backward solution of the difference equation.

Let us note that the Euler method has been many times generalized and in the present days it is not often practically used because it is one of the simplest methods, unstable for many problems. However, we can say that it still has been one of the fundamental building blocks of numerical mathematics.

Furthermore, one can focus on the analysis of boundary value problems for differential equations. The method of finite differences (see, e.g., R. J. LeVeque [53]) is one of standard numerical methods for these problems. This leads to boundary value problems for difference equations which are another main topic of this thesis. For example, consider the Dirichlet boundary value problem for second-order ODE

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in (0,1), \quad f : (0,1) \times \mathbb{R} \to \mathbb{R}, \\ u(0) = u(1) = 0. \end{cases}$$

The method of finite differences is based on the decomposition of the interval $[0,1]$ similarly as above into the set $\{0, h, 2h, \dots, 1-h, 1\}$ with the discretization step $h = 1/k$, $k \in \mathbb{N}$, and an approximation of the solution u of the continuous problem by the solution of the following discrete problem

$$\begin{cases} -\Delta_h^2 v(t-h) = f(t, v(t)), \quad t = h, 2h, \dots, 1-h, \\ v(0) = v(1) = 0, \end{cases}$$

where $\Delta_h^2 v(t-h) = \frac{1}{h^2}\left(u(t-h) - 2u(t) + u(t+h)\right)$.[2] Therefore, one could study these two problems together and for example, establish a general condition for the nonlinearity f such that both problems has a unique solution. Then, the question of convergence of solutions v to the solution u provided $h \to 0+$ is very interesting and one of the main topics in numerical mathematics.

Let us note that the reader can found the systematic theory of difference equations, e.g., in the monographs W. G. Kelley, A. C. Peterson [46] or S. Elaydi [30].

1.2 Population models

As Fibonacci's example shows, difference (later also differential) equations have been often used for the prediction of a future state of some biological system. Main problems studied in the thesis have often the motivation from biological or chemical modeling. Thus, let us briefly introduce basic models of population dynamics in their continuous and discrete form. The summary is based on the monograph of J. D. Murray [60].[3] Let us start with continuous models.

Example 1.1. At the end of 18th century T. R. Malthus in [58] presented the exponential growth model of population dynamics. It is based on a simple idea that the birth rate and mortality are proportional to the size of population linearly. In the continuous setting, denoting by $N(t)$ the population size at time t, the change of $N(t)$ is given by

$$N'(t) = bN(t) - dN(t),$$

where $b > 0$ is the parameter describing the birth rate and $d > 0$ describes the death rate. The solution of this simple linear ODE is $N(t) = N_0 \mathrm{e}^{(b-d)t}$, where $N_0 \geq 0$ is the initial size of population at time $t = 0$. Therefore,

- if $b < d$, then $N(t) \to 0+$ for $t \to +\infty$ and the population dies out,
- if $b = d$, then $N(t) \equiv N_0$,
- if $b > d$, then $N(t) \to +\infty$ exponentially for $t \to +\infty$.

This model seems to be unrealistic, since it is very simple and does not involve the influence the environment (e.g., the spatial capacity, the amount of sources, etc.). However, sometimes it can describe the evolution satisfactory. For example, the world's population has increased exponentially until now and thus, one can use this model for a prediction of future progress. If it would be done for immediate future, this prediction will be plausible. However, it is hard to estimate when the world resources will start to be deficient.

Example 1.2. Interestingly, in several particular countries (e.g., in the Western Europe) the population growth rate has started to decrease. This can motivate us for another model which was introduced by P. F. Verhulst in [83] in the middle of 19th century – the logistic model. It has the following form

[2] Let us note that in the theory of difference equations, boundary value problems are usually re-scaled, i.e., for example in the case of homogeneous Dirichlet boundary conditions the following problem is considered

$$\begin{cases} -\Delta^2 w(n-1) = g(n, w(n)), & n = 1, 2, \dots, N-1, \\ w(0) = w(N) = 0, \end{cases}$$

where $N = 1/h$, $w(n) = u(nh)$, $\Delta^2 w(n-1) = w(n-1) - 2w(n) + w(n+1)$, $g(n,w) = h^2 f(nh, w)$.

[3] We restrict ourselves to models of one population, since we do not study systems of equations further in the thesis (i.e., we omit famous models of cooperation, conflict, predator-prey models of Lotka-Volterra type, etc.)

$$N'(t) = rN(t)\left(1 - \frac{N(t)}{K}\right), \quad r > 0, \quad K > 0.$$

The birth rate per capita in this case is given by $r\left(1 - \frac{N(t)}{K}\right)$ and thus, it depends on the population size. Here, the parameter $K > 0$ describes the carrying capacity of the environment. If the size $N(t)$ approaches the capacity K, then the birth rate per capita tends to zero. The explicit solution of this equation is given by

$$N(t) = \frac{N_0 K e^{rt}}{K + N_0\left(e^{rt} - 1\right)}.$$

One can observe that the logistic equation has two stationary (constant in time) solutions $N(t) \equiv 0$ and $N(t) \equiv K$. Solution for any positive initial condition $N_0 > 0$ tends to the value K provided $t \to +\infty$ and

- if $N_0 < \frac{K}{2}$, then the solution is strictly increasing to K and has well-known sigmoid shape, which for small time instances exponentially grows up, then the growth acceleration stops and decreases,
- if $\frac{K}{2} \leq N_0 < K$, then the solution is strictly increasing to K, however, it is everywhere concave,
- if $N_0 > K$, then the solution is strictly decreasing to K and it is everywhere convex.

This yields that the stationary solution $N(t) \equiv K$ is globally asymptotically stable and $N(t) \equiv 0$ is unstable.

There are many generalizations of logistic model, e.g., spruce budworm model in which the hysteresis effect occurs, model with harvesting, etc. (see J. D. Murray [60]). Let us mention one of these possible modifications.

Example 1.3. In 1930's, W. C. Allee studied the effect in the population dynamics when the population has better conditions for reproduction provided it is larger (see W. C. Allee [1]). Moreover, there can exist a positive survival threshold for the population size below which the population dies out. Biological mechanism evoking these procedures can be, e.g., collective feeding, collective defense against predators, the frequency of meeting for reproduction, etc. This effect is called after its founder – the Allee effect. The mathematical model describing the strong Allee effect (with the survival threshold), uses a type of cubic nonlinearity which arises by the modification of logistic function

$$N'(t) = rN(t)\left(1 - \frac{N(t)}{K}\right)\left(\frac{N(t)}{A} - 1\right), \quad r > 0, \quad K > A > 0.$$

It has three stationary solutions $N(t) \equiv 0$, $N(t) \equiv A$ and $N(t) \equiv K$. Basically,

- for $0 < N_0 < A$ the solution is strictly decreasing and $N(t) \to 0+$ for $t \to +\infty$, i.e., the population dies out,
- for $A < N_0 < K$ the solution is strictly increasing and $N(t) \to K-$ for $t \to +\infty$,
- for $N_0 > K$ the solution is strictly decreasing and $N(t) \to K+$ for $t \to +\infty$,

i.e., the stationary solutions $N(t) \equiv 0$, $N(t) \equiv K$ are locally asymptotically stable, $N(t) \equiv A$ is unstable.[4] This yields that the population actually dies out provided the initial size of population is less than the survival threshold A.

[4]These cubic nonlinearities are often called bistable nonlinearities.

Let us focus on discrete population models. It is reasonable to apply difference equations rather than differential equations, for example, when the population generations do not overlap (e.g., for the reproduction of one-year plants). We denote the population size at generation t by N_t.

Example 1.4. Let us start with the discrete Malthus' model

$$N_{t+1} = rN_t,$$

where $r > 0$ is the rate of reproduction, e.g., the average of descendants which individuals have. The solution is $N(t) = N_0 r^t$ where N_0 is again the initial size of population for $t = 0$. It is obvious that it has similar behavior as the continuous Malthus' model

- if $r < 1$, the solution exponentially decreases $N_t \to 0+$ for $t \to +\infty$ and the population dies out,
- if $r = 1$, the solution is constant $N_t \equiv N_0$,
- if $r < 1$, the solution exponentially grows up $N_t \to +\infty$ for $t \to +\infty$.

Example 1.5. The second and very interesting model is the discrete logistic equation

$$N_{t+1} = rN_t\left(1 - N_t\right), \quad r > 0,$$

(we consider the carrying capacity $K = 1$ for the brevity, the results below can be easily rescaled for $K \neq 1$). Let us focus on stationary solutions, i.e., on solutions for which $N_{t+1} = N_t$ for all $t = 0, 1, 2, \ldots$ Firstly, observe that $N_t \equiv 1 = K$ is not a solution. After simple calculations, we obtain that there are two stationary solutions $N_t \equiv 0$ and $N_t \equiv \frac{r-1}{r}$. From the discrete stability theory, one could derive that

- $N_t \equiv 0$ is stable for $r \in (0, 1)$,
- $N_t \equiv \frac{r-1}{r}$ is stable for $r \in (1, 3)$.

Moreover, for $r > 3$ there are no stable stationary solutions. However, one can prove that there exist two $2-$periodic solutions which are stable for $r \in (3, r_2)$, where $r_2 > 3$ is a constant. For $r > r_2$ the $2-$periodic solutions become unstable and there appear four stable $4-$periodic solutions for $r \in (r_2, r_4)$. One could continue in this way and analyze the behavior of numbers r_i, $i = 2, 4, 8, \ldots$ There exists a limiting point $r_c > 0$ such that $r_i \to r_c$ and for $r > r_c$ every $2n-$periodic solution becomes unstable. But there exist odd-periodic solutions. A. N. Sharkovski showed in 1960's in [70] that for $r > r_c$ there exist even chaotic solution. For more detailed investigation of the discrete logistic equations, see also S. Elaydi [30]. This shows again that a discrete counterpart of well-posed and often simply solvable continuous problem can be very hard to analyze and have actually different properties.

As the last example of population models we describe a model involving partial differential equations (PDEs). Contrary to ODE models where only the time-evolution of the population number is described, in the PDE model we want to include also spatial movement of the population.

Example 1.6. Let $u(x, t)$ denote the population density at time t and position x in the space. Classical models assume that the population spreads in space for increasing time and describe it by the diffusion process.[5] Consequently, the models have the form of a reaction-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = k\Delta u(x,t) + f(x,t,u(x,t)), \quad x \in \Omega \subset \mathbb{R}^N, \quad t \geq 0, \quad k > 0,$$

where $\Delta u(x,t) = \sum_{i=1}^{N} \frac{\partial^2 u(x,t)}{\partial x_i^2}$ denotes the Laplace operator. This equation is then joined with an appropriate boundary condition on $\partial\Omega$. The function f models a local reaction which can be biologically interpreted as a birth rate and it could be, e.g.,

[5]Let us note that there are also models describing an opposite process – the so-called chemotaxis, which causes that the population agglomerates and moves to separated places with high density of population.

- $f(x,t,u) = ru$, $r \in \mathbb{R}$, (exponential growth),
- $f(x,t,u) = ru\left(1 - \frac{u}{K}\right)$, $r > 0$, $K > 0$, (logistic nonlinearity),
- $f(x,t,u) = ru\left(1 - \frac{u}{K}\right)\left(\frac{u}{A} - 1\right)$, $r > 0$, $0 < A < K$, (bistable nonlinearity).

We analyze the discrete-space analogues of these problems, i.e., we assume that the environment is spatially structured (e.g., biological cells, crystal lattices, qualitatively different areas in nature, cities). In one spatial dimension, we consider the following problem lying on the borderline of differential and difference equations

$$\frac{\mathrm{d}u(x,t)}{\mathrm{d}t} = k(u(x-1,t) - 2u(x,t) + u(x+1,t)) + f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \geq 0, \quad k > 0. \tag{1.2}$$

Let us note that we often call the case with discrete spatial variable and continuous time variable as semidiscrete equation. We study also entirely discrete version

$$\frac{u(x,t+h) - u(x,t)}{h} = k(u(x-1,t) - 2u(x,t) + u(x+1,t)) + f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in h\mathbb{N}_0, \tag{1.3}$$

where $h\mathbb{N}_0 = \{hn, n \in \mathbb{N}_0\}$, $h > 0$. These problems are generally called lattice differential equations and we devote the whole following section to the survey about them.

Since we deal with problems which combine properties of differential and difference equations, we have to emphasize that there was a big effort to unify continuous and discrete calculus and thus, differential and difference equations into one setting. In 1988, Stefan Hilger in his dissertation thesis [41] has come up with the so-called time scale calculus which presents some kind of this unification. By this invention he has laid the foundations for the so-called dynamic equations on time scales.[6]

Time scales are a nice mathematical tool. Their beauty and mathematical elegance arise especially when it is used for emphasizing the differences of discrete and continuous world. If one has a property that, e.g., simply holds for a continuous problem and it does not for its discrete version, nice question is when the moment of this change is. In this case, time scales provide a great mathematical way how to investigate this question. For example, in Subsection 3.2.3 we analyze the validity of strong maximum principle for lattice reaction-diffusion dynamic equation. We show that in the case of continuous time variable (1.2) the strong maximum principle holds. However, for the discrete equation (1.3) it does not. Analyzing the time-scale version

$$u^{\Delta}(x,t) = k(u(x-1,t) - 2u(x,t) + u(x+1,t)) + f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{T},$$

where $\mathbb{T}$ is a time scale and $u^{\Delta}(x,t)$ is the delta-derivative with respect to t, we show that the strong maximum principle holds when the time scale $\mathbb{T}$ contains at least one dense point.

1.3 Lattice differential equations

As we mentioned above, lattice differential equations (LDEs) are basically analogues of evolutionary PDEs which are formulated on discrete spatial domains called lattices. One of the simplest examples is the equation (1.2) which is formulated for $x \in \mathbb{Z}$. However, more complicated lattices (e.g., $\mathbb{Z}^N$) are also studied and one can consider problems defined even on non-symmetric lattices assumed only to be graphs. These equations can be understood as discretizations of PDEs, however, the study of LDEs have

[6] We do not present any introduction into this theory, since it is rather technical and everyone could find it in S. Hilger [40] or in the survey books M. Bohner, A. C. Peterson [13, 14].

started not as numerical models of continuous problems but as original models of binary alloys (see, e.g., J. W. Cahn [17], H. E. Cook et al. [24]). Besides material engineering, LDEs origin naturally in many other areas, in biology (see, e.g., J. Bell [6], J. Bell, C. Cosner [7], J. P. Keener [45]), chemistry (e.g., T. Erneux, G. Nicolis [31], J. P. Laplante, T. Erneux [50]), image processing (e.g., W. J. Firth [33]), etc. For detailed information about LDEs and other references, see the monograph S.-N. Chow et al. [20]. In the following paragraphs we expose only a brief summary about main ideas based on [20].

One of the most known examples is lattice Allen-Cahn equation applied for modeling of motion of the interface of binary alloy

$$\frac{\mathrm{d}u(\mathbf{x},t)}{\mathrm{d}t} = k\Delta_L u(\mathbf{x},t) + f(u(\mathbf{x},t)), \quad k > 0, \tag{1.4}$$

where $t \geq 0$, $\mathbf{x} \in \mathbb{Z}^N$,

$$f(u) = ru(1-u^2),$$

is the symmetric bistable nonlinearity and $\Delta_L u$ denotes a lattice/discrete Laplace operator on $\mathbb{Z}^N$, e.g.,

- on $\mathbb{Z}$, it is the standard second central spatial difference $\Delta_L u(x,t) = u(x-1) - 2u(x,t) + u(x+1,t)$, i.e., (1.4) is in fact the equation (1.2),
- on $\mathbb{Z}^2$ it can be (denoting $u(\mathbf{x},t) = u(x,y,t)$, $x,y \in \mathbb{Z}$)

  $$\begin{aligned}\Delta_L u(x,y,t) = \Delta_+ u(x,y,t) = {} & u(x-1,y,t) + u(x,y-1,t) - 4u(x,y,t) \\ & + u(x+1,y,t) + u(x,y+1,t),\end{aligned}$$

  or

  $$\begin{aligned}\Delta_L u(x,y,t) = \Delta_\times u(x,y,t) = {} & u(x-1,y-1,t) + u(x+1,y-1,t) - 4u(x,y,t) \\ & + u(x-1,y+1,t) + u(x+1,y+1,t),\end{aligned}$$

  i.e., $\Delta_L u$ is two-dimensional discrete Laplacian based on $+$-shaped or $\times$-shaped stencil, respectively.

One can consider the equation (1.4) with general nonlinearity f and distinguish LDEs on finite or infinite lattices. LDEs on finite lattices (e.g., finite discrete interval $[a,b]_\mathbb{Z} = [a,b] \cap \mathbb{Z}$ or finite graphs) can be reformulated as a finite system of ODEs. Equations on infinite lattices (e.g., $\mathbb{Z}^N$, infinite graphs) then correspond to an infinite system of ODEs. In that case, the initial condition $u(\mathbf{x},0) = u_0(\mathbf{x})$ is classically taken bounded (e.g., $\{u_0(\mathbf{x})\}_{\mathbf{x}\in\mathbb{Z}^N} \in \ell^\infty(\mathbb{Z}^N)$) and we investigate bounded solutions. Denoting $U(t) = \{u(\mathbf{x},t)\}_{\mathbf{x}\in\mathbb{Z}^N}$, the equation (1.4) can be rewritten as a differential equation on the space ℓ^∞

$$\begin{cases} U'(t) = F(U(t)), \quad F: \ell^\infty(\mathbb{Z}^N) \to \ell^\infty(\mathbb{Z}^N), \\ U(0) = \{u_0(\mathbf{x})\}_{\mathbf{x}\in\mathbb{Z}^N}, \end{cases} \tag{1.5}$$

where $U: \mathbb{R}_0^+ \to \ell^\infty(\mathbb{Z}^N)$ and $F(U) = \{k\Delta_L u(\mathbf{x}) + f(u(\mathbf{x}))\}_{\mathbf{x}\in\mathbb{Z}^N}$.

In both cases, under standard continuity assumptions on the function f, one can obtain the existence and uniqueness of at least one local solution applying contraction principle in the standard way. First interesting question then arises – the global existence. Maximum principles are one of the possible ways how to prove the global existence, since they guarantee that the local solution cannot blow up (see Section 3.2).

Stationary solutions and their stability is another property that is often studied. In [20] there is shown the application of nonlinear semigroup theory for an analysis of invariant sets and global attractors for Allen-Cahn LDE. Furthermore, the existence of special type solutions which have been observed in numerical experiments, the so-called traveling wave solutions, have taken lot of attention. For example, for $x \in \mathbb{Z}$ the traveling wave solution is a solution in the form

$$u(x,t) = \varphi(x - ct),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a function and $c \in \mathbb{R}$ is a speed of wave propagation. There is often additionally required that $\varphi(-\infty) = 0$ and $\varphi(+\infty) = 1$ (where $\varphi(\pm\infty)$ are limits at $\pm\infty$) with a monotonicity assumption on φ. Let us mention the work of H. F. Weinberger [88] in which he established a very general result about the existence of traveling waves for a wide class of problems including LDEs. Let us note that in Weinberger's result a maximum principle appears as a general hypotheses which can be another motivation for the study of a priory bounds for LDEs that we present in the thesis. The existence and stability of wavefront solutions for LDEs with $x \in \mathbb{Z}$ are proved in another pioneering works B. Zinner [90, 91].

Furthermore, in biology, there have been investigated motionless waves, especially, the property called propagation failure. J. P. Keener in [45] proved for the equation (1.4) with

$$f(u) = (u - a)\left(u^2 - 1\right),$$

that there exists a parameter $\alpha \in (0,1)$ such that for $-\alpha \le a \le \alpha$ there is a unique wavefront solution φ and it has $c = 0$. On the contrary, for analogous PDE there is $c \neq 0$ for all $a \neq 0$, i.e., any non-symmetric f causes a non-negative speed of propagation.

Generally speaking, this thesis is concerned with transport and reaction-diffusion type LDEs. We investigate equations in which the time variable appears continuous, discrete as well as equations in which the time variable is assumed to be from a general time scale. We call this setting partial dynamic equations with discrete space or lattice dynamic equations. We are fundamentally focused on local existence and uniqueness of solutions and on weak and strong maximum principles with their consequences, especially, the global existence, boundedness and uniform stability. Moreover, we investigate several special properties of solutions such as sign and integral preservation.

1.4 Interesting moments in history of PDEs

We study modifications of standard continuous PDEs and therefore, we present a short historical overview about PDEs based on the paper of H. Brézis, F. Browder [15].[7] Let us note that an introduction to the theory of PDEs can be found, e.g., in the monographs P. Drábek, G. Holubová [28] (linear problems), L. C. Evans [32] (linear and nonlinear problems), J. D. Logan [56] and T. Roubíček [69] (nonlinear problems).

The invention of infinitesimal calculus started the ecstatic boom of the study of differential equations together with whole mathematical analysis. The studied problems such as mechanics of continua were very complex and hence, mathematicians came very briefly with an exposition of first PDEs. The pioneering works go to J. R. d'Alembert, L. Euler, D. Bernoulli and P. S. Laplace into the middle of 18th century. It started in 1750's by the work of J. R. d'Alembert who established one-dimensional wave equation describing a vibrating string. This was followed in 1760's by L. Euler and D. Bernoulli who generalized it into two- and three-dimensional case. Then the stationary problem, today known as Laplace equation, was formulated by P. S. Laplace in 1780's for the study of gravitational field. The first diffusion model goes back to J. Fourier who modeled the heat transfer on the beginning of 19th century.

[7] Let us readily disclose that we present a summary of several interesting and important milestones in the evolution of PDEs which is substantially simplified. Moreover, the evolution of PDEs has not been so linear as we perform and has gone in many various ways and directions which have often co-operated. The reader is invited to see the paper of H. Brézis and F. Browder [15] for better imagination about that.

Besides these three major linear equations (generalized and classified later as hyperbolic, elliptic and parabolic linear problems), also nonlinear PDEs arose in 18th century. In 1750's, L. Euler studied the first nonlinear model of incompressible fluid dynamics. This was followed later, in the first half of 19th century, by C. L. Navier and G. G. Stokes who established their famous model for compressible viscous fluid which was called after them – the Navier-Stokes equations. The analysis of this problem has been one of the most challenging problems in the theory of PDEs until today.

Firstly, the problems involving PDEs were studied via establishing direct methods for finding their solutions. Major examples of these techniques were separation of variables (which led later to the detailed study of Fourier series), method of the Green's function or power series method. In the middle of 19th century, B. Riemann studied the existence of solution for the Laplace equation

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad \Omega \subset \mathbb{R}^3, \\ u = \varphi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.6}$$

via the Dirichlet principle (discovered by G. Green and G. F. Gauss), i.e., via the minimization of the Dirichlet (or also energy) integral

$$\int_\Omega \sum_{i=1}^{3} \left(\frac{\partial v}{\partial x_i} \right)^2 \mathrm{d}x,$$

by a smooth function over all functions v which satisfy $v = \varphi$ on $\partial\Omega$. However, B. Riemann presented no mathematically regular proof of existence of such a minimizer. This and others motivated K. Weierstrass to introduce a program for development of correct and mathematically accurate proofs in the mathematical analysis and specifically, for rigorous proofs of existence for differential equations.

One of the greatest persons in the solvability theory for PDEs was H. Poincaré. He was the first mathematician who proved in 1890's the existence and uniqueness of a solution to Dirichlet boundary value problem for Laplace equation (1.6) with any continuous boundary data (see H. Poincaré [65]). He did it using an iterative method, a maximum principle and the Harnack inequality. Poincaré's work presented many others deep theoretical results. Let us mention two of them. Firstly, he and E. I. Fredholm stated the basis and initial fundamental results of spectral theory. Next, following the work of É. Picard, H. Poincaré in [64] introduced the so-called continuity method which was based on joining the studied problem with a simpler one via a one-parameter family of problems. Consequently, this idea led later to the invention of bifurcation theory, to the application of homotopy mappings and topological methods (e.g., the degree of mapping), etc.

There was another milestone on turn of 19th and 20th century when D. Hilbert presented his list of challenging open problems, the so-called Hilbert program. Besides others, he recalled the B. Riemann's problem of minimizing energy integrals. He presented in [39] an idea of taking a minimizing sequence and proving that it converges to a minimizer. B. Levy proved few years later in [54] that a general minimizing sequence is a Cauchy sequence and therefore, it converges in an appropriate completion space to a generalized function. Consequently, this idea and the development of Lebesgue integral together with Lebesgue spaces later led to the definition of Sobolev spaces.

In the first half of the 20th century, together with the study of fundamentals of functional analysis, PDEs were reformulated as abstract operator equations on appropriate function spaces. This brought a new way of thinking about differential equations. Many famous results are based on the work of S. Banach and his colleagues. He systematically studied linear functional analysis on which many famous tools rely (Fredholm alternative, Lax-Milgram theorem, etc.). Importantly, he proved in [4] in 1920's the famous contraction principle by generalizing the É. Picard's idea of successive approximations to a general complete metric space. This essential result still has been applied for the direct proofs of existence and uniqueness but also for the proofs of other functional-analytic tools as inverse and implicit function theorems, bifurcation theorems, etc.

From another point of view, in the first half of 20th century two main approaches (which naturally complemented each other) took attention – variational and topological methods. The above mentioned Dirichlet principle unsuccessfully studied by B. Riemann and recalled by D. Hilbert was a great motivation for the detailed study of variational methods (i.e., finding conditions under which the energy integrals have minima or other types of critical points). From this field, besides the problem of minimization, we point out the work of L. A. Ljusternik and L. G. Schnirelmann who provided in [55] in 1930's the lower bound for the number of critical points for a functional defined on finite-dimensional manifold. They used a minimax technique together with a topological instrument named later the Ljusternik-Schnirelmann category. M. Morse analyzed before in 1920's the type of critical points and developed the so-called Morse theory (see M. Morse [59]). The restriction to finite-dimensional problems (some compactness was needed) was the big disadvantage of these principles. This was resolved later in the 1960's by R. Palais and S. Smale in [63] who, instead of finite dimension, presented a compactness condition on choosing convergent subsequences which was applicable even in infinite-dimensional spaces and which is now named after these two great mathematicians. These works were followed later in 1970's by A. Ambrosetti and P. Rabinowitz who expanded the minimax technique in the simpler geometrical setting and proved famous nonlinear tools – the mountain pass theorem in [2], saddle point theorem in [68] and linking theorem in [67].

In topological methods, the generalization of the Brouwer topological degree and the Brouwer fixed point theorem (invented by L. E. J. Brouwer on the beginning of 20th century) into infinite-dimensional spaces was one of the most significant turning points. It was done in 1930's by J. Leray and J. Schauder in [52] using the compactness of appropriate operators. These principles provide, compared to the S. Banach's contraction principle, only the existence of a solution. Therefore, it can be applied for different class of problems for which, e.g., the corresponding operators are not contractive. The idea of Leray-Schauder degree of mapping was also many times generalized. Let us mention the work of I. V. Skrypnik [72] from the end of 1980's who established the notion of degree also for operators that does not map a Banach space into itself, but a Banach space into its dual space. It provides a possible application of this topological argument for a class of quasilinear equations.

As we mentioned above, the development of Lebesgue integral in 1900's by H. Lebesgue in [51] and later introduction of Sobolev spaces in 1930's were another important moments. Motivated, e.g., by the convergence of minimizing sequence of Dirichlet integral studied by B. Levy, mathematicians came with the definition of a solution in a generalized sense – so-called weak solution. This started a new period of solvability theory. From that moment, standard proofs of existence have consisted of two steps – firstly, one shows the existence of a weak solution and then a regularity result which analyses the smoothness of the weak solution, in the best way, if the weak solution is also classical.[8]

In the theory of initial-boundary value problems for evolutionary PDEs (e.g., wave or diffusion equation), there is another interesting and useful tool – the semigroup theory. It was established in the middle of 20th century generally for linear operator initial value problems on Banach spaces independently by E. Hille in [42] and K. Yosida in [89]. This is a very efficient tool for proving the existence and interesting properties of solutions (e.g., the existence of backward solutions). The semigroup theory was also generalized for nonlinear problems, usually via the concept of the so-called mild solution.

We conclude this historical survey by the remark about maximum principles and a priori bounds. We mentioned that a maximum principle was one of the key ingredients in H. Poincaré's first proof of existence for the Laplace equation. A maximum principle for these problems was firstly introduced in the form of the so-called Harnack inequality by A. Harnack in the end of 1880's.[9] This was later followed by A. Paraf, É. Picard, H. Hopf and others who established the classical notion of maximum principle for

[8]Interestingly, although the existence of a weak solution for three dimensional Navier-Stokes equations was proved by J. Leray in 1930's, the regularity and uniqueness for this problem has been still unsolved and become one of the millennium open problems.

[9]Let us mention that a version of the Harnack inequality was applied on the beginning of 21st century by G. Perelman in the proof of Poincaré's conjecture, the first and only millennium problem which has been solved.

elliptic PDEs. These assertions were later applied, e.g., by E. DeGiorgi in [25] in the proof of regularity for general elliptic problems with measurable data and by J. F. Nash in [61] for parabolic problems. These principles present one of the fundamental tools in the theory of PDEs.

1.5 Continuous conservation laws, constitutive relations

Conservation laws are a natural way how to derive mathematical models of a real system. These are equalities describing balances of state and flux magnitudes during the time progress. For example, there are well-known balance laws in physics – conservation of mass, energy, the first law of thermodynamics, etc. However, conservation laws appear also in population dynamics since the change of population size is influenced only by sources (the birth rate, mortality) and by migration. Basically, some version of conservation law occurs in many other sciences. Let us recall the derivation of standard continuous conservation law (we refer to P. Drábek, G. Holubová [28] and J. D. Logan [56]).

We start with integral version of balance. We model the motion of fluid in an open region $\Omega \in \mathbb{R}^3$. Let us denote by $u(\mathbf{x},t)$ the density of fluid at position $\mathbf{x} = (x,y,z)$ and time t and consider an arbitrary open bounded set $D \subset \Omega$ with a smooth boundary. If we compare the amount of the fluid in D at time $t = t_1$ and $t = t_2$, $t_1 < t_2$, we obtain the difference[10]

$$\int_D u(\mathbf{x},t_2)\mathrm{d}\mathbf{x} - \int_D u(\mathbf{x},t_1)\mathrm{d}\mathbf{x}.$$

This change can be influenced only by the flux of fluid through the boundary ∂D and by internal sources. Let $\boldsymbol{\phi}(\mathbf{x},t)$ denote the density of flux and $f(\mathbf{x},t)$ the density of sources at position $\mathbf{x}$ and time t. Total amount of fluid which passes through the boundary ∂D out of D during the time interval $[t_1,t_2]$ is given by the following integral

$$\int_{t_1}^{t_2} \int_{\partial D} \boldsymbol{\phi}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x})\mathrm{d}S\mathrm{d}t,$$

where $\mathbf{n}(\mathbf{x})$ is the unit vector of outer normal to ∂D. Total contribution of sources in D is given by

$$\int_{t_1}^{t_2} \int_D f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t.$$

Consequently, we obtain the above mentioned balance as

$$\int_D u(\mathbf{x},t_2)\mathrm{d}\mathbf{x} - \int_D u(\mathbf{x},t_1)\mathrm{d}\mathbf{x} = -\int_{t_1}^{t_2} \int_{\partial D} \boldsymbol{\phi}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x})\mathrm{d}S\mathrm{d}t + \int_{t_1}^{t_2} \int_D f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t, \tag{1.7}$$

which is called the integral form of conservation law.

Now we formulate some additional assumptions on the functions $u(\mathbf{x},t)$ and $\boldsymbol{\phi}(\mathbf{x},t)$ to obtain the required differential form of conservation law. Let $u(\mathbf{x},t)$ be continuously differentiable with respect to the time variable t. Then using the fundamental theorem of calculus we get

$$u(\mathbf{x},t_2) - u(\mathbf{x},t_1) = \int_{t_1}^{t_2} \frac{\partial u(\mathbf{x},t)}{\partial t}\mathrm{d}t,$$

and hence, we rewrite (1.7) as

$$\int_D \int_{t_1}^{t_2} \frac{\partial u(\mathbf{x},t)}{\partial t}\mathrm{d}t\mathrm{d}\mathbf{x} = -\int_{t_1}^{t_2} \int_{\partial D} \boldsymbol{\phi}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x})\mathrm{d}S\mathrm{d}t + \int_{t_1}^{t_2} \int_D f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t. \tag{1.8}$$

[10] We assume in the following that all mathematical formulas make sense, i.e., the appearing functions satisfy all appropriate assumptions.

Applying the Fubini theorem we can interchange the order of integration on the left-hand side of (1.8) provided the mentioned integral exists,

$$\int_{t_1}^{t_2}\int_D \frac{\partial u(\mathbf{x},t)}{\partial t}\mathrm{d}\mathbf{x}\mathrm{d}t = -\int_{t_1}^{t_2}\int_{\partial D}\boldsymbol{\phi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})\mathrm{d}S\mathrm{d}t + \int_{t_1}^{t_2}\int_D f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t. \tag{1.9}$$

If we assume moreover that the flux density $\boldsymbol{\phi}(\mathbf{x},t)$ is continuously differentiable with respect to the space variable $\mathbf{x}$, then we can apply the divergence theorem (also known as the Gauss-Ostrogradskii theorem) for the surface integral

$$\int_{\partial D}\boldsymbol{\phi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})\mathrm{d}S = \int_D \mathrm{div}\boldsymbol{\phi}(\mathbf{x},t)\mathrm{d}\mathbf{x},$$

where $\mathrm{div}\boldsymbol{\phi}(\mathbf{x},t) = \nabla\cdot\boldsymbol{\phi}(\mathbf{x},t) = \frac{\partial\phi_1(\mathbf{x},t)}{\partial x} + \frac{\partial\phi_2(\mathbf{x},t)}{\partial y} + \frac{\partial\phi_3(\mathbf{x},t)}{\partial z}$ is the divergence operator. Therefore, we can rewrite (1.9) into

$$\int_{t_1}^{t_2}\int_D \left(\frac{\partial u(\mathbf{x},t)}{\partial t} + \mathrm{div}\boldsymbol{\phi}(\mathbf{x},t) - f(\mathbf{x},t)\right)\mathrm{d}\mathbf{x}\mathrm{d}t = 0.$$

Since this equality has to be satisfied for all time intervals $[t_1,t_2]$ and all choices of $D\subset\Omega$ and assuming that the integrated function is continuous we obtain that the integrand has to vanish everywhere, i.e.,

$$\frac{\partial u(\mathbf{x},t)}{\partial t} + \mathrm{div}\boldsymbol{\phi}(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x}\in\Omega, \quad t\in[t_1,t_2]. \tag{1.10}$$

This is called the differential form of conservation law and it is the origin of many PDEs. The analogue of (1.10) in one spatial dimension has the form

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial\phi(x,t)}{\partial x} = f(x,t), \quad x\in I\subset\mathbb{R}, \quad t\in[t_1,t_2], \tag{1.11}$$

and can be obtained in the same way as (1.10).

Let us note that integral form (1.7) can be satisfied under weaker assumptions on the functions $u(\mathbf{x},t)$, $\boldsymbol{\phi}(\mathbf{x},t)$ and $f(\mathbf{x},t)$ than the equality (1.10) (or (1.11)), since there suffices if they are only integrable, i.e., they do not have to be even continuous.

There appear two unknown functions $u(\mathbf{x},t)$ and $\boldsymbol{\phi}(\mathbf{x},t)$ in the equation (1.10) (or (1.11)). Hence, we have to establish an additional coupling relation. First possibility is to add a so-called constitutive law for $u(\mathbf{x},t)$ and $\boldsymbol{\phi}(\mathbf{x},t)$ dependently on the modeled problem. Problems studied in this thesis (transport equations, diffusion equations, reaction-diffussion equations) arise actually from the constitutive laws. Second possibility is to add another conservation law for the flux magnitude (e.g., the Navier-Stokes equations for the fluid dynamics use the conservation of momentum).

Since we proceed via the former way, let us present basic constitutive laws. For the sake of clarity, we use the index notation for partial derivatives, i.e., $u_t(x,t)$ for $\frac{\partial u(\mathbf{x},t)}{\partial t}$, $u_x(\mathbf{x},t)$ for $\frac{\partial u(\mathbf{x},t)}{\partial x}$, $u_{xx}(\mathbf{x},t)$ for $\frac{\partial^2 u(\mathbf{x},t)}{\partial x^2}$, etc.

Example 1.7. Consider the one-dimensional conservation law (1.11) and let the flux $\phi(x,t)$ depend linearly on the density $u(x,t)$, i.e.,

$$\phi(x,t) = ku(x,t), \quad k>0. \tag{1.12}$$

Consequently, we get from (1.11) the transport (or advection) equation

$$u_t(x,t) + ku_x(x,t) = f(x,t), \tag{1.13}$$

which describes, e.g., transport of chemicals dissolved in a fluid which runs through a tube (neglecting the diffusion process).

Example 1.8. The analogy of transport equation in three spatial dimensions (and similarly in two dimensions) can be obtained assuming

$$\boldsymbol{\phi}(\mathbf{x},t) = \mathbf{v}u(\mathbf{x},t),$$

where $\mathbf{v} \in \mathbb{R}^3$ describes the direction and velocity of motion. Then three dimensional transport equation follows from (1.10) as

$$u_t(\mathbf{x},t) + v_1 u_x(\mathbf{x},t) + v_2 u_y(\mathbf{x},t) + v_3 u_z(\mathbf{x},t) = f(\mathbf{x},t).$$

Note that many generalizations can be obtained for transport equations. One can suppose that the parameter k (or $\mathbf{v}$ in three dimensions) also depends on x and t and investigate the equation with variable coefficients. Furthermore, one can consider that k is dependent on the density $u(x,t)$ or more generally, that the flux is a nonlinear function density, i.e., $\phi(x,t) = \phi\left(u(x,t)\right)$, $\phi : \mathbb{R} \to \mathbb{R}$,[11] and analyze nonlinear transport equations (see, e.g., J. D. Logan [56] for more information).

Example 1.9. Let us derive the diffusion equation as an example of equation of second order. Again, consider the problem in one spatial dimension firstly. The constitutive law coupling $u(x,t)$ with $\phi(x,t)$ in this case is the so-called Fick law

$$\phi(x,t) = -k u_x(x,t), \quad k > 0, \tag{1.14}$$

which means that the flux is larger at points where the gradient of density is larger and it is directed oppositely to the gradient. This occurs, e.g., in the heat transfer, movement of micro-particles into others, etc. We obtain from (1.11) the one-dimensional diffusion equation

$$u_t(x,t) = k u_{xx}(x,t) + f(x,t). \tag{1.15}$$

Example 1.10. For the diffusion equation in three dimensions, we use the multi-dimensional version of the Fick law

$$\boldsymbol{\phi}(\mathbf{x},t) = -k\nabla u(\mathbf{x},t), \quad k > 0,$$

and applying the definition of the Laplace operator

$$\Delta u(\mathbf{x},t) = u_{xx}(\mathbf{x},t) + u_{yy}(\mathbf{x},t) + u_{zz}(\mathbf{x},t) = \operatorname{div}\left(\nabla u(\mathbf{x},t)\right),$$

we obtain from (1.10)

$$u_t(\mathbf{x},t) = k\Delta u(\mathbf{x},t) + f(\mathbf{x},t),$$

the diffusion equation in three spatial dimensions.

Let us mention also a possible nonlinear generalization of diffusion equation since it is of our great interest – the reaction-diffusion equations, regarding the case when the source function f depends also on the density $u(x,t)$.

Example 1.11. One could verify that the derivation of conservation law works even when the source function f appearing on the right-hand side of conservation laws (1.10) and (1.11) is also a function of density $u(x,t)$, i.e., $f = f(x,t,u(x,t))$ (e.g., the birth rate in population dynamics). Then applying the Fick law we obtain one-dimensional RDE

[11] We denote the nonlinear function also by ϕ as the flux itself, because we use this notation also in Section 2.2 as well as in Conclusion where we introduce problems involving the so-called ϕ-Laplacian. However, at this place it is formally incorrect.

$$u_t(x,t) = ku_{xx}(x,t) + f(x,t,u(x,t)), \tag{1.16}$$

or its three-dimensional analogy

$$u_t(\mathbf{x},t) = k\Delta u(\mathbf{x},t) + f(\mathbf{x},t,u(\mathbf{x},t)),$$

which are generally nonlinear problems provided f is nonlinear in the third variable.

Let us conclude the section with the stationary counterpart of diffusion equation – the Poisson equation (nonhomogeneous version of the Laplace equation).

Example 1.12. Suppose for the simplicity that the source function is not a function of time, i.e., $f = f(x)$ for the diffusion equation or $f = f(x, u(x,t))$ for RDE. Further, denote $g(x) = \frac{1}{k}f(x)$ or $g(x,u(x,t)) = \frac{1}{k}f(x,u(x,t))$. Stationary solutions of the diffusion or reaction-diffusion equation are constant in time, i.e., have zero partial time derivatives. Therefore, these solutions have to satisfy

$$-u_{xx}(x) = g(x), \quad \text{or} \quad -\Delta u(\mathbf{x}) = g(\mathbf{x}),$$

in the case of diffusion equation which is called the Poisson equation, or

$$-u_{xx}(x) = g(x,u(x)), \quad \text{or} \quad -\Delta u(\mathbf{x}) = g(\mathbf{x},u(\mathbf{x})),$$

in the case of RDE which is the nonlinear Poisson equation or semilinear elliptic equation.[12]

1.6 Semidiscrete conservation laws

In this section we present a possible way how to establish a conservation law for equations with discrete spatial variable $x \in \mathbb{Z}$. Hence, we derive the discrete-space counterpart for the equality (1.11).

Note that in future research we want to analyze also problems considered generally on graphs. As a motivation, we present conservation laws for equations with discrete variable being from a directed graph in Conclusion (see Section 5.3).

Let us assume that $x \in \mathbb{Z}$ and $t \in [t_1, t_2]$. Recall that we call problems with this structure of variables semidiscrete equations. Denote by $u(x,t)$ the amount of modeled substance at the point x and time t (for example, $[a,b] \cap \mathbb{Z}$ models a line segment of cells, e.g., neurons, see J. Bell [6], J. Bell, C. Cosner [7]).

Analogically as in the continuous case, we investigate the total change of amount of u at one fixed discrete point $x \in \mathbb{Z}$ during time interval $t \in [t_1, t_2]$ [13]

$$u(x,t_2) - u(x,t_1).$$

This change is again influenced only by the total amount of u that passes into and out of $x \in \mathbb{Z}$ during $t \in [t_1, t_2]$ and by total supply of internal sources. Let us denote by $\phi(x,t)$ the flux quantifying the amount of u that passes between points x and $x+1$ at time t,[14] and by $f = f(x,t)$ the source function expressing the production of sources at x and at time t.

Therefore, we obtain the following equality establishing the above mentioned balance

$$u(x,t_2) - u(x,t_1) = \int_{t_1}^{t_2} \left(\phi(x-1,t) - \phi(x,t)\right) \mathrm{d}t + \int_{t_1}^{t_2} f(x,t)\mathrm{d}t. \tag{1.17}$$

[12] In one-spatial dimension, the stationary problems are obviously ODEs.

[13] Note that the discrete spatial structure allows us to formulate the balance immediately locally at one point $x \in \mathbb{Z}$.

[14] Correctly, we should denote the flux between the point x and its neighbor $x+1$ by $\phi(x, x+1, t)$ because for a general graph the flux function acts on the set of edges of the graph. However, in the case of $\mathbb{Z}$, which can be interpreted as a simple undirected graph (a path), each edge has the form $\{x, x+1\}$. Therefore, we use the shortened notation $\phi(x,t)$ for the brevity.

Let us assume that $u(x,t)$ is continuously differentiable with respect to time variable t, then

$$u(x,t_2) - u(x,t_1) = \int_{t_1}^{t_2} u_t(x,t)\mathrm{d}t.$$

Consequently, we get from (1.17) (denoting by $\nabla_x \phi(x,t) = \phi(x,t) - \phi(x-1,t)$ the left spatial difference)

$$\int_{t_1}^{t_2} \left(u_t(x,t) + \nabla_x \phi(x,t) - f(x,t)\right) \mathrm{d}t = 0.$$

Since $[t_1,t_2]$ is arbitrary and assuming that the integrand is continuous with respect to the variable t, we obtain the semidiscrete conservation law

$$u_t(x,t) + \nabla_x \phi(x,t) = f(x,t), \quad x \in \mathbb{Z}, \quad t \in [t_1,t_2]. \tag{1.18}$$

Finally, we have to add to (1.18) a constitutive law relating the flux $\phi(x,t)$ with $u(x,t)$ again. In the following chapters of the thesis, we investigate two versions of constitutive laws and, consequently, two arising problems – lattice transport equations and lattice RDEs.

CHAPTER 2

Transport equations

This section is devoted to transport equations on discrete-space domains. Continuous linear homogeneous transport PDE in one spatial dimension

$$u_t(x,t) + k u_x(x,t) = 0, \quad k > 0,$$

arises from the differential form of conservation law (1.11) applying the constitutive relation $\phi(x,t) = ku(x,t)$ and the vanishing source density function $f(x,t) = 0$ (see Example 1.7). It is well-known (see, e.g., P. Drábek, G. Holubová [28] or J. D. Logan [56]) that the solution of an initial value problem for this equation with $u(x,0) = \varphi(x)$, $\varphi \in C^1(\mathbb{R},\mathbb{R})$, has the form of the so-called traveling wave

$$u(x,t) = \varphi(x - kt).$$

This means that the initial distribution of the density propagates along the characteristic lines $x - kt = c$, $c \in \mathbb{R}$. Generally, assuming that k is not a constant but a continuous function $k = k(x,t)$, the solution of the initial value problem has also the form of traveling wave provided it exists. However, the characteristics become curves in the xt-plane, not the straight lines.

The nonlinear transport PDE follows from the conservation law (1.11) if we suppose that the flux is a nonlinear function of density $\phi(x,t) = \phi(u(x,t))$, $\phi \in C^1(\mathbb{R},\mathbb{R})$.[1] These models have essentially different behavior than the linear ones. They can lead to interesting (but in some sense undesirable) effects like shocks, etc. (see J. D. Logan [56]).

Importance of transport equations arises from the following facts. Firstly, they are usually applied for modeling of advection of some fluid in one spatial direction. Secondly, from the theoretical point of view it is the first step in the study of hyperbolic PDEs since, e.g., the simplest linear wave equation

$$u_{tt}(x,t) - k^2 u_{xx}(x,t) = 0, \quad k > 0,$$

can be reformulated as a system of two transport equations (see P. Drábek, G. Holubová [28] or J. D. Logan [56] again).

[1] We denote the nonlinear function in the constitutive law formally incorrectly by ϕ as the flux itself.

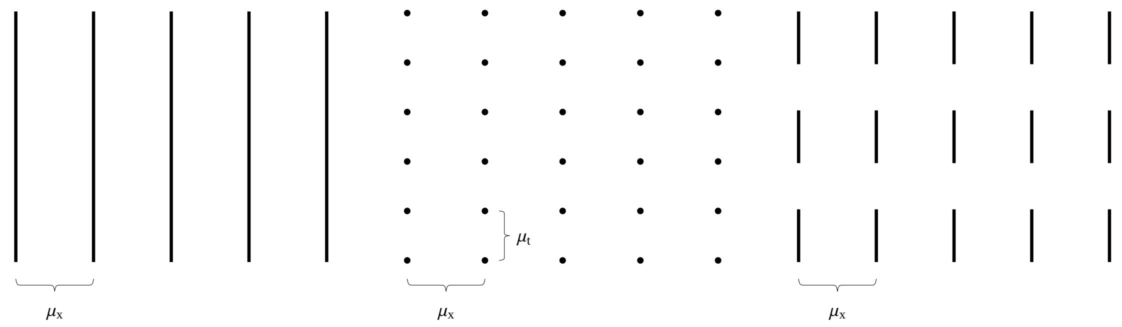

Figure 2.1: Examples of various discrete-space and general (continuous, discrete and time scale) time domains.

2.1 Linear transport equations

In this section we analyze the linear homogeneous transport equation with $x \in \mathbb{Z}$ and continuous, discrete or time scale time structure (see Figure 2.1). It is an overview of the results of the paper P. Stehlík, J. Volek [79].

The problems with continuous time variable follow from the semidiscrete conservation law (1.18) assuming that the flux $\phi(x,t)$ is directed from the point x to $x+1$ and depends linearly on the density $u(x,t)$ at the point x, i.e.,

$$\phi(x,t) = ku(x,t), \quad k > 0.$$

Then (1.18) yields the semidiscrete equation

$$u_t(x,t) + k\nabla_x u(x,t) = 0, \tag{2.1}$$

where $\nabla_x u(x,t) = u(x,t) - u(x-1,t)$. The other two considered time structures, discrete time or time scale time, are obtained interchanging the partial time derivative in (2.1) by partial time difference or partial time Δ-derivative, respectively.

We firstly find closed formulas of solutions for these problems. Then we show their relationship with stochastic processes, since we study the sign and integral/sum preservation. Particularly, we prove that the solutions of transport equations on discrete-space domains form Poisson-Bernoulli-type counting processes which are widely applied for modeling of the waiting time for occurence of events (e.g., defects, phone calls, customers' arrivals).

Before we start with our analysis let us summarize, for the comparison, the essential properties of the classical transport PDE

$$\begin{cases} u_t(x,t) + ku_x(x,t) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \quad k \in \mathbb{R}, \\ u(x,0) = \varphi(x), \end{cases} \tag{2.2}$$

with $\varphi \in C^1(\mathbb{R},\mathbb{R})$. The properties whose counterparts we study in the following are:

- the unique solution of (2.2) is given by $u(x,t) = \varphi(x - kt)$,
- if $\varphi(x) \geq 0$, then $u(x,t) \geq 0$ for all $x \in \mathbb{R}$, $t \in \mathbb{R}_0^+$, i.e., the solution $u(x,t)$ preserves sign,

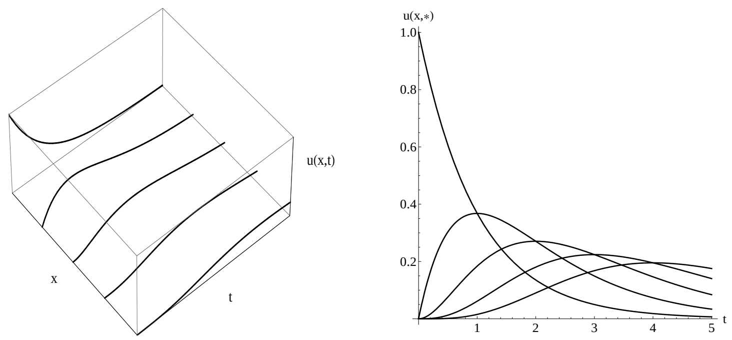

Figure 2.2: Solution (2.4) of the transport equation with discrete space and continuous time variable (2.3) with $A = k = 1$.

- if $\int_{-\infty}^{\infty} \varphi(x)\mathrm{d}x = K$, then

$$\int_{-\infty}^{\infty} u(x,t)\mathrm{d}x = K \quad \text{for all} \quad t \in \mathbb{R}_0^+,$$

  i.e., the solution preserves integral in spatial sections,

- for $k > 0$ and fixed $x \in \mathbb{Z}$ we get

$$\int_0^{\infty} u(x,t)\mathrm{d}t = \frac{1}{k}\int_{-\infty}^{x} \varphi(s)\mathrm{d}s.$$

  and consequently, if $\varphi(x) = 0$ for $x \geq x_0$, then the integral along time sections is constant for all $x \geq x_0$.

2.1.1 Semidiscrete equation

Consider the linear transport equation with discrete space and continuous time[2,3]

$$\begin{cases} u_t(x,t) + k\nabla_x u(x,t) = 0, \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}^+, \quad k > 0, \\ u(x,0) = \begin{cases} A > 0, \quad x = 0, \\ 0, \quad x \neq 0. \end{cases} \end{cases} \tag{2.3}$$

Using variation of constants and mathematical induction we could obtain that the unique locally bounded solution of (2.3) is given by (see P. Stehlík, J. Volek [79, Lem. 4.1])

$$u(x,t) = \begin{cases} A\dfrac{k^x}{x!}t^x e^{-kt}, \quad x \in \mathbb{Z}, \quad x \geq 0, \quad t \in \mathbb{R}_0^+, \\ 0, \quad x \in \mathbb{Z}, \quad x < 0, \quad t \in \mathbb{R}_0^+, \end{cases} \tag{2.4}$$

(see Figure 2.2).

Now, applying the explicit formula for the solution $u(x,t)$ one can prove the following statement about the sign and integral/sum preservation (P. Stehlík, J. Volek [79, Lem. 4.2]).

[2]We solve the one-point initial condition at first, since later one can use the superposition principle (see A. Slavík, P. Stehlík [76, Cor. 3.8]) for a general initial condition.

[3]For the sake of brevity, we assume that the spatial discretization step $\mu_x = 1$ but all our results are valid for arbitrary $\mu_x > 0$ by considering $\bar{k} = \frac{k}{\mu_x}$ instead of k in (2.3).

Lemma 2.1. *Let $u(x,t)$ be the solution of* (2.3) *given by* (2.4). *Then:*

- $u(x,t) \geq 0$ *for all* $x \in \mathbb{Z}$ *and* $t \in \mathbb{R}_0^+$,
- $\int_0^\infty u(x,t)\mathrm{d}t = \frac{A}{k}$ *for all* $x \in \mathbb{Z}$,
- $\sum\limits_{x=0}^{\infty} u(x,t) = A$ *for all* $t \in \mathbb{R}_0^+$.

The following consequences give the first relationship of our problems with the stochastic processes and follow immediately from the detailed analysis of parameters A and k in Lemma 2.1 (see Figure 2.2 again):

- If $A = k$, then the time sections $u(x,\cdot)$ of solution (2.4) generate the probability density functions of the Erlang distribution which is the special case of the Gamma distribution (for $x = 0$ we get the exponential distribution).
- If $A = 1$, then the spatial sections $u(\cdot,t)$ of solution (2.4) form the probability mass functions of the Poisson distribution.
- Consequently, if $A = k = 1$, then the solution $u(x,t)$ describes the Poisson stochastic process.

We conclude with the corollary about the solution of the general initial value problem

$$\begin{cases} u_t(x,t) + k\nabla_x u(x,t) = 0, \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}^+, \quad k > 0, \\ u(x,0) = \varphi(x), \end{cases} \tag{2.5}$$

where $\varphi : \mathbb{Z} \to \mathbb{R}$ is bounded. Using (2.4) and the superposition principle (see A. Slavík, P. Stehlík [76, Cor. 3.8]) we get the following assertion.

Corollary 2.2. *The unique locally bounded solution of* (2.5) *is given by*

$$u(x,t) = \sum_{i=-\infty}^{x} \varphi(i) \frac{(kt)^{x-i}}{(x-i)!} e^{-kt}. \tag{2.6}$$

2.1.2 Discrete equation

In this paragraph we study the properties of transport difference equation, i.e., of the problem with discrete spatial and discrete time variable (for a general survey about partial difference equations see, e.g., the monograph S. S. Cheng [19]),

$$\begin{cases} \Delta_t u(x,t) + k\nabla_x u(x,t) = 0, \quad x \in \mu_x\mathbb{Z}, \quad t \in \mu_t\mathbb{N}_0, \\ u(x,0) = \begin{cases} A, & x = 0, \\ 0, & x \neq 0, \end{cases} \end{cases} \tag{2.7}$$

where $\mu_x, \mu_t > 0$, $\mu_x\mathbb{Z} = \{m\mu_x : m \in \mathbb{Z}\}$, $\mu_t\mathbb{N}_0 = \{n\mu_t : n \in \mathbb{N}_0\}$, $A > 0$, $k > 0$ and the partial differences are defined as

$$\Delta_t u(x,t) = \frac{u(x,t+\mu_t) - u(x,t)}{\mu_t} \quad \text{and} \quad \nabla_x u(x,t) = \frac{u(x,t) - u(x-\mu_x,t)}{\mu_x}. \tag{2.8}$$

We can easily rewrite the equation in (2.7) into

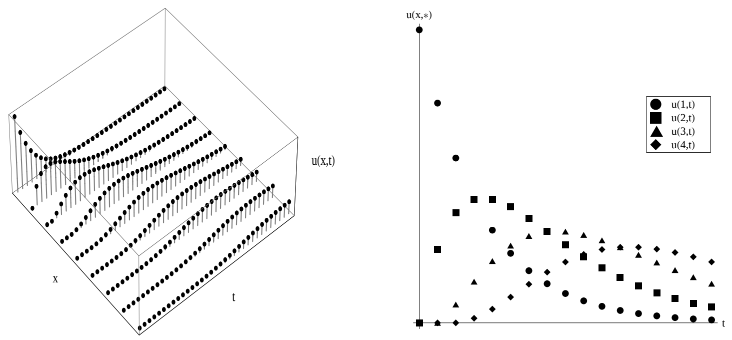

Figure 2.3: Solution (2.9) of the transport equation with discrete space and discrete time variable (2.7) with $A = k = 1$, $\mu_t = \frac{1}{4}$ and $\mu_x = 1$.

$$u(x, t + \mu_t) = \left(1 - \frac{k\mu_t}{\mu_x}\right) u(x, t) + \frac{k\mu_t}{\mu_x} u(x - \mu_x, t),$$

and using the mathematical induction and properties of the so-called falling factorials (see, e.g., W. G. Kelley, A. C. Peterson [46]) derive that the unique solution of (2.7) is given by (see P. Stehlík, J. Volek [79, Lem. 5.1])

$$u(m\mu_x, n\mu_t) = \begin{cases} A\binom{n}{m}\left(1 - \frac{k\mu_t}{\mu_x}\right)^{n-m}\left(\frac{k\mu_t}{\mu_x}\right)^m, & n \geq m \geq 0, \\ 0, \quad 0 \leq n < m \quad \text{or} \quad m < 0, \end{cases} \tag{2.9}$$

(see Figure 2.3).

From the explicit formula (2.9) one can prove the following lemma about the sign a sum preservation (P. Stehlík, J. Volek [79, Lem. 5.2]).

Lemma 2.3. *Assume that*

$$1 - \frac{k\mu_t}{\mu_x} > 0. \tag{2.10}$$

Then the unique solution $u(x, t)$ of (2.7) satisfies:

- $u(x, t) \geq 0$ *for all* $x \in \mu_x\mathbb{Z}$ *and* $t \in \mu_t\mathbb{N}_0$,
- $\sum_{n=0}^{\infty} u(x, n\mu_t) = \frac{A\mu_x}{k\mu_t}$ *for all* $x \in \mu_x\mathbb{Z}$,
- $\sum_{m=-\infty}^{\infty} u(m\mu_x, t) = A$ *for all* $t \in \mu_t\mathbb{N}_0$.

Again, analyzing Lemma 2.3 in detail we deduce conditions under which the solution $u(x, t)$ of (2.7) forms probability distributions (see P. Stehlík, J. Volek [79, Cor. 5.4]).

Corollary 2.4. *Let $u(x, t)$ be the unique solution of (2.7). Then the spatial and time sections $\mu_x u(\cdot, t)$ and $\mu_t u(x, \cdot)$ form probability mass functions if and only if $k = 1$, $\mu_t < \mu_x$ and $A = \frac{1}{\mu_x}$.*

	$u(\cdot,t)$	$u(0,\cdot)$	$u(x,\cdot), x \geq 0$	$u(x,t)$
$\mathbb{Z} \times \mathbb{R}_0^+$	Poisson dist.	exponential dist.	Erlang (Gamma) dist.	Poisson process
$\mathbb{Z} \times p\mathbb{N}_0$	binomial dist.	geometric dist.	negative binomial dist.	Bernoulli process

Table 2.1: Correspondence of time and spatial sections of solution $u(x,t)$ with probability distributions.

Particularly, if we put $A = k = \mu_x = 1$ and $\mu_t = p \in (0,1)$ then

$$u(m,np) = \binom{n}{m}(1-p)^{n-m}p^m, \quad n \geq m,$$

and therefore (see Figure 2.3 again):

- For each fixed $n \in \mathbb{N}_0$, $u(\cdot, np)$ forms a probability mass function of the binomial distribution.
- For each fixed $m \in \mathbb{N}_0$, $pu(m,\cdot p)$ forms a probability mass function of the negative binomial distribution.
- Consequently, the solution $u(x,t)$ describes the Bernoulli stochastic process.

The relationship of transport equation with discrete spatial and continuous/discrete time variable with the probability distributions and moreover, with the stochastic processes is summarized in Table 2.1.

2.1.3 Dynamic equation

In this part we generalize the results from the previous Subsections 2.1.1 and 2.1.2 for the time variable being from a general time scale $\mathbb{T}$ (note that for $\mathbb{T} = \mathbb{R}_0^+$ we get continuous case (2.3) and for $\mathbb{T} = \mu_t\mathbb{N}_0$ the discrete case(2.7)).

Consider the following problem[4]

$$\begin{cases} u^\Delta(x,t) + k\nabla_x u(x,t) = 0, & x \in \mu_x\mathbb{Z}, \quad t \in [0,\infty)_\mathbb{T}, \\ u(x,0) = \begin{cases} A, & x = 0, \\ 0, & x \neq 0, \end{cases} \end{cases} \tag{2.11}$$

where $\mathbb{T}$ is a time scale such that $0 \in \mathbb{T}$, $[0,\infty)_\mathbb{T} = [0,\infty) \cap \mathbb{T}$, u^Δ is the Δ-derivative with respect to the variable t, $\nabla_x u(x,t)$ is the backward spatial difference defined in (2.8) and $A > 0$, $k > 0$. We rewrite the equation in (2.11) into

$$u^\Delta(x,t) = -\frac{k}{\mu_x}\left(u(x,t) - u(x-\mu_x,t)\right). \tag{2.12}$$

Let us state the positive regressivity condition

$$1 - \frac{k\mu_t(t)}{\mu_x} > 0, \tag{2.13}$$

which is important for the positivity of time scale exponential function $e_{-\frac{k}{\mu_x}}(t,0)$.[5]

[4] We present results for one-point initial condition again, since for a general initial value problem the superposition principle A. Slavík, P. Stehlík [76, Cor. 3.8] also holds.

[5] We denote by $e_p(t,t_0)$ the time scale exponential function defined as a unique solution of the initial value problem

$$\begin{cases} v^\Delta(t) = p(t)v(t), \\ v(t_0) = 1, \end{cases}$$

(see, e.g., M. Bohner, A. C. Peterson [13]).

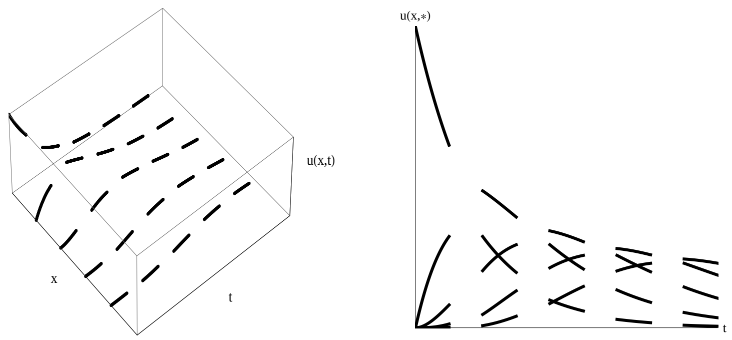

Figure 2.4: Solution of the transport equation with discrete spatial variable and time variable being from a time scale (2.11) with $A = k = 1$, $\mu_x = 1$ and $\mathbb{T} = \bigcup\limits_{i=0}^{\infty} \left[i, i + \frac{1}{2}\right]$.

One can prove that $u(x, \cdot) \equiv 0$ is the unique locally bounded solution for all $x \in \mathbb{Z}$, $x < 0$. Assuming $u(-\mu_x, \cdot) \equiv 0$, the equality (2.12) yields that $u^{\Delta}(0, t) = -\frac{k}{\mu_x} u(0, t)$. Immediately, we obtain

$$u(0, t) = Ae_{-\frac{k}{\mu_x}}(t, 0).$$

Then we can continue inductively via the time scale variation of constants (see Figure 2.4). Since these computations depend strongly on particular time scale we cannot derive the closed form of the unique locally bounded solution at all $x \in \mathbb{Z}$, $x > 0$, on general time scale $\mathbb{T}$. A. Slavík and P. Stehlík later derived in [75] the abstract infinite series representation of the bounded solution

$$u(x, t) = \sum_{i=0}^{\infty} \binom{i}{x} (-1)^{i+x} k^i h_i(t, 0),$$

where $h_i(t, t_0)$ are the time scale polynomials given recursively

$$h_{i+1}(t, t_0) = \int_{t_0}^{t} h_i(\tau, t_0) \Delta\tau, \quad i \in \mathbb{N}, \qquad h_0(t, t_0) = 1.$$

Under the assumption (2.13) we can prove the following statement about the sign/integral/sum preservation (P. Stehlík, J. Volek [79, Thm. 6.3, 6.5, 6.6]).

Theorem 2.5. *Assume* (2.13) *and let* $u(x, t)$ *be the unique locally bounded solution of* (2.11). *Then:*

- $u(x, t) \geq 0$ *for all* $x \in \mu_x\mathbb{Z}$ *and* $t \in \mathbb{T}$,
- $\int_0^{\infty} u(x, t)\Delta t = A\frac{\mu_x}{k}$ *for all* $x \in \mu_x\mathbb{Z}$,
- $\sum\limits_{m=0}^{\infty} u(m\mu_x, t) = A$ *for all* $t \in \mathbb{T}$.

2.2 Nonlinear semidiscrete transport equation

In this section we are interested in the nonlinear semidiscrete transport equation which arises from the conservation law (1.18). We present the results from the paper J. Volek [85]. Namely, we study the following initial-boundary value problem with discrete space and continuous time[6]

$$\begin{cases} u_t(x,t) + \nabla_x \phi(x,t,u(x,t)) = 0, \quad x \in \mathbb{N}, \quad t \in \mathbb{R}_0^+, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{N}, \\ u(0,t) = \xi(t), \quad t \in \mathbb{R}_0^+, \end{cases} \tag{2.14}$$

where $\phi : \mathbb{N} \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$, $\varphi : \mathbb{N} \to \mathbb{R}$ is an initial condition, $\xi \in C^1(\mathbb{R}_0^+)$ is a left boundary condition at $x = 0$ and the left spatial difference $\nabla_x \phi(x,t,u(x,t))$ is given by[7]

$$\nabla_x \phi(x,t,u(x,t)) = \phi(x,t,u(x,t)) - \phi(x-1,t,u(x-1,t)).$$

2.2.1 Maximum principle, existence and uniqueness

In this subsection we present a maximum principle for (2.14). For a survey about maximum principles for completely continuous PDEs see, e.g., the monograph M. H. Protter, H. F. Weinberger [66]. These results are applied as a priori bounds for proving the existence and uniqueness of a global solution.

Let $N \in \mathbb{N}$, $T \in \mathbb{R}_0^+$ and define the following two constants for the brevity,

$$M_{N,T} = \max_{x \in [1,N]_{\mathbb{N}}, t \in [0,T]} \{\varphi(x), \xi(t)\}, \quad m_{N,T} = \min_{x \in [1,N]_{\mathbb{N}}, t \in [0,T]} \{\varphi(x), \xi(t)\}.$$

The following maximum principle (see J. Volek [85, Thm. 3.2, 3.3]) is based on the monotonicity properties of the nonlinear function ϕ. More precisely, we show a slightly stronger assertion than in [85]. However, the proof based on the so-called stairs method works in the same way even for this more general case.

Theorem 2.6. *Let the function ϕ be independent on x and strictly increasing in u. Assume that u is a solution of* (2.14) *and $N \in \mathbb{N}$, $T \in \mathbb{R}_0^+$ are arbitrary. Then for all $x \in [0,N]_{\mathbb{N}_0}$ and $t \in [0,T]$ there is*

$$m_{N,T} \le u(x,t) \le M_{N,T}.$$

Theorem 2.6 can be used for proving the following global existence and uniqueness result (see J. Volek [85, Thm. 4.2]) by the application of the Picard-Lindelöf theorem (see, e.g., P. Drábek, J. Milota [29, Theorem 2.3.4]).

Theorem 2.7. *Assume that:*

- *the initial-boundary conditions φ, ξ are bounded,*
- *ϕ is independent on x and strictly increasing in u,*
- *ϕ is a continuous function and locally Lipschitz in u,*

[6] Let us note that besides (2.14), there is also the following initial-boundary value problem with the difference inside the nonlinearity contained in J. Volek [85]

$$\begin{cases} u_t(x,t) + \phi(x,t,\nabla_x u(x,t)) = 0, \quad x \in \mathbb{N}, \quad t \in \mathbb{R}_0^+, \\ u(x,0) = \varphi(x), \\ u(0,t) = \xi(t), \end{cases}$$

with $\varphi : \mathbb{N} \to \mathbb{R}$ and $\xi \in C^1(\mathbb{R}_0^+)$. However, this problem does not directly arise from the conservation law (1.18). Therefore, it is omitted in this summary. The reader can see J. Volek [85, Sec. 6] for details.

[7] Again, we assume the spatial discretization step $\mu_x = 1$. However, all our results hold even for general $\mu_x > 0$ redefining the nonlinear function $\bar{\phi} = \frac{1}{\mu_x}\phi$.

- *there exist $r, q > 0$ such that for all $t \in \mathbb{R}_0^+$ and $u \in \mathbb{R}$ there is*

$$|\phi(t,u)| \le r|u| + q.$$

Then (2.14) *possesses a unique solution and it exists for all $x \in \mathbb{N}_0$ and $t \in \mathbb{R}_0^+$.*

If we analyze the proof of Theorem 2.7 in [85, Thm. 4.2] in detail we realize that the statement can be improved. We can omit the assumptions that the initial-boundary conditions are bounded and mainly, that the nonlinear function ϕ has a sublinear growth. It is possible, since the maximum principle from Theorem 2.6 can be applied better. Let us firstly present some preliminary facts from the theory of ODEs for the initial value problem

$$\begin{cases} v'(t) = g(t, v(t)), & t \in \mathbb{R}_0^+, \\ v(0) = v_0, \end{cases} \tag{2.15}$$

where $g : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}^N$ is continuous and $v_0 \in \mathbb{R}$. For the following definition and theorem we refer to W. G. Kelley, A. C. Peterson [47, Def. 8.31, Thm. 8.33].

Definition 2.8. Let g be continuous and v be a solution of (2.15) defined on $[0, \eta)$. Then we say $[0, \eta)$ is a maximal interval of existence for v if there does not exist $\eta_1 > \eta$ and a solution w defined on $[0, \eta_1)$ such that $v(t) = w(t)$ for $t \in [0, \eta)$.

Theorem 2.9. *Let g be continuous and v be a solution of* (2.15) *defined on $[0, \omega)$. Then v can be extended to a maximal interval of existence $[0, \eta)$, $0 < \eta \le +\infty$. Furthermore, there is*

$$\textit{either} \quad \eta = +\infty \quad \textit{or} \quad |v(t)| \overset{t\to\eta-}{\longrightarrow} +\infty.$$

Now we can prove the stronger result about the existence and uniqueness of a global solution for (2.14).

Theorem 2.10. *Assume that:*

- *ϕ is independent on x and strictly increasing in u,*
- *ϕ is continuous function and it is locally Lipschitz in u,*

Then (2.14) *possesses a unique solution and it exists for all $x \in \mathbb{N}_0$ and $t \in \mathbb{R}_0^+$.*

Proof. We prove the statement by induction on $x \in \mathbb{N}_0$.

- For $x = 0$ we have $u(0,t) = \xi(t)$.
- Assume that $x \in \mathbb{N}_0$ is fixed and we have the unique solution $u(x, \cdot)$ which exists for all $t \in \mathbb{R}_0^+$. Then we obtain from (2.14) that the function $u(x+1, \cdot)$ has to be the solution of the following initial value problem

$$\begin{cases} u_t(x+1,t) = -\phi(t, u(x+1,t)) + \phi(t, u(x,t)), \\ u(x+1, 0) = \varphi(x+1), \end{cases} \tag{2.16}$$

where $\phi(t, u(x,t))$ is a given function of t from the induction hypothesis. Then the assumptions on ϕ allow us to apply the Picard-Lindelöf theorem to get a uniquely defined local solution $u(x+1, \cdot)$ of (2.16). By Theorem 2.9, the solution $u(x+1, \cdot)$ can be extended to a maximal interval of existence, i.e., for $t \in [0, \eta)$, and

$$\text{either} \quad \eta = +\infty \quad \text{or} \quad |u(x+1,t)| \overset{t\to\eta-}{\longrightarrow} +\infty.$$

Since we know from the maximum principle in Theorem 2.6 that $|u(x+1,t)| \to +\infty$ for $t \to \eta-$ does not occur, there is $\eta = +\infty$, i.e., the solution $u(x+1, \cdot)$ exists for all $t \in \mathbb{R}_0^+$.

The uniqueness could be proved in a similar way as in [85, Thm. 4.2]. □

Remark 2.11. If we omit the assumption of local Lipschitz continuity of ϕ in the variable u in Theorems 2.7 and 2.10, the uniqueness is not guaranteed and one can obtain only the existence result applying the Cauchy-Peano theorem (see, e.g., P. Drábek, J. Milota [29, Prop. 5.2.7]) instead of the Picard-Lindelöf theorem.

Examples of nonlinear functions ϕ that satisfy assumptions in Theorem 2.10 could be:

- $\phi(t,u) = k(t)u$ where $k(t) > 0$ for all $t \in \mathbb{R}_0^+$ (linear equation),
- $\phi(t,u) = k(t)\arctan(u)$ where $k(t) > 0$ for all $t \in \mathbb{R}_0^+$,
- $\phi(t,u) = |u|^{p-1}u$ with $p \geq 1$.

For the following function ϕ we have only existence guaranteed (cf. Remark 2.11):

- $\phi(t,u) = |u|^{p-1}u$ with $p \in (0,1)$.

Maximum principle from Theorem 2.6 has the following two immediate consequences for boundedness and sign preservation of the solution of (2.14) (see J. Volek [85, Cor. 5.1, 5.2]).

Corollary 2.12. *Assume ϕ is independent on x and strictly increasing in u and φ, ξ are bounded. Let u be a solution of* (2.14)*. Then u is bounded.*

Corollary 2.13. *Assume ϕ is independent on x and strictly increasing in u and φ, ξ are nonnegative. Let u be a solution of* (2.14)*. Then u is nonnegative.*

2.2.2 Continuous dependence for linear problems

Another interesting consequence of the maximum principle is the uniform stability for the linear initial-boundary value problem

$$\begin{cases} u_t(x,t) + k\nabla_x u(x,t) = 0, \quad x \in \mathbb{N}, \quad t \in \mathbb{R}_0^+, \quad k > 0, \\ u(x,0) = \varphi(x), \\ u(0,t) = \xi(t), \end{cases} \tag{2.17}$$

where $\varphi : \mathbb{N} \to \mathbb{R}$ and $\xi \in C^1(\mathbb{R}_0^+)$. We can apply Theorem 2.6 to prove the following corollary (see J. Volek [85, Cor. 5.3]).

Corollary 2.14. *Let u_1, u_2 be the unique solutions of* (2.17) *with initial-boundary conditions φ_1, ξ_1 and φ_2, ξ_2, respectively. Then*

$$\sup_{x\in[1,N]_{\mathbb{N}}, t\in[0,T]} |u_1(x,t) - u_2(x,t)| \quad \leq \quad \sup_{x\in[1,N]_{\mathbb{N}}, t\in[0,T]} \{|\varphi_1(x) - \varphi_2(x)|, |\xi_1(t) - \xi_2(t)|\}.$$

Corollary 2.14 immediately implies the continuous dependence on initial-boundary conditions (see J. Volek [85, Cor. 5.4]).

Corollary 2.15. *Let $\{u_n\}_{n=1}^{+\infty}$ be a sequence of unique solutions u_n of* (2.17) *with the initial-boundary conditions φ_n, ξ_n. Let u_0 be the unique of* (2.17) *with the initial-boundary conditions φ_0, ξ_0. Assume that $\varphi_n \rightrightarrows \varphi_0$ on $\mathbb{N}$ and $\xi_n \rightrightarrows \xi_0$ on $\mathbb{R}_0^+$. Then*

$$u_n \rightrightarrows u_0 \quad on \quad \mathbb{N}_0 \times \mathbb{R}_0^+.$$

CHAPTER 3

Reaction-diffusion equations

Linear one-dimensional diffusion PDEs follow from the differential conservation law (1.11) employing the Fick constitutive law $\phi(x,t) = -ku_x(x,t)$. We obtain (see Example 1.9)

$$u_t(x,t) = ku_{xx}(x,t) + f(x,t), \quad k > 0.$$

Diffusion PDEs are widely applied for the modeling of heat transfer, diffusion of a substance in chemical reactions, but also population dynamics, etc.

There are many nonlinear generalization of linear diffusion problems, e.g., non-Fickian flux models such as the Boltzmann equation, nonlinear advection-diffusion problems such as the Burgers equation and others (see J. D. Logan [56]). We are interested in one of these nonlinear generalizations, namely, one-dimensional reaction-diffusion equations (RDEs)

$$u_t(x,t) = ku_{xx}(x,t) + f(x,t,u(x,t)),$$

(see Example 1.11). They arise from the linear diffusion problem assuming that the source density function depends moreover on the density of modeled substance. It is applied to model a local reaction of the substance. These problems are studied by many authors for its rich behavior. They exhibit, e.g., traveling waves solutions, spatial pattern formation, etc. (see J. D. Logan [56] again or V. Volpert [86]).

In many situations, e.g., in biology, chemistry, material modeling, there are natural to consider the spatial variable being from a lattice and investigate these discrete-space problems (see, e.g., J. Bell [6], J. Bell, C. Cosner [7], J. P. Keener [45] or survey monograph S.-N. Chow et. al [20]). Let the discrete spatial domain can be modeled by integers, i.e., $x \in \mathbb{Z}$. We start with the semidiscrete conservation law (1.18) and assume that the source function depends on the density u as well, i.e.,

$$u_t(x,t) + \phi(x,t) - \phi(x-1,t) = f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+. \tag{3.1}$$

Suppose that the density u and the flux ϕ are related with the following discrete analogy of the Fick law

$$\phi(x,t) = -k(u(x+1,t) - u(x,t)), \quad k > 0,$$

(recall that $\phi(x,t)$ denotes the flux between points x and $x+1$). Hence, we obtain from (3.1) the following semidiscrete RDE

$$u_t(x,t) = k(u(x+1,t) - 2u(x,t) + u(x-1,t)) + f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+. \tag{3.2}$$

Many of above mentioned works deal with this semidiscrete problem, i.e., assuming that the time variable is continuous. However, there are also papers dealing with completely discrete analogy of (3.2) (see, e.g., S.-N. Chow, W. Shen [21] or H. Hupkes, E. Van Vleck [44])

$$u(x,t+1) - u(x,t) = k(u(x+1,t) - 2u(x,t) + u(x-1,t)) + f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0.$$

For both discrete-space problems the authors study especially existence of traveling waves (see, e.g., S.-N. Chow, J. Mallet-Paret, W. Shen [23], T. Erneux, G. Nicolis [31] or J. P. Keener [45]), pattern formation (e.g., S.-N. Chow, J. Mallet-Paret [22]) and asymptotic behavior (e.g., T. Caraballo, F. Morillas, J. Valero [18]). Furthermore, these problems are interesting also from the numerical point of view, since they arise from continuous reaction-diffusion PDEs via semi- or full-discretization (see, e.g., H. Hupkes, E. Van Vleck [44]). Particularly, the specific choices of reaction function are studied – the Fisher equation with the logistic nonlinearity, the Nagumo/Allen-Cahn equation with the bistable nonlinearity, etc.

3.1 Maximum principles for RDEs on finite domains

Let us note that linear diffusion equations on infinite lattices are studied in the works of A. Slavík, P. Stehlík [75, 76] and M. Friesl, A. Slavík and P. Stehlík [34]. The authors consider $x \in \mathbb{Z}$ and general time-scale time structure and study essential properties of linear diffusion-type dynamic initial value problems such as existence, uniqueness in the set of bounded solutions, maximum principles, explicit solution formulas and sign- and integral/sum-preservation with consequences to the probability theory.

We study the above mentioned nonlinear modifications of diffusion equations – RDEs. We do not restrict ourselves to the specific nonlinear reaction functions and consider RDEs on finite discrete-space domains with general reaction function f in this section. This is based on the results from the paper P. Stehlík, J. Volek [80]. We present maximum principles for initial-boundary value problems for either completely discrete or semidiscrete RDEs. Whereas the maximum principles in the semidiscrete case exhibit similar features to those of fully continuous reaction-diffusion PDEs, in the discrete case the weak maximum principle holds for a smaller class of functions and the strong maximum principle is valid only in a weaker sense. Then applying the maximum principles we obtain the global existence of solutions which is an essential question for the semidiscrete equation.

Before we focus on the discrete-space problems let us summarize results that are known about the classical reaction-diffusion PDE

$$\begin{cases} u_t(x,t) = ku_{xx}(x,t) + f\left(x,t,u(x,t)\right), & x \in (a,b), \quad t \in \mathbb{R}_0^+, \quad k > 0, \\ u(x,0) = \varphi(x), \quad x \in [a,b], \\ u(a,t) = \xi_a(t), \quad t \in \mathbb{R}_0^+, \\ u(b,t) = \xi_b(t), \quad t \in \mathbb{R}_0^+, \end{cases} \tag{3.3}$$

where $f : (a,b) \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a reaction function and $\varphi : [a,b] \to \mathbb{R}, \xi_a, \xi_b : \mathbb{R}_0^+ \to \mathbb{R}$ are initial-boundary conditions satisfying $\varphi(a) = \xi_a(0)$ and $\varphi(b) = \xi_b(0)$.

The following results about existence and weak and strong maximum principles for (3.3) can be found in H. F. Weinberger [87].

Theorem 3.1. *Let $T > 0$ be arbitrary and f be uniformly Hölder continuous in x and t and Lipschitz in u for $(x,t) \in (a,b) \times (0,T]$. Then for all Hölder continuous initial-boundary conditions φ, ξ_a, ξ_b the problem* (3.3) *has a unique bounded solution which is defined on $[a,b] \times [0,T]$.*

For the sake of brevity we define the following two numbers

$$M_T = \max_{x\in[a,b],t\in[0,T]} \{\varphi(x), \xi_a(t), \xi_b(t)\}, \quad m_T = \min_{x\in[a,b],t\in[0,T]} \{\varphi(x), \xi_a(t), \xi_b(t)\}.$$

Theorem 3.2. *Let $T > 0$ be arbitrary and f be uniformly Hölder continuous in x and t and Lipschitz in u for $(x,t) \in (a,b) \times (0,T]$ and assume that*

$$f(x,t,M_T) \le 0 \le f(x,t,m_T) \quad \text{for all} \quad (x,t) \in (a,b) \times (0,T]. \tag{3.4}$$

Let u be a continuous solution of (3.3) *with Hölder continuous initial-boundary conditions φ, ξ_a, ξ_b. Then*

$$m_T \le u(x,t) \le M_T \quad \text{for all} \quad (x,t) \in [a,b] \times [0,T].$$

Moreover, the strong maximum principle also holds.

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied and u be a solution of* (3.3) *on $[a,b] \times [0,T]$. If $u(x_0,t_0) \in \{m_T, M_T\}$ for some $(x_0,t_0) \in (a,b) \times (0,T]$ then*

$$u(x,t) = u(x_0,t_0) \quad \text{for all} \quad (x,t) \in [a,b] \times [0,t_0].$$

3.1.1 Discrete RDE

After the short overview about the classical reaction-diffusion PDE (3.3), let us consider the initial-boundary problem for the discrete RDE

$$\begin{cases} \Delta_t u(x,t) = k\Delta^2_{xx} u(x-1,t) + f\left(x,t,u(x,t)\right), \quad x \in (a,b) \cap \mathbb{Z} = (a,b)_{\mathbb{Z}}, \quad t \in h\mathbb{N}_0, \quad k > 0, \\ u(x,0) = \varphi(x), \quad x \in (a,b)_{\mathbb{Z}}, \\ u(a,t) = \xi_a(t), \quad t \in h\mathbb{N}_0, \\ u(b,t) = \xi_b(t), \quad t \in h\mathbb{N}_0, \end{cases} \tag{3.5}$$

where $f : (a,b)_{\mathbb{Z}} \times h\mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ is a reaction function, $\varphi : (a,b)_{\mathbb{Z}} \to \mathbb{R}, \xi_a, \xi_b : h\mathbb{N}_0 \to \mathbb{R}$ are initial-boundary conditions, $\Delta_t u(x,t) = \frac{u(x,t+h)-u(x,t)}{h}$, $h > 0$, and $\Delta^2_{xx} u(x-1,t) = u(x-1,t) - 2u(x,t) + u(x+1,t)$.[1]

The existence and uniqueness of a global solution can be easily obtained from (3.5) since $u(x,t+h)$ is given by

$$u(x,t+h) = \begin{cases} u(x,t) + h\left(k\Delta^2_{xx} u(x-1,t) + f\left(x,t,u(x,t)\right)\right), & x \in (a,b)_{\mathbb{Z}}, \\ \xi_a(t+h), & x = a, \\ \xi_b(t+h), & x = b. \end{cases} \tag{3.6}$$

Consequently, the problem (3.5) has a unique solution which is defined on $[a,b]_{\mathbb{Z}} \times h\mathbb{N}_0$.

For $T \in h\mathbb{N}_0$ we define the following two numbers similarly as for the reaction-diffusion PDE (3.3)

$$M_T = \max_{x\in(a,b)_{\mathbb{Z}},t\in[0,T]_{h\mathbb{N}_0}} \{\varphi(x), \xi_a(t), \xi_b(t)\}, \quad m_T = \min_{x\in(a,b)_{\mathbb{Z}},t\in[0,T]_{h\mathbb{N}_0}} \{\varphi(x), \xi_a(t), \xi_b(t)\}.$$

For the brevity of the weak maximum principle we formulate the assumption on the reaction function f:

[1] We denote the time discretization step by $h > 0$ to correspond with P. Stehlík, J. Volek [80]. Again, we assume the space discretization step $h_x = 1$, but all our results are easily extendable to an arbitrary step $h_x > 0$ if we use the diffusion constant $\bar{k} = \frac{k}{h_x^2}$ instead of k.

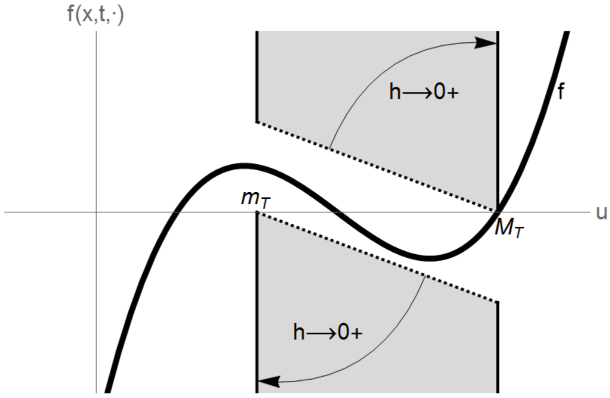

Figure 3.1: Forbidden areas for the function $f(x,t,\cdot)$ in assumption (D_1). The change of these areas as $h \to 0+$.

(D_1) *Let $T \in h\mathbb{N}_0$ and f satisfy*

$$\frac{2hk-1}{h}(u-m_T) \leq f(x,t,u) \leq \frac{2hk-1}{h}(u-M_T),$$

for all $x \in (a,b)_{\mathbb{Z}}$, $t \in [0,T]_{h\mathbb{N}_0}$ and $u \in [m_T, M_T]$.[2]

The inequalities in (D_1) mean that for all fixed x and t the graph of function $f(x,t,\cdot)$ does not intersect forbidden areas depicted in Figure 3.1. Moreover, let us notice that for $h \to 0+$ the slope $\frac{2hk-1}{h}$ goes to $-\infty$, i.e., the forbidden areas are smaller in the sense of inclusion (see Figure 3.1 again).

We present now the weak maximum principle which is essentially dependent on the assumption (D_1) (see P. Stehlík, J. Volek [80, Thm. 9]).[3]

Theorem 3.4. *Let $T \in h\mathbb{N}_0$ be arbitrary, function f satisfy (D_1) and u be the unique solution of* (3.5). *Then*

$$m_T \leq u(x,t) \leq M_T \quad \textit{for all} \quad x \in [a,b]_{\mathbb{Z}}, \quad t \in [0,T]_{h\mathbb{N}_0}. \tag{3.7}$$

As examples of nonlinear reaction functions f that could satisfy (D_1) with suitable initial-boundary conditions, we can mention:

- $f(x,t,u) = -|u|^{p-1}u$ with $p > 1$,
- the logistic function $f(x,t,u) = u(1-u)$,

[2] Note that if $m_T < M_T$ and $h > \dfrac{1}{2k}$, then (D_1) does not hold for any function f since there should be

$$0 < \frac{2hk-1}{h}(u-m_T) \leq f(x,t,u) \leq \frac{2hk-1}{h}(u-M_T) < 0 \quad \text{for} \quad u \in (m_T, M_T).$$

[3] Let us note that we provide in P. Stehlík, J. Volek [80, Thm. 11] even a more general assertion than in Theorem 3.4 which reads as follows. If the assumption (D_1) is not satisfied with m_T, M_T, but there exist constants $r_T < m_T$ and $R_T > M_T$ such that (D_1) holds with them instead of m_T, M_T respectively, then one can prove in the same way that the following a priori bound holds,

$$r_T \leq u(x,t) \leq R_T \quad \text{for all} \quad x \in [a,b]_{\mathbb{Z}}, \quad t \in [0,T]_{h\mathbb{N}_0}.$$

- the bistable nonlinearity $f(x,t,u) = \lambda u(u-a)(1-u)$, $a \in (0,1)$,
- $f(x,t,u) = \lambda u(1-u^p)$ where $p \in \mathbb{N}$,
- $f(x,t,u) = -|x| \arctan(t^2 u)$.

Theorem 3.4 has the following two immediate consequences (see P. Stehlík, J. Volek [80, Cor. 13, 14]).

Corollary 3.5. *Assume that ξ_a, ξ_b are bounded and f satisfies (D_1) for all $T > 0$. Then the unique solution u of* (3.5) *is bounded.*

Corollary 3.6. *Assume that φ, ξ_a, ξ_b are nonnegative and f satisfies (D_1) for all $T > 0$. Then the unique solution u of* (3.5) *is nonnegative.*

Since we know that for classical reaction-diffusion PDE (3.3) even the strong maximum principle holds, we are interested in this important property for discrete RDE (3.5) as well. Unfortunately, the strong maximum principle does not hold for (3.5) in general, which the following example shows (see P. Stehlík, J. Volek [80, Ex. 15])

Example 3.7. Let us consider $x \in [-2,2]_{\mathbb{Z}}$, $t \in \mathbb{N}_0$, $f(x,t,u) \equiv 0$ and $k = \frac{1}{2}$ (note that $h = \frac{1}{2k}$) and let

$$\varphi(x) = M > 0, \quad x \in \{-1,0,1\} \quad \text{and} \quad \xi_{-2}(t) = \xi_2(t) \equiv 0, \quad t \in \mathbb{N}_0.$$

We obtain from (3.6) that $u(0,1) = u(0,0) + ku(-1,0) - 2ku(0,0) + ku(1,0) + f(0,0,0) = M$ and analogously, $u(-1,1) = u(1,1) = \frac{M}{2}$. Hence, the strong maximum principle is not valid.

However, we are able to prove a weaker statement using the domain of dependence and influence defined as follows respectively

$$\mathcal{D}(x_0,t_0) = \left\{(x,t) \in [a,b]_{\mathbb{Z}} \times h\mathbb{N}_0 : \quad t \le t_0, \quad x = x_0 \pm j, \quad j = 0,1,\dots,\frac{t_0 - t}{h}\right\},$$

$$\mathcal{I}(x_0,t_0) = \left\{(x,t) \in [a,b]_{\mathbb{Z}} \times h\mathbb{N}_0 : \quad t \ge t_0, \quad x = x_0 \pm j, \quad j = 0,1,\dots,\frac{t - t_0}{h}\right\}.$$

Furthermore, we apply a modified assumption

(D_2) *Let $T \in h\mathbb{N}_0$ and f satisfy for all $x \in (a,b)_{\mathbb{Z}}$, $t \in [0,T]_{h\mathbb{N}_0}$:*

- $f(x,t,u) < \dfrac{2hk-1}{h}(u - M_T)$ *when* $u \in [m_T, M_T)$,
- $f(x,t,u) > \dfrac{2hk-1}{h}(u - m_T)$ *when* $u \in (m_T, M_T]$,
- $f(x,t,M_T) \le 0 \le f(x,t,m_T)$.

Then one can prove the following statement (see P. Stehlík, J. Volek [80, Thm. 16]).

Theorem 3.8. *Assume that the function f satisfies (D_2) for all $T \in h\mathbb{N}_0$. Let u be the unique solution of* (3.5) *and $(x_0,t_0) \in [a,b]_{\mathbb{Z}} \times h\mathbb{N}_0$.*

1. *If $u(x_0,t_0) \in \{m_T, M_T\}$, then $u(x,t) = u(x_0,t_0)$ on $\mathcal{D}(x_0,t_0)$.*
2. *If $m_T < u(x_0,t_0) < M_T$, then $m_T < u(x,t) < M_T$ on $\mathcal{I}(x_0,t_0)$.*

3.1.2 Semidiscrete RDE

As we mentioned, the discrete RDE (3.5) can be applied, e.g., for modeling of non-overlapping populations. However, we use the results for discrete RDE theoretically for investigating the following semidiscrete problem

$$\begin{cases} u_t(x,t) = k\Delta_{xx}^2 u(x-1,t) + f\left(x,t,u(x,t)\right), \quad x \in (a,b)_{\mathbb{Z}}, \quad t \in \mathbb{R}_0^+, \quad k > 0, \\ u(x,0) = \varphi(x), \quad x \in (a,b)_{\mathbb{Z}}, \\ u(a,t) = \xi_a(t), \quad t \in \mathbb{R}_0^+, \\ u(b,t) = \xi_b(t), \quad t \in \mathbb{R}_0^+, \end{cases} \tag{3.8}$$

where $f : (a,b)_{\mathbb{Z}} \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a reaction function, $\varphi : (a,b)_{\mathbb{Z}} \to \mathbb{R}$ and $\xi_a, \xi_b \in C^1(\mathbb{R}_0^+)$ are initial-boundary conditions.

We start with the local existence and uniqueness result for (3.8) which holds under the following two assumptions:

(C_1) *Let $f(x,t,u)$ be continuous in $(t,u) \in \mathbb{R}_0^+ \times \mathbb{R}$ for all $x \in (a,b)_{\mathbb{Z}}$.*

(C_2) *Let $f(x,t,u)$ be locally Lipschitz with respect to u on $(a,b)_{\mathbb{Z}} \times \mathbb{R}_0^+ \times \mathbb{R}$.*

Since the initial-boundary problem (3.8) can be rewritten as a finite system of ODEs, the local existence follows immediately from the Picard-Lindelöf theorem (see P. Stehlík, J. Volek [80, Thm. 18]).

Theorem 3.9. *Let f satisfy (C_1) and (C_2). Then there exists $\eta > 0$ such that (3.8) has a unique solution defined on $[a,b]_{\mathbb{Z}} \times [0,\eta]$.*

The discrete problem (3.5) arises from (3.8) by the Euler method for discretization of time variable. Thus, the idea of proving the weak maximum principle for (3.8) is the approximation by solutions of (3.5). Let us define numbers M_T and m_T similarly as for the discrete problem

$$M_T = \max_{x \in (a,b)_{\mathbb{Z}}, t \in [0,T]} \left\{\varphi(x), \xi_a(t), \xi_b(t)\right\}, \quad m_T = \min_{x \in (a,b)_{\mathbb{Z}}, t \in [0,T]} \left\{\varphi(x), \xi_a(t), \xi_b(t)\right\}.$$

Consequently, we need the following assumption which is the limit case of (D_1) for $h \to 0+$.

(C_3) *Let $f(x,t,u)$ satisfy $f(x,t,M_T) \le 0 \le f(x,t,m_T)$ for all $x \in (a,b)_{\mathbb{Z}}$, $t \in [0,T]$.*

Therefore applying the convergence of the Euler method one can prove the following statement (see P. Stehlík, J. Volek [80, Thm. 24]).[4]

Theorem 3.10. *Let $T > 0$ be arbitrary, f satisfy (C_1), (C_2) and (C_3) and u be a solution of (3.8) defined on $[a,b]_{\mathbb{Z}} \times [0,T]$. Then*

$$m_T \le u(x,t) \le M_T \quad \textit{for all} \quad x \in (a,b)_{\mathbb{Z}}, \quad t \in [0,T].$$

Applying the weak maximum principle as a priori bound we can prove the existence of a unique global solution (see P. Stehlík, J. Volek [80, Thm. 31]) similarly as Theorem 2.10.

[4] Again as for the discrete RDE, we present in P. Stehlík, J. Volek [80, Thm. 25] a more general statement than in Theorem 3.10, i.e., if (C_3) does not hold with m_T, M_T, but it is satisfied with $r_T < m_T$ and $R_T > M_T$ instead of m_T, M_T respectively, then the following holds,

$$r_T \le u(x,t) \le R_T \quad \text{for all} \quad x \in (a,b)_{\mathbb{Z}}, \quad t \in [0,T].$$

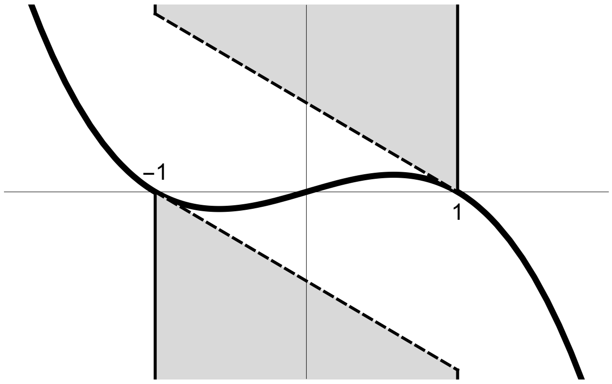

Figure 3.2: The cubic nonlinearity (3.9) with $\lambda > 0$ and the forbidden areas from the assumption (D_1).

Theorem 3.11. *Let f satisfy (C_1), (C_2) and (C_3) for all $T > 0$. Then* (3.8) *has a unique solution defined on $[a,b]_{\mathbb{Z}} \times \mathbb{R}_0^+$ which satisfies*

$$\inf_{x\in(a,b)_{\mathbb{Z}},t\in\mathbb{R}_0^+} \{\varphi(x), \xi_a(t), \xi_b(t)\} \le u(x,t) \le \sup_{x\in(a,b)_{\mathbb{Z}},t\in\mathbb{R}_0^+} \{\varphi(x), \xi_a(t), \xi_b(t)\}$$

for all $x \in [a,b]_{\mathbb{Z}}$ and $t \in \mathbb{R}_0^+$.

Theorems 3.10 and 3.11 have the following two simple consequences (see P. Stehlík, J. Volek [80, Cor. 33, 34]).

Corollary 3.12. *Assume that ξ_a, ξ_b are bounded and f satisfies (C_1), (C_2) and (C_3) for all $T > 0$. Then the unique solution u of* (3.8) *is bounded.*

Corollary 3.13. *Assume that φ, ξ_a, ξ_b are nonnegative and f satisfies (C_1), (C_2) and (C_3) for all $T > 0$. Then the unique solution u of* (3.8) *is nonnegative.*

Finally, in contrast to the discrete case, the strong maximum principle holds for the semidiscrete problem (3.8) similarly as for the classical reaction-diffusion PDE (3.3) (see P. Stehlík, J. Volek [80, Thm. 28]). One can prove it using the Gronwall inequality.

Theorem 3.14. *Let $T > 0$ be arbitrary, f satisfy (C_1), (C_2) and (C_3) and u be a solution of* (3.8) *defined on $[a,b]_{\mathbb{Z}} \times [0,T]$. If $u(x_0,t_0) \in \{m_T, M_T\}$ for some $x_0 \in (a,b)_{\mathbb{Z}}$ and $t_0 \in (0,T]$ then*

$$u(x,t) = u(x_0,t_0) \quad \textit{for all} \quad x \in [a,b]_{\mathbb{Z}}, \quad t \in [0,t_0].$$

As we mentioned in Introduction (see Section 1.1), this is the moment when the interesting question arises – what causes the validity of the strong maximum principle when we pass from the discrete problem to the semidiscrete one? In the next Section 3.2 focused on generalization of results from this section for dynamic RDEs on infinite discrete-space domains, we study this phenomena and claim that only one dense point in the time domain is sufficient for the validity of the strong maximum principle.

3.1.3 Application for Nagumo RDE

In this subsection we apply the previous results to problems involving the cubic nonlinearity

$$f(x,t,u) = \lambda u(1-u^2), \quad \lambda \in \mathbb{R}, \tag{3.9}$$

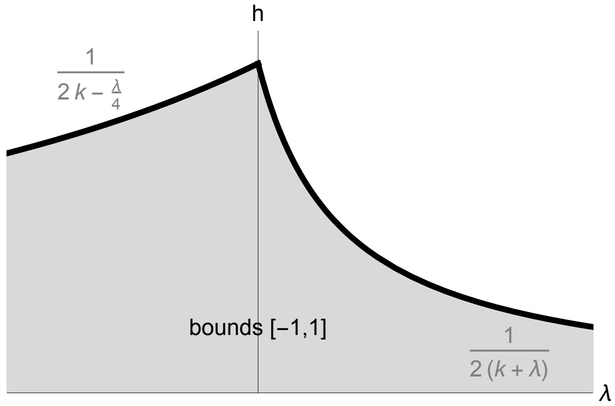

Figure 3.3: The dependence of bounds (3.10) for solutions of the discrete Nagumo equation (3.5)+(3.9) on the values of parameters $\lambda \in \mathbb{R}$ and $h > 0$.

often called the Nagumo equation. Throughout this section we assume that the initial-boundary conditions φ, ξ_a, ξ_b are such that $m_T = -1$ and $M_T = 1$.

Let us start with the semidiscrete case (3.8). We observe immediately that f is locally Lipschitz continuous and satisfies $f(x,t,-1) = 0 = f(x,t,1)$. Consequently, for all $\lambda \in \mathbb{R}$ we can apply Theorem 3.11 to get that there exists a unique global solution such that

$$u(x,t) \in [-1,1] \quad \text{for all} \quad x \in [a,b]_{\mathbb{Z}}, \quad t \in \mathbb{R}_0^+.$$

A more interesting case is the discrete Nagumo equation (3.5). Let us distinguish three cases:

- For $\lambda > 0$ (the bistable case) we observe that $f'(1) = -2\lambda$. Hence the application of Theorem 3.4 is restricted to cases for which the slope of the dashed line in the forbidden areas from assumption (D_1) given by $2k - \frac{1}{h}$ satisfies $2k - \frac{1}{h} \le -2\lambda$ (see Figure 3.2). Consequently, if

  $$h \le \frac{1}{2(k+\lambda)},$$

  we can apply Theorem 3.4 to get $u(x,t) \in [-1,1]$ for all $x \in [a,b]_{\mathbb{Z}}$, $t \in h\mathbb{N}_0$.

  Once $h > \frac{1}{2(k+\lambda)}$, Theorem 3.4 is no longer available. We point out that one can use the more general statement mentioned in Footnote 3 to obtain a weaker a priory bound. We invite the reader to see P. Stehlík, J. Volek [80, Sec. 8] for more details.

- For $\lambda = 0$ the reaction function vanishes and the problem (3.8) reduces to the linear case and we can trivially apply Theorem 3.4 whenever $h \le \frac{1}{2k}$.

- If $\lambda < 0$, then the assumption (D_1) is satisfied as long as the line $\left(2k - \frac{1}{h}\right)(u-1)$ does not intersect for $u < 0$ or is tangential to $f(u) = \lambda u(1-u^2)$. One can easily compute that the tangential case occurs if

  $$2k - \frac{1}{h} = \frac{\lambda}{4}.$$

Let us summarize these results for discrete Nagumo equation depending on values of λ and h. We obtain the following bound for the solution of (3.5) with the bistable nonlinearity (3.9) (see Figure 3.3)

$$u(x,t) \in [-1,1] \quad \text{provided} \quad \lambda \leq 0, \quad h \leq \frac{1}{2k - \frac{\lambda}{4}} \quad \text{or} \quad \lambda > 0, \quad h \leq \frac{1}{2(k+\lambda)}. \tag{3.10}$$

3.2 Dynamic RDEs on infinite domain

In this section, we present a generalization of results about discrete-space RDEs on finite domains from previous Section 3.1. This is an overview of main results from the paper A. Slavík, P. Stehlík, J. Volek [77]. The generalization goes in three ways. Firstly, we consider problems on infinite spatial domain, namely, $x \in \mathbb{Z}$. Next, we do not restrict ourselves to discrete and semidiscrete cases and assume that the time set is generally a subset of a time scale. Finally, we focus on problems which are formulated so that it involves both symmetric or non-symmetric diffusion as well as transport equation, all together.

We study the general nonhomogeneous initial-value problem

$$\begin{cases} u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \quad t \in [t_0,T]^{\kappa}_{\mathbb{T}}, \\ u(x,t_0) = u^0_x, \quad x \in \mathbb{Z}, \end{cases} \tag{3.11}$$

where $\{u^0_x\}_{x\in\mathbb{Z}}$ is a bounded real sequence, $a,b,c \in \mathbb{R}$, $\mathbb{T} \subseteq \mathbb{R}$ is a time scale, u^{Δ} denotes the Δ-derivative of u with respect to the variable t and $t_0, T \in \mathbb{T}$. We use the notation $[\alpha,\beta]_{\mathbb{T}} = [\alpha,\beta] \cap \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$, and

$$[t_0,T]^{\kappa}_{\mathbb{T}} = \begin{cases} [t_0,T]_{\mathbb{T}} & \text{if } T \text{ is left-dense}, \\ [t_0,T)_{\mathbb{T}} & \text{if } T \text{ is left-scattered}. \end{cases}$$

Particularly, if $a = c > 0$ and $b = -2a$ then (3.11) becomes the symmetric reaction-diffusion equation analogical to (3.5) and (3.8). The nonsymmetric case $a \neq c$, $b = -(a+c)$ corresponds to the lattice reaction-advection-diffusion equation. Next, if $a = 0$ and $c = -b > 0$ then (3.11) reduces to the lattice reaction-transport equation. For more details and other special cases see A. Slavík, P. Stehlík [76].

3.2.1 Existence and uniqueness

In this subsection we show existence and uniqueness results for (3.11). We impose the following conditions on the function $f : \mathbb{R} \times \mathbb{Z} \times [t_0,T]_{\mathbb{T}} \to \mathbb{R}$:

(H_1) *f is bounded on each set $B \times \mathbb{Z} \times [t_0,T]_{\mathbb{T}}$, where $B \subset \mathbb{R}$ is bounded.*

(H_2) *f is Lipschitz-continuous in the first variable on each set $B \times \mathbb{Z} \times [t_0,T]_{\mathbb{T}}$, where $B \subset \mathbb{R}$ is bounded.*

(H_3) *For each bounded set $B \subset \mathbb{R}$ and each choice of $\varepsilon > 0$ and $t \in [t_0,T]_{\mathbb{T}}$, there exists a $\delta > 0$ such that if $s \in (t-\delta, t+\delta) \cap [t_0,T]_{\mathbb{T}}$, then $|f(u,x,t) - f(u,x,s)| < \varepsilon$ for all $u \in B$, $x \in \mathbb{Z}$.*

Firstly, we present the statement guaranteeing the existence of a local solution to (3.11). Given a function $U : \mathbb{T} \to \ell^{\infty}(\mathbb{Z})$, the symbol $U(t)_x$ denotes the x-th component of the sequence $U(t)$.[5] The following theorem (see A. Slavík, P. Stehlík, J. Volek [77, Thm. 2.1]) can be proved applying the abstract local existence result M. Bohner, A. C. Peterson [13, Thm. 8.14].

[5] The notation $U(t)_x$ should not be confused with the index notation of the derivative of U with respect to x (which never appears in this section).

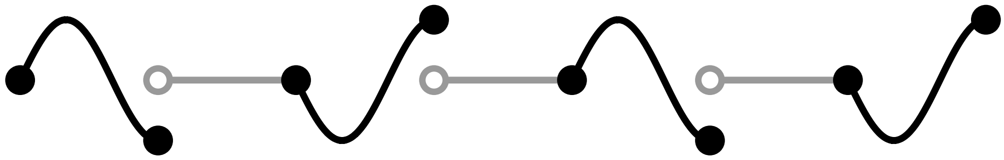

Figure 3.4: Piecewise constant extension x^* (gray) of a function x (black); see (3.14).

Theorem 3.15. *Assume that $f : \mathbb{R} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ satisfies (H1)–(H3). Then for each $u^0 \in \ell^\infty(\mathbb{Z})$, the initial-value problem* (3.11) *has a bounded local solution defined on $\mathbb{Z} \times [t_0, t_0 + \delta]_{\mathbb{T}}$, where $\delta > 0$ and $\delta \geq \mu(t_0)$. The solution is obtained by letting $u(x,t) = U(t)_x$, where $U : [t_0, t_0+\delta]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ is a solution of the abstract dynamic equation*

$$U^\Delta(t) = \Phi(U(t), t), \quad U(t_0) = u^0, \tag{3.12}$$

with $\Phi : \ell^\infty(\mathbb{Z}) \times [t_0, T]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ being given by

$$\Phi(\{u_x\}_{x\in\mathbb{Z}}, t) = \{au_{x+1} + bu_x + cu_{x-1} + f(u_x, x, t)\}_{x\in\mathbb{Z}}.$$

Note that even in the linear case $f \equiv 0$ the solutions of (3.11) are not unique in general (see A. Slavík, P. Stehlík [76]) and the uniqueness can be expected only in the class of bounded solutions. Uniqueness of bounded solutions to the initial-value problem (3.11) follows from the next theorem (see A. Slavík, P. Stehlík, J. Volek [77, Thm. 2.3]).

Theorem 3.16. *Assume that $f : \mathbb{R}\times\mathbb{Z}\times[t_0, T]_{\mathbb{T}} \to \mathbb{R}$ satisfies (H1) and (H2). Then for each $u^0 \in \ell^\infty(\mathbb{Z})$, the initial-value problem* (3.11) *has at most one bounded solution $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$.*

3.2.2 Continuous dependence

This subsection is devoted to the study of continuous dependence of solutions to abstract dynamic equations with respect to the choice of the time scale. These results are applicable to (3.11) whose bounded solutions are obtained as solutions to the abstract dynamic equation (3.12) (see Theorem 3.15). Moreover, we use these results as an essential tool for proving the weak maximum principle for (3.11).

Since we need to compare solutions defined on different time scales (whose intersection might be even empty), we introduce the following definitions.

Consider an interval $[t_0, T] \subset \mathbb{R}$ and a time scale $\mathbb{T}$ with $t_0 \in \mathbb{T}$, $\sup \mathbb{T} \geq T$. Let $g_{\mathbb{T}} : [t_0, T] \to \mathbb{R}$ be given by

$$g_{\mathbb{T}}(t) = \inf\{s \in [t_0, T]_{\mathbb{T}};\, s \geq t\}, \quad t \in [t_0, T]. \tag{3.13}$$

Each function $x : [t_0, T]_{\mathbb{T}} \to X$ can be extended to a function $x^* : [t_0, T] \to X$ by letting

$$x^*(t) = x(g_{\mathbb{T}}(t)), \quad t \in [t_0, T]. \tag{3.14}$$

Note that x^* coincides with x on $[t_0, T]_{\mathbb{T}}$, and is constant on each interval $(u, v]$ where $(u, v) \cap \mathbb{T} = \emptyset$. We refer to x^* as the piecewise constant extension of x, see Figure 3.4.

One can use now the relation between dynamic equations and the so-called measure differential equations (see A. Slavík [74]) to prove an abstract continuous dependence result for measure differential equations (A. Slavík, P. Stehlík, J. Volek [77, Thm. 3.1]). From that we obtain the following result about continuous dependence of solutions to abstract dynamic equations with respect to the choice of the time scale and initial condition (see A. Slavík, P. Stehlík, J. Volek [77, Thm. 3.2]).

Theorem 3.17. *Let X be a Banach space, $\mathcal{B} \subseteq X$. Consider an interval $[t_0, T] \subset \mathbb{R}$ and a sequence of time scales $\{\mathbb{T}_n\}_{n=0}^{\infty}$ such that $t_0 \in \mathbb{T}_n$ and $\sup \mathbb{T}_n \geq T$ for each $n \in \mathbb{N}_0$, $T \in \mathbb{T}_0$, and $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$. Denote $\mathbb{T} = \overline{\bigcup_{n=0}^{\infty} \mathbb{T}_n}$. Suppose that $\Phi : \mathcal{B} \times [t_0, T]_{\mathbb{T}} \to X$ is continuous on its domain and Lipschitz-continuous with respect to the first variable. Let $x_n : [t_0, T]_{\mathbb{T}_n} \to \mathcal{B}$, $n \in \mathbb{N}_0$, be a sequence of functions satisfying*

$$x_n^{\Delta}(t) = \Phi(x_n(t), t), \quad t \in [t_0, T]_{\mathbb{T}_n}^{\kappa}, \quad n \in \mathbb{N}_0,$$

and $x_n(t_0) \to x_0(t_0)$. Then the sequence of piecewise constant extensions $\{x_n^*\}_{n=1}^{\infty}$ is uniformly convergent to the piecewise constant extension x_0^* on $[t_0, T]$. In particular, for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\|x_n(t) - x_0(t)\| < \varepsilon$ for all $n \geq n_0$, $t \in [t_0, T]_{\mathbb{T}_n} \cap [t_0, T]_{\mathbb{T}_0}$.

Furthermore, for the proof of maximum principle for (3.11) we need an assertion guaranteeing that for each time scale $\mathbb{T}_0$ there exists an approximating sequence $\{\mathbb{T}_n\}_{n=1}^{\infty}$ of discrete time scales (in the sense $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$). We introduce the following notation

$$\overline{\mu}_{\mathbb{T}} = \max_{t \in [t_0, T)_{\mathbb{T}}} \mu(t).$$

The following statement provides the desired approximation result (see A. Slavík, P. Stehlík, J. Volek [77, Thm. 3.4]).

Theorem 3.18. *If $\mathbb{T}_0 \subset \mathbb{R}$ is a time scale with $t_0, T \in \mathbb{T}_0$, there exists a sequence of discrete time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ with $\mathbb{T}_n \subseteq \mathbb{T}_0$, $\min \mathbb{T}_n = t_0$, $\max \mathbb{T}_n = T$, and such that $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$.*

Moreover, if $\overline{\mu}_{\mathbb{T}_0} = 0$, then $\lim_{n\to\infty} \overline{\mu}_{\mathbb{T}_n} = 0$; otherwise, if $\overline{\mu}_{\mathbb{T}_0} > 0$, then the sequence $\{\mathbb{T}_n\}_{n=1}^{\infty}$ can be chosen so that $\overline{\mu}_{\mathbb{T}_n} = \overline{\mu}_{\mathbb{T}_0}$ for all $n \in \mathbb{N}$.

Let us recall that in Subsection 2.2.2 there is shown continuous dependence on initial-boundary condition for the linear transport equation which follows from the maximum principle. On the contrary, in this subsection we introduce continuous dependence on initial condition as well as on the choice of underlying time scale for nonlinear abstract dynamic initial value problems on Banach spaces. Moreover, we use continuous dependence on the choice of time scale as fundamental tool in the proof of weak maximum principle for (3.11).

3.2.3 Maximum principles and global existence

We present weak a strong maximum principles for (3.11). For an initial condition $u^0 \in \ell^\infty(\mathbb{Z})$ we denote

$$m = \inf_{x \in \mathbb{Z}} u_x^0, \quad M = \sup_{x \in \mathbb{Z}} u_x^0.$$

Further, we need the following assumptions:

(H_4) *a, b, $c \in \mathbb{R}$ are such that $a, c \geq 0$, $b < 0$, and $a + b + c = 0$.*

(H_5) *$b < 0$ and $\overline{\mu}_{\mathbb{T}} \leq -1/b$.*

(H_6) *One of the following statements holds:*

- *$\overline{\mu}_{\mathbb{T}} = 0$ and $f(M, x, t) \leq 0 \leq f(m, x, t)$ for all $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.*
- *$\overline{\mu}_{\mathbb{T}} > 0$ and $\dfrac{1 + \overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(m - u) \leq f(u, x, t) \leq \dfrac{1 + \overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(M - u)$ for all $u \in [m, M]$, $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.*

Let us note that if (H_4)–(H_6) are not satisfied, the maximum principle also does not hold in general (see A. Slavík, P. Stehlík [76] and P. Stehlík, J. Volek [80]). The assumption (H_6) defines forbidden areas which the nonlinearity f cannot intersect, similarly as in Figure 3.1.

If (H_6) holds for a nonlinear function f in the continuous case $\overline{\mu}_{\mathbb{T}} = 0$, the following lemma shows that (H_6) is also satisfied for all sufficiently fine time scales (specifically, for almost all of the discrete approximating time scales $\mathbb{T}_n$ from Theorem 3.18), (see A. Slavík, P. Stehlík, J. Volek [77, Lem. 4.2]).

Lemma 3.19. *Assume that $\overline{\mu}_{\mathbb{T}} = 0$ and (H_2), (H_6) hold. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the following inequalities hold*

$$\frac{1+\varepsilon b}{\varepsilon}(m-u) \le f(u,x,t) \le \frac{1+\varepsilon b}{\varepsilon}(M-u) \quad \textit{for all} \quad u \in [m,M], \quad x \in \mathbb{Z}, \quad t \in [t_0,T].$$

Applying Theorem 3.17 (continuous dependence of solutions on the choice of time scale), Theorem 3.18 (existence of approximating sequence of discrete time scales) and previous Lemma 3.19 one can prove the weak maximum principle (see A. Slavík, P. Stehlík, J. Volek [77, Thm. 4.4]).[6]

Theorem 3.20. *Assume that (H_1)–(H_6) hold. If $u : \mathbb{Z} \times [t_0,T]_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of* (3.11)*, then*

$$m \le u(x,t) \le M \quad \textit{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0,T]_{\mathbb{T}}.$$

The weak maximum principle usually provides wanted a priory bound needed for the proof of the global existence (see A. Slavík, P. Stehlík, J. Volek [77, Thm. 4.6]).

Theorem 3.21. *If $u^0 \in \ell^\infty(\mathbb{Z})$ and (H_1)–(H_6) hold, then* (3.11) *has a unique bounded solution $u : \mathbb{Z} \times [t_0,T]_{\mathbb{T}} \to \mathbb{R}$.*

Moreover, the solution depends continuously on u^0 in the following sense – for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $v^0 \in \ell^\infty(\mathbb{Z})$, $m \le v^0_x \le M$ for all $x \in \mathbb{Z}$, and $\|u^0 - v^0\|_\infty < \delta$, then the unique bounded solution $v : \mathbb{Z} \times [t_0,T]_{\mathbb{T}} \to \mathbb{R}$ of (3.11) *corresponding to the initial condition v^0 satisfies $|u(x,t) - v(x,t)| < \varepsilon$ for all $x \in \mathbb{Z}$, $t \in [t_0,T]_{\mathbb{T}}$.*

Let us illustrate the application of Theorems 3.20 and 3.21 on (3.11) where f is a nonautonomous logistic function (see A. Slavík, P. Stehlík, J. Volek [77, Ex. 4.9]).

Example 3.22. Consider the initial value problem (3.11) where the nonlinear function f is given by

$$f(u,x,t) = \lambda u(d(x,t) - u), \quad u \in \mathbb{R}, \quad x \in \mathbb{Z}, \quad t \in [t_0,T]_{\mathbb{T}}, \tag{3.15}$$

where $\lambda > 0$ and $d : \mathbb{Z} \times [t_0,T]_{\mathbb{T}} \to \mathbb{R}$. The equation can be interpreted as the logistic population model where the carrying capacity d depends on position and time. Assume that d has the following properties:

- There exists $D \ge 0$ such that $|d(x,t)| \le D$ for all $x \in \mathbb{Z}$ and $t \in [t_0,T]_{\mathbb{T}}$, i.e., d is bounded.
- For each choice of $\varepsilon > 0$ and $t \in [t_0,T]_{\mathbb{T}}$, there exists a $\delta > 0$ such that if $s \in (t-\delta, t+\delta) \cap [t_0,T]_{\mathbb{T}}$, then $|d(x,t) - d(x,s)| < \varepsilon$ for all $x \in \mathbb{Z}$.

[6] Note that again we show in A. Slavík, P. Stehlík, J. Volek [77] a more general assertion than in Theorem 3.20. If the assumption (H_6) does not hold with m, M, but there are $r < m$ and $R > M$ such that (H_6) is satisfied with them, then

$$r \le u(x,t) \le R \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0,T]_{\mathbb{T}}.$$

Then the function f satisfies (H_1)–(H_3). Indeed, if $B \subset \mathbb{R}$ is bounded, it is contained in a ball of radius ρ centered at the origin. Consequently, for all $u, v \in B$, $x \in \mathbb{Z}$, $t, s \in [t_0, T]_{\mathbb{T}}$, we get the estimates

$$\begin{aligned} |f(u,x,t)| &\le \lambda |u|(|d(x,t)| + |u|) \le \lambda\rho(D+\rho), \\ |f(u,x,t) - f(v,x,t)| &= \lambda|(u-v)(d(x,t)-u-v)| \le \lambda|u-v|(D+2\rho), \\ |f(u,x,t) - f(u,x,s)| &= \lambda|u(d(x,t)-d(x,s))| \le \lambda\rho|d(x,t)-d(x,s)|, \end{aligned}$$

which imply that (H_1)–(H_3) hold.

As concrete examples of d we can point out:

- $d(x,t) = e(x - \gamma t)$ with $\gamma > 0$ and $e : \mathbb{R} \to \mathbb{R}$ being continuous, nondecreasing and bounded (e.g., population model with a shifting habitat, see C. Hu, B. Li [43]),
- $d(x,t) = e(t)$ with $e : \mathbb{R} \to \mathbb{R}$ being continuous and periodic (e.g., population model with periodically changing habitat).

Suppose now that $a, c \ge 0$, $b < 0$, $a + b + c = 0$, and $\overline{\mu}_{\mathbb{T}} \le -\frac{1}{b}$, i.e., ($H_4$) and ($H_5$) hold. Consider an arbitrary nonnegative initial condition $u^0 \in \ell^\infty(\mathbb{Z})$, i.e., $m \ge 0$, and assume that $m \le d(x,t) \le M$ for all $x \in \mathbb{Z}$ and $t \in [t_0, T]_{\mathbb{T}}$. Then

$$f(m,x,t) \ge 0 \quad \text{and} \quad f(M,x,t) \le 0 \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0,T]_{\mathbb{T}}.$$

This means that (H_6) holds if $\overline{\mu}_{\mathbb{T}} = 0$. Applying Lemma 3.19 we obtain that (H_6) holds also for $\overline{\mu}_{\mathbb{T}}$ positive and sufficiently small. In these cases, the problem (3.11) with f being defined by (3.15) possesses a unique global solution u and

$$m \le u(x,t) \le R \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0,T]_{\mathbb{T}}.$$

We conclude this section with the strong maximum principle. We need the following stronger versions of (H_4)–(H_6):

(H_7) *a, b, $c \in \mathbb{R}$ are such that $a, c > 0$, $b < 0$, and $a + b + c = 0$.*

(H_8) *$b < 0$ and $\overline{\mu}_{\mathbb{T}} < -1/b$.*

(H_9) *The following statements hold for all $x \in \mathbb{Z}$ and $t \in [t_0, T]_{\mathbb{T}}$:*

- *$f(M,x,t) \le 0 \le f(m,x,t)$.*
- *If $\overline{\mu}_{\mathbb{T}} > 0$, then $f(u,x,t) > \dfrac{1+\overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(m-u)$ for all $u \in (m, M]$.*
- *If $\overline{\mu}_{\mathbb{T}} > 0$, then $f(u,x,t) < \dfrac{1+\overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(M-u)$ for all $u \in [m, M)$.*

The following theorem establishes the strong maximum principle for the initial value problem (3.11) (see A. Slavík, P. Stehlík, J. Volek [77, Thm. 5.3]). It is an example which shows the beauty of time scale calculus because using the language of time scales it finds a moment when (3.11) qualitatively changes its behavior with respect to the underlying structure of time domain (cf. Theorems 3.8 and 3.14).

Theorem 3.23. *Assume that (H_1)–(H_3), (H_7)–(H_9) hold and $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of* (3.11)*. If $u(\bar{x}, \bar{t}) \in \{m, M\}$ for some $\bar{x} \in \mathbb{Z}$ and $\bar{t} \in (t_0, T]_{\mathbb{T}}$, then the following statements hold:*

- *If $[t_0, \bar{t}]_{\mathbb{T}}$ contains only isolated points, i.e., $t_0 = \rho^k_{\mathbb{T}}(\bar{t})$ for some $k \in \mathbb{N}$, and*

$$\mathcal{D}(\bar{x}, \bar{t}) = \left\{(x,t) \in \mathbb{Z} \times [t_0, \bar{t}]_{\mathbb{T}} : \ t = \rho^j_{\mathbb{T}}(\bar{t}), \ j = 0, \ldots, k, \ \text{and } x = \bar{x} \pm i, \ i = 0, \ldots, j\right\},$$

  then $u(x,t) = u(\bar{x}, \bar{t})$ for all $(x,t) \in \mathcal{D}(\bar{x}, \bar{t})$.

- *Otherwise, if $[t_0, \bar{t}]_{\mathbb{T}}$ contains a point which is not isolated, then $m = M$ and $u(x,t) = M$ for all $x \in \mathbb{Z}$ and $t \in [t_0, T]_{\mathbb{T}}$.*

We emphasize that the fact whether a point is isolated or not is considered with respect to the time scale interval $[t_0, \bar{t}]_{\mathbb{T}}$, not the entire time scale $\mathbb{T}$. In other words, Theorem 3.23 distinguishes between the cases in which the interval $[t_0, \bar{t}]_{\mathbb{T}}$ is a finite set (former case) or at least countable (latter case).

The strong maximum principle has the following immediate consequence (see A. Slavík, P. Stehlík, J. Volek [77, Cor. 5.5]).

Corollary 3.24. *Assume that (H_1)–(H_3), (H_7)–(H_9) hold and $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of* (3.11)*. If there is a point $t_d \in [t_0, T)_{\mathbb{T}}$ that is not isolated and if the initial condition u^0 is not constant, then*

$$m < u(x,t) < M \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in (t_d, T]_{\mathbb{T}}.$$

Let us conclude the overview about strong maximum principle with a remark that the strong maximum principle does not hold under the weaker assumptions (H_4)–(H_6). The stronger versions (H_7)–(H_9) are actually needed. The reader can see A. Slavík, P. Stehlík, J. Volek [77, Rem. 5.6, 5.7, 5.8]) where we provide a counterexample to each of (H_4)–(H_6).

3.3 Implicit discrete Nagumo equation

In this section which is an overview about results from the paper P. Stehlík, J. Volek [81], we study a completely discrete RDE on infinite discrete-space domain. Particularly, we focus on the specific problem involving the cubic nonlinearity, the so-called Nagumo equation. Motivated by the numerical mathematics, we consider a fully implicit discretization of this problem.

Numerical methods for RDEs usually consist of two processes. First, a space discretization reduces a partial differential equation into a system of ordinary differential equations. Then a certain time discretization technique is applied (see, e.g., V. Thomée [82]). In the case of RDEs, implicit methods are often used from the stiffness reasons (see, e.g., A. Madzvamuse, A. H. W. Chung [57]).

Many studies considered preservation of various characteristics of RDEs through discretization processes. In contrast to the problem on a finite domain (see, e.g., O. A. Ladyzhenskaya [48]), the problem on an infinite domain is infinite-dimensional and the corresponding dynamics is more complex (see, e.g., A. V. Babin, M. I. Vishik [3], W. J. Beyn, S. Y. Pilyugin [12], H. J. Hupkes, E. S. Van Vleck [44]).

We study the following initial value problem

$$\begin{cases} \Delta_t v(x,t) = k\Delta^2_{xx} v(x-1, t+h) + \lambda v(x, t+h)\left(1 - v^2(x, t+h)\right), & \lambda \in \mathbb{R}, \\ v(x, 0) = \varphi(x), \end{cases} \tag{3.16}$$

where $x \in \mathbb{Z}$, $t \in h\mathbb{N}_0 = \{0, h, 2h, \ldots\}$, $k > 0$, $h > 0$ and the partial differences are defined by $\Delta_t v(x,t) = \frac{v(x,t+h) - v(x,t)}{h}$, $\Delta^2_{xx} v(x-1, t+h) = v(x-1, t+h) - 2v(x, t+h) + v(x+1, t+h)$.[7]

We use variational methods to get existence (and uniqueness for $\lambda \geq 0$) of solutions $u(x,t)$ to the implicit problem (3.16) whose spatial sections $u(\cdot, t)$ lie in $\ell^2 = \ell^2(\mathbb{Z})$ for each time instance $t \in h\mathbb{N}_0$ (note

[7] As in the previous sections, we assume for the sake of brevity, that the space discretization step $h_x = 1$. However, all our results are valid for arbitrary $h_x > 0$ by considering $\bar{k} = \frac{k}{h_x^2}$ instead of k in (3.16).

that A. V. Babin, M. I. Vishik in [3] and W. J. Beyn, S. Y. Pilyugin in [12] study (3.16) in weighted sequence spaces). Our technique provides results for solutions in ℓ^2 in certain cases which have not been studied so far (e.g., when the dissipativity condition used in W. J. Beyn, S. Y. Pilyugin [12] is violated).

3.3.1 Abstract formulation

Since we study the existence and uniqueness of solutions to (3.16) whose spatial sections $v(\cdot,t) = \{v(x,t)\}_{x\in\mathbb{Z}}$ lies in ℓ^2 for each time step $t \in h\mathbb{N}_0$, we assume that $\varphi = \{\varphi(x)\}_{x\in\mathbb{Z}} \in \ell^2$. Moreover, let us define the following two operators for $u = \{u_i\}_{x\in\mathbb{Z}}$ that we use later in an operator reformulation of (3.16),

$$L : \ell^2 \to \ell^2, \quad (Lu)_i = ku_{i-1} - 2ku_i + ku_{i+1}, \quad i \in \mathbb{Z}, \tag{3.17}$$

$$N : \ell^2 \to \ell^2, \quad (N(u))_i = u_i\left(1 - u_i^2\right), \quad i \in \mathbb{Z}. \tag{3.18}$$

Therefore, the problem (3.16) is equivalent to the abstract difference equation on the Hilbert space ℓ^2

$$\begin{cases} \Delta_t v(\cdot,t) = L(v(\cdot,t+h)) + \lambda N(v(\cdot,t+h)), & \lambda \in \mathbb{R}, \\ v(\cdot,0) = \varphi, \end{cases} \tag{3.19}$$

where $\Delta_t v(\cdot,t) = \frac{1}{h}\left(v(\cdot,t+h) - v(\cdot,t)\right)$.

First, we consider the following problem – for a fixed $t \in h\mathbb{N}_0$ and a given $v(\cdot,t) \in \ell^2$ (e.g., for $t = 0$ there is $v(\cdot,0) = \varphi$ an initial condition) we look for a solution $v(\cdot,t+h) \in \ell^2$ of (3.19). We call this a local problem which is later applied in the mathematical induction to prove the global existence.

One can rewrite the equation in (3.19) into

$$v(\cdot,t+h) = v(\cdot,t) + hL(v(\cdot,t+h)) + h\lambda N(v(\cdot,t+h)), \qquad \lambda \in \mathbb{R}.$$

If we denote the fixed known element $b = v(\cdot,t) \in \ell^2$, and the unknown one $u = v(\cdot,t+h) \in \ell^2$, then the problem (3.19) for a fixed $t \in h\mathbb{N}_0$ is equivalent to the fixed-point problem on ℓ^2

$$u = b + hLu + h\lambda N(u), \quad \lambda \in \mathbb{R}. \tag{3.20}$$

Let us note that the operator L defined by (3.17) is linear bounded, self-adjoint, negative and $\|L\|_{\mathcal{L}(\ell^2)} = 4k$ (see P. Stehlík, J. Volek [81, Lem. 2.1, 2.2]). The nonlinear superposition (Nemyckii) operator N given by (3.18) is continuous and $\mathrm{Dom}(N) = \ell^2$ (see P. Stehlík, J. Volek [81, Lem. 2.3]).

Let us introduce the variational setting for (3.20). The fixed point problem (3.20) is equivalent to the operator equation

$$F(u) = u - b - hLu - h\lambda N(u) = o. \tag{3.21}$$

The operator $F : \ell^2 \to \ell^2$ has a potential $\mathcal{F} : \mathrm{Dom}(\mathcal{F}) = \ell^2 \to \mathbb{R}$ given by

$$\begin{aligned} \mathcal{F}(u) &= \frac{1}{2}\sum_{i\in\mathbb{Z}} u_i^2 - \sum_{i\in\mathbb{Z}} b_i u_i - \frac{h}{2}\sum_{i\in\mathbb{Z}} (Lu)_i u_i - \frac{h\lambda}{2}\sum_{i\in\mathbb{Z}} u_i^2 + \frac{h\lambda}{4}\sum_{i\in\mathbb{Z}} u_i^4 \\ &= \frac{1-h\lambda}{2}\|u\|_2^2 - (b,u)_2 - \frac{h}{2}(Lu,u)_2 + \frac{h\lambda}{4}\|u\|_4^4. \end{aligned} \tag{3.22}$$

There is $\mathcal{F} \in C^1(\ell^2,\mathbb{R})$ and its critical points correspond equivalently to solutions of (3.21) (see P. Stehlík, J. Volek [81, Lem. 3.1, 3.5]).

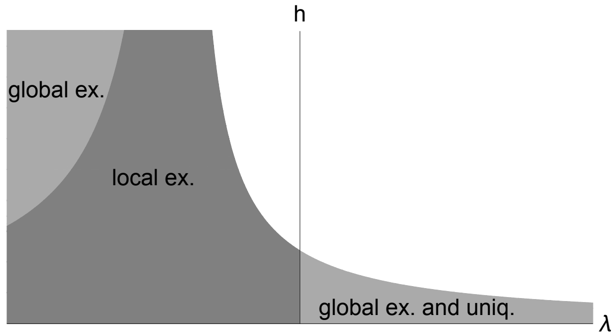

Figure 3.5: Graphical illustration of existence and uniqueness results for implicit Nagumo RDE (3.16). See Table 3.1 for more details.

3.3.2 Existence results

Let us firstly consider the Nagumo equation (3.16) with the bistable setting, i.e., for $\lambda \geq 0$. Since the potential $\mathcal{F}$ has strictly convex and weakly coercive geometry for a certain setting of parameters, one can use the statement about the existence and uniqueness of global minimizer of $\mathcal{F}$ (see, e.g., P. Drábek, J. Milota [29, Thm. 7.2.12, Prop. 7.1.8]) to prove the following lemma.

Lemma 3.25. *Let $\lambda \geq 0$ and $h(\lambda + 4k) < 1$. Then the functional $\mathcal{F}$ given by* (3.22) *has a unique global minimizer $\tilde{u} \in \ell^2$ which is the unique critical point of $\mathcal{F}$.*

Immediately from Lemma 3.25, we can prove via mathematical induction the following result about the global existence and uniqueness (on the set of all functions having spatial sections in the space ℓ^2) result (see P. Stehlík, J. Volek [81, Thm. 4.1]).

Theorem 3.26. *Let $\lambda \geq 0$, $h(\lambda + 4k) < 1$ and assume $\varphi \in \ell^2$. Then the problem* (3.16) *has a unique solution $v(x,t)$ that exists for all $x \in \mathbb{Z}$, $t \in h\mathbb{N}_0$ and satisfies*

$$\left(\sum_{x \in \mathbb{Z}} |v(x,t)|^2 \right)^{\frac{1}{2}} < \infty \quad \textit{for all} \quad t \in h\mathbb{N}_0.$$

We note that for given $\lambda \geq 0$ and $k > 0$ there always exist sufficiently small values of time discretization step $h > 0$ which satisfy $h(\lambda + 4k) < 1$, see Figure 3.5. Obviously, the stronger reaction or the stronger diffusion, the smaller $h > 0$ is required.

For negative values of λ in (3.16) (the monostable case) we lose the globally convex and weakly coercive geometry of the potential $\mathcal{F}$. However, $\mathcal{F}$ is at least locally convex in this case and we can apply the statement about the existence of a local minimizer for $\mathcal{F}$ (see P. Drábek, J. Milota [29, Thm. 7.2.11, Prop. 7.1.8]).

For the sake of brevity, we define the auxiliary real valued function $\xi : \mathbb{R} \to \mathbb{R}$ by

$$\xi(s) = \frac{1 - h\lambda - 4hk}{2} s + \frac{h\lambda}{4} s^3 \tag{3.23}$$

and the positive constant (assuming that $\lambda < 0$ and $h(\lambda + 4k) < 1$)

$$R = \min \left\{ \left(\frac{h\lambda - 1}{3h\lambda} \right)^{\frac{1}{2}}, \left(\frac{2(4hk + h\lambda - 1)}{3h\lambda} \right)^{\frac{1}{2}} \right\}. \tag{3.24}$$

The following lemma provides the existence of at least one critical point of $\mathcal{F}$ (see P. Stehlík, J. Volek [81, Lem. 5.1]).

Lemma 3.27. *Let $\lambda < 0$, $h(\lambda + 4k) < 1$ and*

$$\|b\|_2 < \xi(R). \tag{3.25}$$

Then the potential $\mathcal{F}$ given by (3.22) *has a local minimizer $\tilde{u} \in \ell^2$ which is the unique critical point of $\mathcal{F}$ with the property*

$$\|\tilde{u}\|_2 < R. \tag{3.26}$$

Applying Lemma 3.27 one can use the correspondence of critical points of $\mathcal{F}$ to local ℓ^2-solutions to (3.16) to show immediately the following local existence result for (3.16) (see P. Stehlík, J. Volek [81, Thm. 5.2]).

Theorem 3.28. *Let $\lambda < 0$, $h(\lambda + 4k) < 1$ and assume $v(x,t)$ is a solution of* (3.16) *at a fixed time $t \in h\mathbb{N}_0$ such that*

$$\left(\sum_{x\in\mathbb{Z}} |v(x,t)|^2\right)^{\frac{1}{2}} < \xi(R).$$

Then there exists a solution $v(x,t+h)$ of the problem (3.16) *at time $t+h$ such that*

$$\left(\sum_{x\in\mathbb{Z}} |v(x,t+h)|^2\right)^{\frac{1}{2}} < R.$$

However, we cannot apply Lemma 3.27 directly in mathematical induction because we do not know if the solution at next time step $v(\cdot, t+h)$ satisfies the assumption (3.25) as well as $v(\cdot, t)$. Combining (3.25) and (3.26) together, there has to be satisfied

$$R \leq \xi(R).$$

This leads to the stronger assumption on parameters and one can prove the following global existence result via mathematical induction (see P. Stehlík, J. Volek [81, Thm. 5.3]).

Theorem 3.29. *Let $\lambda < 0$, $h(\lambda + 4k) \leq -2$ and assume that $\varphi \in \ell^2$ satisfies*

$$\|\varphi\|_2 < \xi(R).$$

Then the problem (3.16) *has a solution $v(x,t)$ that exists for all $x \in \mathbb{Z}$, $t \in h\mathbb{N}_0$ and is unique with the property*

$$\left(\sum_{x\in\mathbb{Z}} |v(x,t)|^2\right)^{\frac{1}{2}} < R \quad \textit{for all} \quad t \in h\mathbb{N}.$$

For the illustration of admissible values of $h > 0$ and $\lambda < 0$ in Theorems 3.28 and 3.29 see Figure 3.5 again.

λ	$\lambda < 0$		$\lambda \geq 0$	
	$\left(-\infty, -\frac{2}{h} - 4k\right]$	$\left(-\frac{2}{h} - 4k, \frac{1}{h} - 4k\right)$	$\left[0, \frac{1}{h} - 4k\right)$	$\left[\frac{1}{h} - 4k, \infty\right)$
Geometry of $\mathcal{F}$	mountain pass	mountain pass	convex, w. coerc.	?
Existence	global (Thm. 3.29)	local (Thm. 3.28)	global (Thm. 3.26)	?
Uniqueness ($v(\cdot, t) \in \ell^2$)	? (Conj. 3.32)	? (Conj. 3.32)	yes (Thm. 3.26)	?

Table 3.1: Summary of results for implicit Nagumo RDE (3.16), see also Figure 3.5.

3.3.3 Conjectures about multiplicity

In Subsection 3.3.2 we claim that the potential is locally convex in the case $\lambda < 0$. Actually, the functional $\mathcal{F}$ given by (3.22) has the mountain pass geometry provided $\lambda < 0$ (see P. Stehlík, J. Volek [81, Lem. 6.3]).

Lemma 3.30. *Let $\lambda < 0$, $h(\lambda + 4k) < 1$ and assume that $b \in \ell^2$ satisfy (3.25). Then there exist $e \in \ell^2$ and $\rho > 0$ such that $\|e\|_2 > \rho$ and the functional $\mathcal{F}$ given by (3.22) satisfies*

$$\inf_{\|u\|_2 = \rho} \mathcal{F}(u) > \mathcal{F}(o) \geq \mathcal{F}(e).$$

Consequently, there is a natural question if we can apply the mountain pass theorem (see A. Ambrosetti, P. H. Rabinowitz [2] or P. H. Rabinowitz [67]) to prove the existence of another critical point. Unfortunately, we are not able to verify the Palais-Smale compactness condition. The difficulty arises from the consideration of infinite spatial domain $x \in \mathbb{Z}$. It causes in the abstract formulation that the underlying function space (ℓ^2 in our case) is infinite-dimensional. If we solved the initial-boundary value problem assuming $x \in [a, b] \cap \mathbb{Z}$ involving boundary conditions at points $x = a$ and $x = b$, the abstract problem would be finite-dimensional and the verification of the Palais–Smale compactness condition would be restricted to the proof of boundedness of the Palais-Smale sequence (which we are able to show, see P. Stehlík, J. Volek [81, Lem. 6.4]). We list the initial-boundary value problem on finite domain as one of our future works (see Section 5.2).

Therefore, the following conjectures has remained still open (see P. Stehlík, J. Volek [81, Conj. 6.2, 6.3]).

Conjecture 3.31. *Let $\lambda < 0$, $h(\lambda + 4k) < 1$ and assume that $b \in \ell^2$ satisfy (3.25). Then the functional $\mathcal{F}$ given by (3.22) has at least two critical points.*

Conjecture 3.32. *Let $\lambda < 0$, $h(\lambda + 4k) < 1$ and $v(x, t)$ be a solution of (3.16) at a fixed time $t \in h\mathbb{N}_0$ such that*

$$\left(\sum_{x \in \mathbb{Z}} |v(x, t)|^2 \right)^{\frac{1}{2}} < \xi(R).$$

Then the problem (3.16) has at least two solutions $v_1(x, t + h)$, $v_2(x, t + h)$ at time $t + h$ such that

$$\left(\sum_{x \in \mathbb{Z}} |v_j(x, t + h)|^2 \right)^{\frac{1}{2}} < \infty, \quad j = 1, 2.$$

We sum up all results and conjectures in Table 3.1.

CHAPTER 4

Stationary problems

We study boundary value problems for nonlinear difference equations of second order in this section. Firstly, these problems can be interpreted as stationary counterparts of the previous evolutionary problems on finite domains. From another point of view, an analysis of stationary difference equations can be important also from numerical reasons, since they arise from differential equations (both ODEs and PDEs) via the finite difference method (see, e.g., B. L. Buzbee, G. H. Golub, C. W. Nielson [16] or R. J. LeVeque [53]).

4.1 Landesman-Lazer conditions for discrete Neumann and periodic BVPs

C. Bereanu and J. Mawhin in [9, 10] use the Brouwer degree and the theory of lower and upper solutions (see, e.g., C. De Coster, P. Habets [26] for a survey about the lower and upper solutions for differential equations) for the proof of existence and uniqueness/multiplicity results to discrete boundary value problems. Besides, e.g., interesting Ambrosetti-Prodi type results they establish Landesman-Lazer conditions for the studied problems (these type of conditions were firstly studied by E. M. Landesman, A. C. Lazer in [49] for elliptic PDEs).

We present a summary of main results from the paper J. Volek [84]. We also study Landesman-Lazer type conditions, namely, for discrete Neumann and periodic problems. Our approach is based on the reformulation of such type of equations into an algebraic system

$$Au = G(u), \quad u \in \mathbb{R}^N, \tag{4.1}$$

where $A \in \mathbb{R}^{N\times N}$ is a matrix satisfying common fundamental properties for all considered problems. This allows us to investigate Neumann and periodic problems as well as ordinary and partial difference equations at once and obtain general existence and uniqueness results for all these alternatives.

Generally speaking, Dirichlet problems correspond to (4.1) with regular and positive definite matrices A (see, e.g., M. Galewski, J. Smejda [35]). On the contrary, Neumann and periodic problems can be rewriten as (4.1) with singular and only positive semi-definite matrices A (see, e.g., P. Stehlík [78]). Precisely, the matrices corresponding to Neumann and periodic problems satisfy the following essential properties:

(A_1) *A is a symmetric and positive semi-definite matrix.*

(A_2) *$\lambda_1 = 0$ is an eigenvalue of A with the multiplicity one.*

(A_3) *$\varphi_1 = [1, 1, \dots, 1]^{\mathrm{T}} \in \mathbb{R}^N$ is the eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$.*

In the following examples we show the mentioned reformulation of Neumann and periodic problems for ordinary difference equations onto (4.1) (see J. Volek [84, Ex. 2.1, 2.2]).

Example 4.1. Consider the discrete Neumann problem

$$\begin{cases} -\Delta^2 u(t-1) = g(t, u(t)), & t = 1, 2, \dots, N, \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \tag{4.2}$$

where $u : \{0, 1, \dots, N, N+1\} \to \mathbb{R}$, $\Delta^2 u(t-1) = u(t-1) - 2u(t) + u(t+1)$ is the second central difference of u, $\Delta u(t) = u(t+1) - u(t)$ is the first forward difference of u and $\tilde{g} : \{1, 2, \dots, N\} \times \mathbb{R} \to \mathbb{R}$ is a given function.[1]

We use the values $u(t)$ for $t = 1, 2, \dots, N$ to define a vector

$$u = [u(1), u(2), \dots, u(N)]^{\mathrm{T}} \in \mathbb{R}^N. \tag{4.3}$$

If we write the equation in (4.2) for each $t = 1, \dots, N$ and employing the boundary conditions, we find out that (4.2) is equivalent to the algebraic problem (4.1) with the vector u defined by (4.3) and

$$A = \begin{bmatrix} 1 & -1 & 0 & & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}, \quad G(u) = \begin{bmatrix} g(1, u(1)) \\ g(2, u(2)) \\ g(3, u(3)) \\ \vdots \\ g(N-1, u(N-1)) \\ g(N, u(N)) \end{bmatrix}. \tag{4.4}$$

The matrix A in (4.4) satisfies (A_1)–(A_3).

Example 4.2. Consider the discrete periodic problem

$$\begin{cases} -\Delta^2 u(t-1) = g(t, u(t)), & t = 1, 2, \dots, N, \\ u(0) = u(N), \\ \Delta u(0) = \Delta u(N). \end{cases} \tag{4.5}$$

Analogously as in Example 4.1, the problem (4.5) can be rewritten as the algebraic problem (4.1) with A defined by

[1] We can consider even the nonhomogeneous problem

$$\begin{cases} -\Delta^2 u(t-1) = g(t, u(t)), & t = 1, 2, \dots, N, \\ \Delta u(0) = c_1, \\ \Delta u(N) = c_2, \end{cases}$$

with $c_1, c_2 \in \mathbb{R}$, since it can be reformulated as the homogeneous problem (4.2) involving the modified function

$$\tilde{g}(t, u) = \begin{cases} g(1, u) - c_1, & t = 1, \\ g(t, u), & t = 2, 3, \dots, N-1, \\ g(N, u) + c_2, & t = N. \end{cases}$$

instead of g. It can be done also for the following Examples 4.2 – 4.4 similarly.

$$A = \begin{bmatrix} 2 & -1 & 0 & & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

and G being a superposition vector function defined in the same way as in (4.4). The matrix A also satisfies (A_1)–(A_3).

In the following two examples we show that also partial difference equations with Neumann or periodic boundary conditions can be considered (see J. Volek [84, Ex. 2.3, 2.4]).

Example 4.3. Consider the Neumann problem for the difference Poisson equation

$$\begin{cases} -\Delta_s^2 u(s-1,t) - \Delta_t^2 u(s,t-1) = g(s,t,u(s,t)), \quad s = 1,2,\dots,M, \quad t = 1,2,\dots,N, \\ \Delta_s u(0,t) = 0 \quad \text{and} \quad \Delta_s u(M,t) = 0 \quad \text{for all} \quad t = 1,2,\dots,N, \\ \Delta_t u(s,0) = 0 \quad \text{and} \quad \Delta_t u(s,N) = 0 \quad \text{for all} \quad s = 1,2,\dots,M, \end{cases} \tag{4.6}$$

where $u : \{0,1,\dots,M,M+1\} \times \{0,1,\dots,N,N+1\} \to \mathbb{R}$, $\Delta_s^2 u(s-1,t)$, $\Delta_t^2 u(s,t-1)$ are the second partial central differences of u, $\Delta_s u(s,t)$, $\Delta_t u(s,t)$ are the first partial forward differences of u with respect to s and t and $g : \{1,2,\dots,M\} \times \{1,2,\dots,N\} \times \mathbb{R} \to \mathbb{R}$.

Following the approach, e.g., from B. L. Buzbee, G. H. Golub, C. W. Nielson [16] or J. Otta, P. Stehlík [62], we define a vector

$$u = [u(1,1),\dots u(1,N), u(2,1),\dots,u(2,N),\dots,u(M,1),\dots,u(M,N)]^{\mathrm{T}} \in \mathbb{R}^{MN}. \tag{4.7}$$

Consequently, we obtain that (4.6) is equivalent to the algebraic problem (4.1) on $\mathbb{R}^{MN}$ with the vector u defined by (4.7) and with a block matrix $A \in \mathbb{R}^{MN\times MN}$ given by

$$A = \begin{bmatrix} B_1 & -I & 0 & & 0 & 0 \\ -I & B_2 & -I & \dots & 0 & 0 \\ 0 & -I & B_2 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & B_2 & -I \\ 0 & 0 & 0 & \dots & -I & B_1 \end{bmatrix},$$

where $I \in \mathbb{R}^{N\times N}$ is the identity matrix and $B_1, B_2 \in \mathbb{R}^{N\times N}$ are given by

$$B_1 = \begin{bmatrix} 2 & -1 & 0 & & 0 & 0 \\ -1 & 3 & -1 & \dots & 0 & 0 \\ 0 & -1 & 3 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 3 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 & -1 & 0 & & 0 & 0 \\ -1 & 4 & -1 & \dots & 0 & 0 \\ 0 & -1 & 4 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 4 & -1 \\ 0 & 0 & 0 & \dots & -1 & 3 \end{bmatrix}.$$

The nonlinear function G can be established analogously as in Example 4.1. The matrix A satisfies (A_1)–(A_3).

Example 4.4. Consider the periodic problem for the difference Poisson equation

$$\begin{cases} -\Delta_s^2 u(s-1,t) - \Delta_t^2 u(s,t-1) = g(s,t,u(s,t)), \quad s = 1,2,\dots,M, \quad t = 1,2,\dots,N, \\ u(0,t) = u(M,t) \quad \text{and} \quad \Delta_s u(0,t) = \Delta_s u(M,t) \quad \text{for all} \quad t = 1,2,\dots,N, \\ u(s,0) = u(s,N) \quad \text{and} \quad \Delta_t u(s,0) = \Delta_t u(s,N) \quad \text{for all} \quad s = 1,2,\dots,M. \end{cases} \tag{4.8}$$

Analogously as in Example 4.3, we find out that (4.8) is equivalent to the algebraic problem (4.1) on $\mathbb{R}^{MN}$ with a block matrix $A \in \mathbb{R}^{MN \times MN}$ given by

$$A = \begin{bmatrix} B & -I & 0 & & 0 & -I \\ -I & B & -I & \dots & 0 & 0 \\ 0 & -I & B & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & B & -I \\ -I & 0 & 0 & \dots & -I & B \end{bmatrix},$$

where $B \in \mathbb{R}^{N \times N}$ is defined by

$$B = \begin{bmatrix} 4 & -1 & 0 & & 0 & -1 \\ -1 & 4 & -1 & \dots & 0 & 0 \\ 0 & -1 & 4 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 4 & -1 \\ -1 & 0 & 0 & \dots & -1 & 4 \end{bmatrix}.$$

The matrix A satisfies (A_1)–(A_3).

The last example shows the possible application of our results to general difference equations on undirected graphs (see J. Volek [84, Ex. 2.5]).

Example 4.5. Let $\mathcal{G} = (V, E)$ be an undirected graph with a set of vertices $V = \{1, 2, \dots, N\}$ and a set of edges $E \subset \{\{s,t\} : s,t \in V, s \neq t\}$. The set $\mathcal{N}(t) = \{i \in V : \{i,t\} \in E\}$ is the neighborhood of the vertex $t \in V$ and the number $d_{\mathcal{G}}(t) = |\mathcal{N}(t)|$ is the degree of vertex $t \in V$ (see, e.g., C. Godsil, G. Royle [37] for details about the graph theory).

Let $u : V \to \mathbb{R}$ be a function defined on the set of vertices V and define a difference operator on the graph $\mathcal{G}$ by

$$\Delta_{\mathcal{G}} u(t) = d_{\mathcal{G}}(t) u(t) - \sum_{i \in \mathcal{N}(t)} u(i) = \sum_{i \in \mathcal{N}(t)} (u(t) - u(i)). \tag{4.9}$$

Consequently, we consider the nonlinear difference equation on the graph $\mathcal{G}$

$$\Delta_{\mathcal{G}} u(t) = g(t, u(t)), \quad t \in V, \tag{4.10}$$

with $g : V \times \mathbb{R} \to \mathbb{R}$. The problem (4.10) is equivalent to the algebraic system (4.1) with A being the so-called Laplace matrix of $\mathcal{G}$ with the entries of A given by

$$A(s,t) = \begin{cases} d_{\mathcal{G}}(t), & s = t, \\ -1, & s \neq t \quad \text{and} \quad \{s,t\} \in E, \\ 0, & s \neq t \quad \text{and} \quad \{s,t\} \notin E. \end{cases}$$

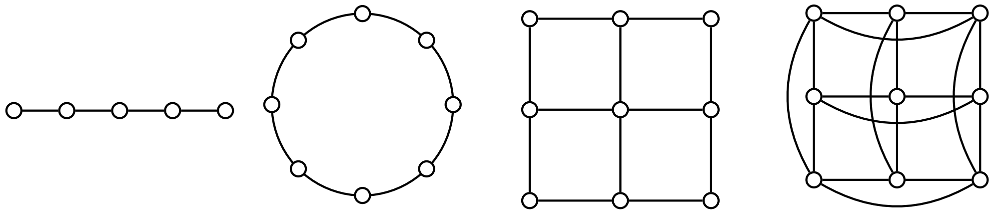

Figure 4.1: The graphs $\mathcal{G}$ from Example 4.5 that are related with Neumann and periodic discrete boundary value problems (4.2), (4.5), (4.6) and (4.8).

If $\mathcal{G}$ is a connected graph then A satisfies (A_1)–(A_3) (see C. Godsil, G. Royle [37] again).

Let us conclude the example with an interesting relationship of difference equations on graphs with Neumann and periodic boundary value problems for difference equations from Examples 4.1 – 4.4. One can show that:

- the algebraic formulation of the Neumann problem for ordinary difference equation (4.2) is equivalent to the algebraic formulation of (4.10) with $\mathcal{G}$ being a path (see Figure 4.1(a)),
- the algebraic formulation of the periodic boundary value problem for ordinary difference equation (4.5) corresponds to the algebraic formulation of (4.10) with $\mathcal{G}$ being a cycle (see Figure 4.1(b)),
- the algebraic formulation of the Neumann problem for the difference Poisson equation (4.6) (for the sake of simplicity let $N = 3$) corresponds to the algebraic formulation of (4.10) with $\mathcal{G}$ given in Figure 4.1(c),
- the algebraic formulation of the periodic problem for the difference Poisson equation (4.8) (again let $N = 3$) is equivalent to the algebraic formulation of (4.10) with $\mathcal{G}$ given in Figure 4.1(d).

One can observe that we do not have to restrict ourselves to discrete problems of second order. The reformulation into (4.1) also works for problems of $2n$-th order ($n \in \mathbb{N}$), see P. Stehlík [78].

Therefore, we study the general algebraic problem for $u \in \mathbb{R}^N$, $N \geq 2$,

$$Au = G(u), \tag{4.11}$$

where $A \in \mathbb{R}^{N\times N}$ is an $N \times N$ matrix satisfying (A_1)–(A_3) and $G : \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear superposition vector function given by

$$G(u) = [g(1, u(1)), g(2, u(2)), \ldots, g(N, u(N))]^{\mathrm{T}},$$

where $g : \{1, 2, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$.

From (A_1)–(A_2), there is $\lambda_1 = 0$ the minimal eigenvalue of A. Thus, the problem (4.11) is a problem at resonance. This motivates us to find a Landesman-Lazer type condition for a class of nonlinear functions G to prove the existence. The function $G : \mathbb{R}^N \to \mathbb{R}^N$ is defined via the function $g : \{1, 2, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$ and hence, we formulate essential needed conditions on G via the function g as well:

(G_1) *The functions $g(t, \cdot)$ are continuous on $\mathbb{R}$ for each $t = 1, 2, \ldots, N$.*

(G_2) *There exist $\alpha, \beta \in [0, 1)$ such that for each $t = 1, 2, \ldots, N$ there exist limits*

$$g_{-\infty}(t) = \lim_{u\to-\infty} \frac{g(t, u)}{|u|^{\alpha}} \quad \text{and} \quad g_{+\infty}(t) = \lim_{u\to+\infty} \frac{g(t, u)}{|u|^{\beta}}.$$

(LL) *The function g satisfies*

$$\sum_{t=1}^{N} g_{-\infty}(t) < 0 < \sum_{t=1}^{N} g_{+\infty}(t).$$

The condition (LL) represents the above mentioned Landesman-Lazer type condition. It is a type of an orthogonality relation, since the inequalities in (LL) are equivalent to

$$(g_{-\infty}, \varphi_1) < 0 < (g_{+\infty}, \varphi_1),$$

where the vectors $g_{\pm\infty} \in \mathbb{R}^N$ are defined by $g_{\pm\infty} = [g_{\pm\infty}(1), g_{\pm\infty}(2), \ldots, g_{\pm\infty}(N)]^{\mathrm{T}}$ and the symbol

$$(u, v) = \sum_{t=1}^{N} u(t)v(t)$$

denotes the scalar product on $\mathbb{R}^N$.

4.1.1 Existence results

We apply variational methods to obtain a sufficient existence condition for (4.11). The associated potential $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}$ to (4.11) is given by

$$\mathcal{F}(u) = \frac{1}{2}(Au, u) - \sum_{t=1}^{N} \int_0^{u(t)} g(t, s)\mathrm{d}s.$$

One can show that $\mathcal{F} \in C^1(\mathbb{R}^N, \mathbb{R})$ and satisfies the assumptions of the saddle point theorem (see P. H. Rabinowitz [67, 68]) provided (A_1)–(A_3), (G_1)–(G_2) and (LL) are satisfied. Therefore, the following existence result holds (see J. Volek [84, Thm. 4.9]).

Theorem 4.6. *Let A satisfy (A_1)–(A_3) and g satisfy (G_1)–(G_2) and (LL). Then there exists a solution of* (4.11).

Considering the boundary value problems (4.2), (4.5), (4.6), (4.8) or (4.10), one can apply Theorem 4.6 for example for the following nonlinear functions:

- $g(t,u) = \begin{cases} |u|^{p-2}u + f(t), & u < 0, \quad p \in (1,2), \\ f(t), & u = 0, \quad f : \{1, 2, \ldots, N\} \to \mathbb{R} \text{ arbitrary}, \\ |u|^{q-2}u + f(t), & u > 0, \quad q \in (1,2), \end{cases}$

- $g(t,u) = \begin{cases} |u|^{p-2}u + f(t), & u \leq -1, \quad p \in (1,2), \\ -\sin\left(\frac{3\pi}{2}u\right) + f(t), & u \in (-1,1), \quad f : \{1, 2, \ldots, N\} \to \mathbb{R} \text{ arbitrary}, \\ |u|^{q-2}u + f(t), & u \geq 1, \quad q \in (1,2), \end{cases}$

- $g(t,u) = \begin{cases} |u-t|^{p-2}(u-t) + \dfrac{\sin(u-t)}{u-t}, & u \neq t, \quad p \in (1,2), \\ 1, & u = t. \end{cases}$

We can also consider bounded nonlinearities, e.g.:

- $g(t,u) = \mathrm{e}^{-u^2} + \tanh(u) + f(t)$ with $-1 < \frac{1}{N}\sum\limits_{t=1}^{N} f(t) < 1$,

- $g(t,u) = (t-2)\arctan(u - \log(t))$ with $N \geq 4$.

Furthermore, Theorem 4.6 has the following two immediate consequences for problems involving special class of nonlinearities in a separated form (see J. Volek [84, Cor. 4.11, 4.12]).

Corollary 4.7. *Let A satisfy (A_1)–(A_3) and g be defined by*

$$g(t,u) = h(u) + f(t),$$

where $f : \{1,2,\ldots,N\} \to \mathbb{R}$ is arbitrary and $h : \mathbb{R} \to \mathbb{R}$ satisfies (G_1)–(G_2) and (LL) with $\alpha, \beta \in (0,1)$. Then there exists a solution of (4.11).

Corollary 4.8. *Let A satisfy (A_1)–(A_3) and g be defined by*

$$g(t,u) = h(u) + f(t),$$

where $f : \{1,2,\ldots,N\} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ satisfies (G_1)–(G_2) and (LL) with $\alpha = \beta = 0$ and $h_{\pm\infty} = \lim_{u\to\pm\infty} h(u)$. If f satisfies

$$-h_{+\infty} < \frac{1}{N}\sum_{t=1}^{N} f(t) < -h_{-\infty}, \tag{4.12}$$

then there exists a solution of (4.11).

Note that, the inequalities in (4.12) are equivalent to $-h_{+\infty} < \frac{1}{N}(f, \varphi_1) < -h_{-\infty}$ with $f \in \mathbb{R}^N$ being defined by $f = [f(1), f(2), \ldots, f(N)]^{\mathrm{T}}$.

Analyzing (4.11) with bounded nonlinear functions g into detail, we find out that the Landesman-Lazer type condition (*LL*) is also necessary under the following additional hypotheses:

(G_3) *The function g satisfies*

$$g_{-\infty}(t) < g(t,u) < g_{+\infty}(t) \quad \textit{for all} \quad t = 1,2,\ldots,N \quad \textit{and} \quad u \in \mathbb{R}.$$

Let us emphasize that if (G_1) and (G_3) hold together, the functions $g(t,\cdot)$ are necessarily bounded for each $t = 1,2,\ldots,N$. This yields that (G_2) is satisfied for all $\alpha, \beta \in [0,1)$. However, the strict inequalities in (G_3) implies that $g(t,\cdot)$ have to be bounded by the limits

$$g_{-\infty}(t) = \lim_{u\to-\infty} g(t,u) \quad \text{and} \quad g_{+\infty}(t) = \lim_{u\to+\infty} g(t,u).$$

Consequently, we can show the following necessary and sufficient condition for the exstence of a solution of (4.11) (see J. Volek [84, Thm. 5.2]).

Theorem 4.9. *Let A satisfy (A_1)–(A_3) and g satisfy (G_1)–(G_3). Then* (4.11) *has a solution if and only if (LL) holds.*

Again, for the class of separated nonlinear functions, the following consequence related to Corollary 4.8 holds (see J. Volek [84, Cor. 5.4]).

Corollary 4.10. *Let A satisfy (A_1)–(A_3) and g be defined as*

$$g(t,u) = h(u) + f(t),$$

where $f : \{1,2,\ldots,N\} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ satisfies (G_1)–(G_3). Then (4.11) *has a solution if and only if* (4.12) *holds.*

Example 4.11. Consider the boundary value problems (4.2), (4.5), (4.6), (4.8) or (4.10) with the nonlinear function g defined by

$$g(t,u) = a\arctan(u) + f(t), \quad a > 0,$$

with a given $f : \{1,2,\ldots,N\} \to \mathbb{R}$. The function g satisfies (G_1)–(G_3) with $\alpha = \beta = 0$ and $g_{\pm\infty}(t) = \pm\frac{a\pi}{2} + f(t)$. Therefore, (4.12) is satisfied if and only if

$$-\frac{a\pi}{2} < \frac{1}{N}\sum_{t=1}^{N} f(t) < \frac{a\pi}{2}.$$

Consequently, Corollary 4.10 yields that:

- for $a > \frac{2}{\pi N}\left|\sum\limits_{t=1}^{N} f(t)\right|$ there exists a solution of considered problems,
- for $a \leq \frac{2}{\pi N}\left|\sum\limits_{t=1}^{N} f(t)\right|$ there does not exist any solution.

As we claimed on the beginning of this section, the Landesman-Lazer type conditions for difference equations have already been studied (among other things), e.g., by C. Bereanu and J. Mawhin in [9, 10]. Our findings complement these results in the following way:

- Our approach via the algebraic formulation (4.11) is general in the sense that Neumann/periodic problems, ordinary/partial difference equations are considered at once, whereas in [9, 10] specific boundary value problems are studied. On the other hand, in [9] more general problems involving discrete ϕ-Laplacian are investigated.
- The Landesman-Lazer conditions in [9, 10] are assumed to be sufficient. We show that for a certain class of bounded nonlinearities (LL) is even necessary and therefore, we obtain the nonexistence result as well.
- Both papers [9, 10] formulate the Landesman-Lazer conditions for nonlinear functions in separated form $g(t,u) = h(u) + f(t)$. We also study functions in general nonseparated form $g(t,u)$.

4.1.2 Uniqueness results

In the paper J. Volek [84] we investigate also the uniqueness for (4.11). We need again additional assumptions:

(G_4) *The functions $g(t,\cdot)$ are continuously differentiable on $\mathbb{R}$ for each $t = 1,2,\ldots,N$.*

(G_5) *Let $A \in \mathbb{R}^{N\times N}$ and $\lambda_s(A)$, $s = 1,2,\ldots,N$, be eigenvalues of A. The function g satisfies*

$$g_u(t,u) \neq \lambda_s(A) \quad \textit{for all} \quad t = 1,2,\ldots,N, \quad u \in \mathbb{R}, \quad \textit{and} \quad s = 1,2,\ldots,N.$$

Involving (G_4)–(G_5) we can prove the following statement by the application of the mean value theorem together with a special spectral result for commuting matrices (see J. Volek [84, Thm. 6.2]).

Theorem 4.12. *Let A be arbitrary and g satisfy (G_4)–(G_5). Then* (4.11) *has at most one solution.*

Putting Theorems 4.6 or 4.9 together with Theorem 4.12 we obtain the following two consequences (see J. Volek [84, Thm. 6.4, 6.5]).

Theorem 4.13. *Let A satisfy (A_1)–(A_3) and g satisfy (G_2), (G_4)–(G_5) and (LL). Then there exists a unique solution of* (4.11).

Theorem 4.14. *Let A satisfy (A_1)–(A_3) and g satisfy (G_2)–(G_5). Then* (4.11) *has a solution if and only if (LL) holds. Moreover, the solution has to be unique provided it exists.*

The following example shows the applications of Theorems 4.13 and 4.14 (see J. Volek [84, Ex. 6.6]).

Example 4.15. Consider the Neumann problem (4.2) from Example 4.1 with $N = 3$. There is $\lambda_1(A) = 0$, $\lambda_2(A) = 1$ and $\lambda_3(A) = 3$. Let the function g be given by

$$g(t,u) = \left(\frac{t}{3} - a\right) \arctan(u) + bt, \quad a > 0, \quad b \in \mathbb{R}.$$

Investigating the assumptions of Theorems 4.14 or 4.13 into detail we can show that:

- For $a \in \left(0, \frac{1}{3}\right)$ the problem (4.2) with $N = 3$ has a solution if and only if b satisfies

$$|b| < \pi \left(\frac{1}{6} - \frac{a}{4}\right). \tag{4.13}$$

  Moreover, the solution is unique provided it exists.

- For $a \in \left(\frac{1}{3}, \frac{2}{3}\right)$ the problem (4.2) with $N = 3$ has a unique solution at least for b satisfying (4.13).

CHAPTER 5

Conclusion and future study

The submitted work is our initial step into the analysis of equations with discrete spatial domains. Generally, we have considered problems on the simplest types of lattices (finite and infinite discrete intervals), studied primary questions for these problems as existence, uniqueness and focused on the maximum principles. We point out that there are many open questions, possible generalizations or unsolved problems. Such a situation offers many directions of future research work.

Although there exist many of possible future ways, we present in detail several of them in which we are interested. We subdivide them into sections for the lucidity.

5.1 Equations with discrete ϕ-Laplacian

First possibility how to generalize problems studied in this thesis is to stay, for now, with equations on finite/infinite discrete spatial intervals and consider equations arising from more complicated and nonlinear constitutive laws. We have something done for transport equations (see Section 2.2). However, for diffusion equations we assume solely the discrete version of the Fick law and from that arising problems with linear spatial diffusion.

Let us consider semidiscrete conservation law (1.18) for problems with $x \in \mathbb{Z}$ and the source function depending also on the density u as in the introduction to Chapter 3

$$u_t(x,t) + \nabla_x \phi(x,t) = f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+,$$

where $\phi(x,t)$ denotes the flux between the points x and $x+1$ and $\nabla_x \phi(x,t) = \phi(x,t) - \phi(x-1,t)$. Moreover, let us assume that u and ϕ are related by the nonlinear constitutive law[1]

$$\phi(x,t) = -k\phi(\Delta_x u(x,t)), \quad \phi : \mathbb{R} \to \mathbb{R}, \quad k > 0,$$

where $\Delta_x u(x,t) = u(x+1,t) - u(x,t)$. Therefore the conservation law provides the following generalized RDE

[1] We denote the nonlinear function in the constitutive law incorrectly by ϕ as the flux itself (as in Section 1.5 and Section 2.2), because we are interested in problems involving the so-called ϕ-Laplacian. Hence, we use this incorrect notation to correspond with the literature.

$$u_t(x,t) - k\nabla_x \phi(\Delta_x u(x,t)) = f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+, \tag{5.1}$$

which involves the so-called discrete ϕ-Laplacian $\nabla_x \phi(\Delta_x u(x,t))$.

The standard choice of the function ϕ is an odd homeomorphism such that $\phi : (-a,a) \to (-b,b)$, $0 < a,b \leq \infty$, (see, e.g., C. Bereanu, P. Jebelean, J. Mawhin [8]). There are three qualitatively different choices of the parameters a,b which are studied and we present them in the following examples.

Example 5.1. Let $a = \infty$ and $b = \infty$. Then the function ϕ and also its inverse is everywhere defined. Well-known example of such situation is

$$\phi(s) = |s|^{p-2} s, \quad p > 1,$$

which yields together with (5.1) the discrete-space RDE involving the discrete p-Laplacian operator

$$u_t(x,t) + \nabla_x \left(|\Delta_x u(x,t)|^{p-2} \Delta_x u(x,t) \right) = f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+. \tag{5.2}$$

The stationary counterpart of (5.2) is studied, e.g., in J. Otta, P. Stehlík [62].

Example 5.2. Let $a = \infty$ and $b < \infty$. Then the function ϕ is defined everywhere but bounded (i.e., its inverse has bounded domain). This situation is in fully continuous case connected with the mean curvature operator in Euclidian spaces taking (see, e.g., D. Gilbarg, N. S. Trudinger [36]),

$$\phi(s) = \frac{s}{\sqrt{1 + |s|^2}}.$$

In the discrete-space situation, we get from (5.1) the following problem

$$u_t(x,t) + \nabla_x \left(\frac{\Delta_x u(x,t)}{\sqrt{1 + |\Delta_x u(x,t)|^2}} \right) = f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+. \tag{5.3}$$

Again, the stationary counterpart of (5.3) is studied, e.g., in C. Bereanu, H. B. Thompson [11].

Example 5.3. Let $a < \infty$ and $b = \infty$. Then the function ϕ has a bounded domain but it takes all real values. The corresponding ϕ-Laplacian is then called singular. In completely continuous case this situation is related to the so-called mean extrinsic curvature operators in Minkowski spaces taking (see, e.g., C. Bereanu, P. Jebelean, J. Mawhin [8] or R. Bartnik, L. Simon [5]),

$$\phi(s) = \frac{s}{\sqrt{1 - |s|^2}}.$$

In the discrete-space case, we obtain from (5.1)

$$u_t(x,t) + \nabla_x \left(\frac{\Delta_x u(x,t)}{\sqrt{1 - |\Delta_x u(x,t)|^2}} \right) = f(x,t,u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{R}_0^+. \tag{5.4}$$

The stationary counterpart of (5.4) is studied, e.g., in C. Bereanu, J. Mawhin [9].

Consequently, we want to study in the future the evolutionary RDEs (5.2), (5.3) and (5.4) involving nonlinear discrete ϕ-Laplacian operators. More generally, we can suppose only the general odd homeomorphism ϕ and corresponding problems.

5.2 Implicit discrete equations on finite domains

We present in Section 3.3 results about the fully implicit discretization of the Nagumo equation (3.16). We show that in the bistable case and for small time discretization steps there exists a global solution which is unique in the set of functions having spatial sections in the space $\ell^2(\mathbb{Z})$. Further, we investigate the monostable case and show the global existence as well. However, we conclude Section 3.3 with several conjectures about multiplicity of these solutions, since the associated potential has the mountain pass geometry.

We come into troubles in the application of the Ambrosetti-Rabinowitz mountain pass theorem while we want to verify the Palais-Smale compactness condition on infinite dimensional sequence space $\ell^2(\mathbb{Z})$.

Therefore, one can ask what happens if we consider a similar implicit problem with the spatial variable being from a finite discrete interval and with boundary conditions at the end points. Since the underlying function space is then finite-dimensional, the Palais-Smale condition is restricted to the proof of boundedness of an appropriate sequence which could be done similarly as in P. Stehlík, J. Volek [81, Lem. 6.4].

Consequently, if we consider the homogeneous Dirichlet boundary conditions for a start, we study the following analogue to (3.16) for $\lambda < 0$ (the monostable case)

$$\begin{cases} \frac{u(x,t+h)-u(x,t)}{h} = k\left(u(x-1,t+h) - 2u(x,t+h) + u(x+1,t+h)\right) + \lambda u(x,t+h)\left(1-u^2(x,t+h)\right), \\ u(x,0) = \varphi(x), \\ u(0,t) = 0, \\ u(N+1,t) = 0, \end{cases} \tag{5.5}$$

where $x \in [1,N] \cap \mathbb{Z} = [1,N]_{\mathbb{Z}}$, $t \in h\mathbb{N}_0$, $h > 0$ and $\varphi : [1,N]_{\mathbb{Z}} \to \mathbb{R}$.

Assume that we know the solution at time t. Similarly as in Subsection 3.3.1 we denote

$$u = [u(1,t+h), u(2,t+h), \dots, u(N,t+h)]^{\mathrm{T}} \in \mathbb{R}^N,$$
$$b = [u(1,t), u(2,t), \dots, u(N,t)]^{\mathrm{T}} \in \mathbb{R}^N.$$

Then the problem of finding a solution at time $t+h$ is equivalent to the following fixed point problem for $u \in \mathbb{R}^N$

$$u = b + hLu + h\lambda N(u), \tag{5.6}$$

where (applying the boundary conditions) $L \in \mathbb{R}^{N\times N}$ and $N : \mathbb{R}^N \to \mathbb{R}^N$ are given by

$$L = \begin{bmatrix} -2 & 1 & 0 & & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 1 & -2 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & -2 & 1 \\ 0 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}, \quad N(u) = \begin{bmatrix} u_1\left(1-u_1^2\right) \\ u_2\left(1-u_2^2\right) \\ u_3\left(1-u_3^2\right) \\ \vdots \\ u_{N-1}\left(1-u_{N-1}^2\right) \\ u_N\left(1-u_N^2\right) \end{bmatrix}.$$

The matrix L is symmetric, negatively definite and the mapping N is continuous.

Consequently, using the symmetry of the matrix L we obtain that the associated potential $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\mathcal{F}(u) = \frac{1-h\lambda}{2}\|u\|_2^2 - (b,u)_{\mathbb{R}^N} - \frac{h}{2}(Lu,u)_{\mathbb{R}^N} + h\lambda\|u\|_4^4. \tag{5.7}$$

One can immediately observe that the potential given by (5.7) has the same form as the one in (3.22), however, it is defined on the finite-dimensional $\mathbb{R}^N$. Therefore, we expect that $\mathcal{F}$ given by (5.7) would have the mountain pass geometry as in Section 3.3. Moreover, in the finite dimension it could be easier to prove the Palais-Smale condition, which we have not proved in the infinite dimension. Then one can show the existence of at least two solutions to the fixed point problem (5.6). This would yield a mutiplicity result for (5.5).

We still have not done this simpler case in detail and thus, it remains open as one of possible future works.

5.3 Equations on graphs

As we mentioned, we consider only problems on finite and infinite discrete intervals, beside the results of Section 4.1 which hold also for equations on undirected graphs. However, motivated exactly by Example 4.5 in Section 4.1, let us formulate problems on more complicated spatial structures such as graphs.

Since our problems arise from conservation laws, let us try to formulate conservation laws on graphs. Let us generally assume a directed graph, because (as we know from the continuous conservation laws) the flux is a directional magnitude.[2] This setting provides a possibility to model problems on discrete spatial structures where the flux of investigated substance between two adjacent places can flow only in one direction for a natural reason.

For example, consider the migration of animals between two adjacent areas in nature. If the borderline between these places is formed, e.g., by a steep cliff, then it is permeable only in one direction. Hence, the flux of animals is directed also in this permeable way only.

Therefore, let $\vec{\mathcal{G}} = (V, E)$ be a directed graph with a set of vertices V (which could be generally finite or infinite) and a set of oriented edges $E \subset \{(x, y) \in V \times V : \ x \neq y\}$. Denote by

$$\begin{aligned} \mathcal{N}_+(x) &= \{y \in V : \ (x, y) \in E\} \quad \text{the out-neighborhood of the vertex } x, \\ \mathcal{N}_-(x) &= \{y \in V : \ (y, x) \in E\} \quad \text{the in-neighborhood of the vertex } x, \end{aligned}$$

and

$$\begin{aligned} d_+(x) &= |\mathcal{N}_+(x)| \quad \text{the out-degree of the vertex } x, \\ d_-(x) &= |\mathcal{N}_-(x)| \quad \text{the in-degree of the vertex } x. \end{aligned}$$

For the details about the directed graphs we refer, e.g., to a general book about the graph theory R. Diestel [27].

Let $u : V \times R_0^+ \to \mathbb{R}$ where $u(x, t)$ represents the amount of a modeled substance at vertex x and time t. Moreover, consider the flux $\phi : E \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ where $\phi(x, y, t)$ represents the flux through the edge (x, y) at time t. Note that it takes only nonnegative values, since the flux in opposite direction from y to x is realized through the edge (y, x) (provided it appears in the graph $\vec{\mathcal{G}}$). Next, let $f : V \times \mathbb{R}_0^+ \to \mathbb{R}$ be the function that describes the contribution of sources.

Let us proceed immediately to the derivation of the balance at an arbitrary vertex $x \in V$. The change of $u(x, t)$ during any time interval $[t_1, t_2]$ is caused only by the flux in and out of x and by the sources at x, i.e.,

$$u(x, t_2) - u(x, t_1) = \sum_{y \in \mathcal{N}_-(x)} \int_{t_1}^{t_2} \phi(y, x, t) \mathrm{d}t - \sum_{y \in \mathcal{N}_+(x)} \int_{t_1}^{t_2} \phi(x, y, t) \mathrm{d}t + \int_{t_1}^{t_2} f(x, t) \mathrm{d}t.$$

[2] Recall that in the continuous multi-dimensional conservation law (1.10) the flux density $\boldsymbol{\phi}$ is a vector-valued function.

Therefore, if we assume for the simplicity that $u(x,\cdot) \in C^1(\mathbb{R}_0^+, \mathbb{R})$, then using the fundamental theorem of calculus and the fact that the sums are finite we obtain

$$\int_{t_1}^{t_2} \left(u_t(x,t) + \sum_{y \in \mathcal{N}_+(x)} \phi(x,y,t) - \sum_{y \in \mathcal{N}_-(x)} \phi(y,x,t) - f(x,t) \right) \mathrm{d}t = 0.$$

Since it holds for all time intervals $[t_1, t_2]$ and we assume that all appearing functions are continuous, there has to be

$$u_t(x,t) + \sum_{y \in \mathcal{N}_+(x)} \phi(x,y,t) - \sum_{y \in \mathcal{N}_-(x)} \phi(y,x,t) = f(x,t), \quad x \in V, \quad t \in \mathbb{R}_0^+, \tag{5.8}$$

which we call the graph conservation law.

Again, we have to add an constitutive law that relates the function u with the flux ϕ. Let us firstly assume that the following relation holds for the flux

$$\phi(x,y,t) = ku^+(x,t), \quad k > 0, \quad (x,y) \in E, \tag{5.9}$$

where $u^+(x,t) = \max\{u(x,t), 0\}$ is the positive part of u (recall that the flux is nonnegative). Then (5.8) yields that the following equality has to be satisfied

$$u_t(x,t) + k \left(d_+(x)u^+(x,t) - \sum_{y \in \mathcal{N}_-(x)} u^+(y,t) \right) = f(x,t), \quad x \in V, \quad t \in \mathbb{R}_0^+. \tag{5.10}$$

We prove the following statement about the preservation of sign for solutions of (5.10).

Lemma 5.4. *Assume that f is nonnegative. Let u be a solution of* (5.10) *such that $u(x,0) \geq 0$ for all $x \in V$. Then*

$$u(x,t) \geq 0 \quad \textit{for all} \quad x \in V, \quad t \in \mathbb{R}_0^+.$$

Proof. It is obvious that $0 \leq d_+(x) \leq |V| - 1$ for all $x \in V$. Let $x \in V$ be arbitrary and fixed, then

$$u_t(x,t) = -kd_+(x)u^+(x,t) + k \sum_{y \in \mathcal{N}_-(x)} u^+(y,t) + f(x,t) \geq -k\left(|V| - 1\right) u^+(x,t). \tag{5.11}$$

Assume now by contradiction that there exists $t_1 > 0$ such that $u(x,t_1) < 0$. Let us define

$$t_0 = \inf\left\{ s \in [0, t_1] : \ u(x,\cdot) < 0 \text{ on } [s, t_1] \right\}. \tag{5.12}$$

Since u is a solution of (5.10), there is $u(x,\cdot) \in C(\mathbb{R}_0^+, \mathbb{R})$ and thus, $t_0 < t_1$. Indeed, $u(x,t_1) < 0$ and $u(x,\cdot) \in C(\mathbb{R}_0^+, \mathbb{R})$ yield together that $u(x,\cdot) < 0$ on $[t_1 - \delta_1, t_1]$ at least for small $\delta_1 > 0$. Furthermore, we claim that $u(x,t_0) = 0$. Assume that $u(x,t_0) > 0$, then the continuity of $u(x,\cdot)$ implies that $u(x,\cdot) > 0$ on $[t_0, t_0 + \delta_0]$ for small $\delta_0 > 0$, a contradiction with the definition of t_0 in (5.12). On the contrary, if $u(x,t_0) < 0$, then $t_0 > 0$ from the assumption on initial condition. Again, $u(x,\cdot) \in C(\mathbb{R}_0^+, \mathbb{R})$ yields that $u(x,\cdot) < 0$ on $[t_0 - \delta_0', t_0]$ for a small $\delta_0' > 0$, a contradiction with (5.12).

Consequently, there is $u(x,t_0) = 0$, $u(x,\cdot) < 0$ on $(t_0, t_1]$ and thus, $u_t(x,t) \geq 0$ on $[t_0, t_1]$ from (5.11). Applying the Gronwall inequality to the problem

$$\begin{cases} u_t(x,t) \geq 0, & t \in [t_0, t_1], \\ u(x,t_0) = 0, \end{cases}$$

we obtain $u(x, t_1) \geq 0$, a contradiction.[3] □

Consequently, if we consider a nonnegative initial condition for u and nonnegative sources f (e.g., a source-free model), we obtain applying Lemma 5.4 that $u^+(x,t) = u(x,t)$ for all $x \in V$ and $t \in \mathbb{R}_0^+$. Thus, a solution of (5.10) is nonnegative and actually a solution of

$$u_t(x,t) + k\left(d_+(x)u(x,t) - \sum_{y \in \mathcal{N}_-(x)} u(y,t)\right) = f(x,t), \quad x \in V, \quad t \in \mathbb{R}_0^+. \tag{5.13}$$

Let us note that the converse is not true, since (5.13) can have the sign-changing solutions even with the vanishing initial condition (see, e.g., A. Slavík, P. Stehlík [76]). However, the nonnegative solutions of (5.13) are also solutions of (5.10).

Example 5.5. Let $\vec{\mathcal{G}} = (V, E)$ where

$$V = \mathbb{Z}, \quad E = \{(x, x+1) : \ x \in V\},$$

(i.e., one-sided oriented integers). Then for all $x \in V$ there is

$$d_+(x) = 1, \quad \mathcal{N}_-(x) = \{x - 1\}.$$

Therefore, the problem (5.13) becomes

$$u_t(x,t) + k\left(u(x,t) - u(x-1,t)\right) = f(x,t), \quad x \in V, \quad t \in \mathbb{R}_0^+,$$

i.e., the linear transport equation with $x \in \mathbb{Z}$, which is actually studied in Section 2.1.

Example 5.6. Let $\vec{\mathcal{G}} = (V, E)$ where

$$V = \mathbb{Z}, \quad E = \{(x, x+1), (x, x-1) : \ x \in V\},$$

(i.e., two-sided oriented integers, or equivalently, undirected integers). Hence, we get for all $x \in V$ that

$$d_+(x) = 2, \quad \mathcal{N}_-(x) = \{x - 1, x + 1\}.$$

Thus, the problem (5.13) is equivalent to

$$u_t(x,t) = k\left(u(x-1,t) - 2u(x,t) + u(x+1,t)\right) + f(x,t), \quad x \in V, \quad t \in \mathbb{R}_0^+,$$

i.e., the linear diffusion equation with $x \in \mathbb{Z}$, which is investigated in A. Slavík, P. Stehlík [75, 76] and M. Friesl, A. Slavík, P. Stehlík [34].

As another example of the constitutive law we can mention more complicated relation

$$\phi(x,y,t) = k\left(u(x,t) - u(y,t)\right)^+, \quad k > 0, \quad (x,y) \in E. \tag{5.14}$$

This combined with the graph conservation law (5.8) yields the following problem

[3] The Gronwall inequality reads as follows. Let $\alpha \in C([a,b], \mathbb{R})$, $u \in C([a,b], \mathbb{R})$ be differentiable on (a,b) and satisfy

$$u'(t) \leq \alpha(t)u(t) \quad \text{for} \quad t \in (a,b).$$

Then

$$u(b) \leq u(a)\mathrm{e}^{\int_a^b \alpha(t)\mathrm{d}t}.$$

$$u_t(x,t) + k\left(\sum_{y\in\mathcal{N}_+(x)} (u(x,t)-u(y,t))^+ - \sum_{y\in\mathcal{N}_-(x)} (u(y,t)-u(x,t))^+\right) = f(x,t), \quad x\in V, \quad t\in\mathbb{R}_0^+. \tag{5.15}$$

Primarily, let us assume for the simplicity that $\vec{\mathcal{G}}$ is such that each edge $(x,y)\in E$ has its opposite, i.e., $(y,x)\in E$ (actually, this is equivalent to undirected graphs). Thus, for every fixed $x\in V$ and for all $y\in\mathcal{N}_+(x)=\mathcal{N}_-(x)$ there is

$$\text{either} \quad u(x,t)\geq u(y,t) \quad \text{or} \quad u(x,t)\leq u(y,t),$$

i.e.,

$$\text{either} \quad \phi(x,y,t)=k(u(x,t)-u(y,t)),\ \phi(y,x,t)=0 \quad \text{or} \quad \phi(x,y,t)=0,\ \phi(y,x,t)=k(u(y,t)-u(x,t)).$$

Hence, the equation (5.15) becomes

$$u_t(x,t) + k\left(d_+(x)u(x,t) - \sum_{y\in\mathcal{N}_-(x)} u(y,t)\right) = f(x,t), \quad x\in V, \quad t\in\mathbb{R}_0^+, \tag{5.16}$$

which is again the linear diffusion equation, at this moment on symmetric (or undirected) graph spatial structure. Let us note that we have studied the stationary counterpart of (5.16) in Example 4.5.

Consequently, there are several possible ways of future work. We are going to study problems (5.10) or (5.15):

- either for more complicated specific choices of graphs $\vec{\mathcal{G}}$,
- or for special classes of graphs,
- or for general graphs if it is possible.

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- P. Stehlík, J. Volek, *Transport equation on semidiscrete domains and Poisson-Bernoulli processes*, Journal of Difference Equations and Applications 19 (2013), 439–456.[4]
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- A. Slavík, P. Stehlík, J. Volek, *Well-posedness and maximum principles for lattice reaction-diffusion equations*, submitted (2016).

[4] The paper P. Stehlík, J. Volek [79] has been published before the doctoral study. We include it into the thesis for the sake of compactness.

Author's talks on conferences

- Equadiff 13, Prague, Czech Republic, August 26–30, 2013, *Transport equation on semidiscrete domains.*
- XXIX Seminar in Differential Equations, Monínec, Czech Republic, April 14–18, 2014, *Maximum principles for nonlinear transport equations on discrete-space domains.*
- Seminar on Differential Equations and Integration Theory, Mathematical Institute, Czech Academy of Sciences, Prague, April 24, 2014, *Transport equations on discrete-space domains.*
- Conference on Differential and Difference Equations and Applications, Jasná, Demänovská Dolina, Slovakia, June 23–27, 2014, *Maximum principles for nonlinear transport equations on discrete-space domains.*
- International Conference on Differential and Difference Equations and Applications, Lisbon, Portugal, May 18–22, 2015, *Maximum principles for discrete reaction-diffusion equations.*
- VII. Symposium on Nonlinear Analysis 2015, Toruń, Poland, September 14–18, 2015, *Maximum principles for discrete reaction-diffusion equations.*
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APPENDIX A

Transport equation on semidiscrete domains and Poisson-Bernoulli processes

[79] P. Stehlík, J. Volek, *Transport equation on semidiscrete domains and Poisson-Bernoulli processes*, Journal of Difference Equations and Applications 19(3) (2013), 439–456.

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Transport equation on semidiscrete domains and Poisson–Bernoulli processes

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In this paper, we consider a scalar transport equation with constant coefficients on domains with discrete space and continuous, discrete or general time. We show that on all these underlying domains, solutions of the transport equation can conserve sign and integrals both in time and space. Detailed analysis reveals that, under some initial conditions, the solutions correspond to counting stochastic processes and related probability distributions. Consequently, the transport equation could generate various modifications of these processes and distributions and provide some insights into corresponding convergence questions. Possible applications are suggested and discussed.

Keywords: transport equation; conservation law; time scales; semidiscrete method; Poisson process; Bernoulli process

AMS Subject Classification: 34N05; 35F10; 39A14; 65M06

1. Introduction

Scalar transport equation with constant coefficients $u_t + ku_x = 0$ belongs among the simplest partial differential equations. Its importance is based on the following facts. Firstly, it describes advective transport of fluids, as well as one-way wave propagation. Secondly, it serves as a base for a study of hyperbolic partial differential equations (and is consequently analysed also in numerical analysis). Thirdly, its nonlinear modifications model complex transport of fluids, heat or mass. Finally, its study is closely connected to conservation laws (see [9] or [13]).

Properties and solutions of partial difference equations have been studied mainly from numerical (e.g. [13]) and also from analytical point of view (e.g. [5]). Meanwhile, in one dimension, there has been a wide interest in the problems with mixed timing, which has recently been clustered around the time scales calculus and the so-called dynamic equations (see [4,11]). Nevertheless, there is only limited literature on partial equations on time scales (see [1,3,18]). These papers indicate the complexity of such settings and the necessity to analyse basic problems such as transport equation. Our analysis is also closely related to numerical semidiscrete methods (e.g. [13], Section 10.4) or analytical Rothe method (e.g. [17]).

In this paper, we consider a transport equation on domains with discrete space and general (continuous, discrete and time scale) time (see Figure 1). We show that the solutions of transport equation does not propagate along characteristics lines as in the

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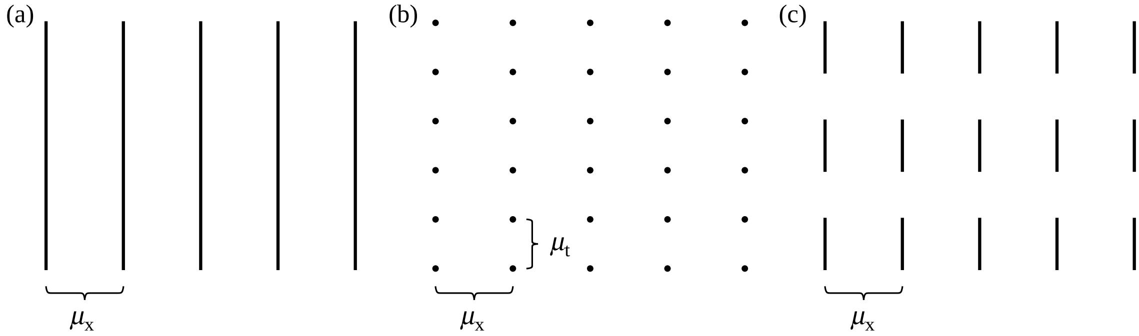

Figure 1. Examples of various domains considered in this paper. We study domains with discrete space and continuous (Section 4), discrete (Section 5) and general time (Section 6).

classical case and feature behaviour close to the classical diffusion equation. Our analysis of sign and integral conservation discloses interesting relationship between the solutions on such domains and probability distributions related to Poisson and Bernoulli stochastic processes. These counting processes are used to model waiting times for occurrence of certain events (defects, phone calls, customers' arrivals, etc.), see [2,10] or [15] for more details. Consequently, considering domains with general time, we are able not only to generalize these standard processes but also to generate transitional processes of Poisson–Bernoulli type and corresponding distributions. Moreover, our analysis provides a different perspective on some numerical questions (numerical diffusion) and relate it to analytical problems (relationship between the Courant-Friedrich-Lewy (CFL) condition and regressivity). Finally, it also establishes relationship between the time scales calculus and heterogeneous and mixed probability distributions in the probability theory.

In Section 3, we summarize well-known features of the classical transport equation. In Section 4, we consider a transport equation with discrete space and continuous time. In Section 5, we solve the problem on domains with discrete time. In Section 6, we generalize those results to domains with a general time and prove the necessary and sufficient conditions which ensure that the sign and both time and space integrals are conserved (Theorem 6.9). Finally, in Section 7, we discuss convergence issues, applications to probability distributions and stochastic processes and provide two examples.

2. Preliminaries and notation

The sets $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ denote real, integer and natural numbers. Furthermore, let us introduce $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ = [0, \infty)$. Finally, we use multiples of discrete number sets, e.g. a-multiple of integers is denoted by $a\mathbb{Z}$ and defined by $a\mathbb{Z} = \{\dots, -2a, -a, 0, a, 2a, \dots\}$.

Partial derivatives are denoted by $u_t(x,t)$ and $u_x(x,t)$ and partial differences by

$$\Delta_t u(x,t) = \frac{u(x, t+\mu_t) - u(x,t)}{\mu_t} \quad \text{and} \quad \nabla_x u(x,t) = \frac{u(x,t) - u(x-\mu_x, t)}{\mu_x}, \tag{1}$$

where μ_t and μ_x denote step sizes in time and space.

In Section 6, we consider time to be a general time scale $\mathbb{T}$, i.e. an arbitrary closed subset of $\mathbb{R}$. Time step could be variable, described by a graininess function $\mu_t : \mathbb{T} \to \mathbb{R}_0^+$. We use the partial delta derivative u^{Δ_t} which reduces to u_t in points in which $\mu_t(t) = 0$ or to $\Delta_t u$ in those t in which $\mu(t) > 0$. Similarly, we work with the so-called delta integral which corresponds to standard integration if $\mathbb{T} = \mathbb{R}$ or to summation if $\mathbb{T} = \mathbb{Z}$. Finally, the dynamic exponential function $e_p(x, x_0)$ is defined as a solution of the initial value problem (under the regressivity condition $1 + p(t)\mu(t) \neq 0$)

$$\begin{cases} x^{\Delta}(t) = p(t)x(t), \\ x(0) = 1. \end{cases}$$

For more details concerning the time scale calculus, we refer the interested reader to the survey monograph [4].

Given function $u(x,t)$, by $u(x,\,\cdot\,)$ we mean functions of one variable having the form $u(0,t)$, $u(1,t)$, etc. Similarly, by $u(\,\cdot\,,t)$ we understand one-dimensional sections of $u(x,t)$ having the form $u(x,0)$, $u(x,1)$, etc.

3. Classical transport equation

Let us briefly summarize essential properties of the classical transport equation

$$\begin{cases} u_t(x,t) + ku_x(x,t) = 0, \quad t \in \mathbb{R}_0^+, \ x \in \mathbb{R}, \\ u(x,0) = \phi(x), \quad x \in \mathbb{R}, \end{cases} \tag{2}$$

with $\phi \in C^1$. Typical features whose counterparts are studied in this paper include:

- the unique solution $u(x,t) = \phi(x-kt)$ could be obtained via the *method of characteristics*, the solution is constant on the *characteristic lines* where $x - kt = C$,
- consequently, the solution *conserves sign*, i.e. if $\phi(x) \geq 0$ then $u(x,t) \geq 0$,
- moreover, the solution *conserves integral* in space sections, i.e. if $\int_{-\infty}^{\infty} \phi(x)\mathrm{d}x = K$, then

$$\int_{-\infty}^{\infty} u(x,t)\mathrm{d}x = K \quad \text{for all} \ \ t \geq 0,$$

- finally, the solution *conserves integral* in time sections in the following sense. For $k > 0$, we have that

$$\int_0^{\infty} u(x,t)\mathrm{d}t = \frac{1}{k}\int_{-\infty}^{x} \phi(s)\mathrm{d}s.$$

Consequently, if $\phi(x) = 0$ for $x \geq x_0$, then the integral along time sections is constant for all $x \geq x_0$.

4. Discrete space and continuous time

In contrast to the classical problem (2), we consider a domain with discrete space and the problem

$$\begin{cases} u_t(x,t) + k\nabla_x u(x,t) = 0, \quad t \in \mathbb{R}_0^+, \ x \in \mathbb{Z}, \\ u(x,0) = \begin{cases} A, & x = 0, \\ 0, & x \neq 0, \end{cases} \end{cases} \tag{3}$$

where $A > 0$, $k > 0$ and $\nabla_x u$ reduces to[1]

$$\nabla_x u(x,t) = u(x,t) - u(x-1,t).$$

One could rewrite the equation in (3) into

$$u_t(x,t) = -ku(x,t) + ku(x-1,t),$$

which implies that problem (3) could be viewed as an infinite system of differential equations.

LEMMA 4.1. *The unique solution of problem* (3) *has the form:*

$$u(x,t) = \begin{cases} A\frac{k^x}{x!}t^x\,\mathrm{e}^{-kt}, & t \in \mathbb{R}_0^+, \ x \in \mathbb{N}_0, \\ 0, & t \in \mathbb{R}_0^+, \ x \in \mathbb{Z}, \ x < 0. \end{cases} \tag{4}$$

Proof. First, let us observe that $u(x,t) = 0$ for all $t \in \mathbb{R}_0^+$, $x < 0$. The uniqueness of the trivial solution for $x < 0$ follows, e.g. from [16] (Corollary 1) or more generally from [6] (Theorem 3.1.3). Let us prove the rest (i.e. $x \geq 0$ by mathematical induction). Obviously, we have that $u(0,t) = A\,\mathrm{e}^{-kt}$, since $u_t(0,t) = -ku(0,t) + ku(-1,t) = -ku(0,t)$ and $u(0,0) = A$.

Moreover, if we assume that $x \in \mathbb{N}_0$ and $u(x,t) = A(k^x/x!)t^x\,\mathrm{e}^{-kt}$, then $u(x+1,t)$ satisfies

$$\begin{cases} u_t(x+1,t) = -ku(x+1,t) + A\frac{k^x}{x!}t^x\,\mathrm{e}^{-kt}, & t \in \mathbb{R}_0^+, \\ u(x,0) = 0. \end{cases}$$

One could use the variation of parameters to show that the unique solution is $u(x+1,t) = A(k^{x+1}/(x+1)!)t^{x+1}\,\mathrm{e}^{-kt}$, which proves the inductive step and consequently finishes the proof. □

Let us analyse the sign and integral preservation of (3).

LEMMA 4.2. *The solution of problem* (3) *conserves the sign, the integral in time and the sum in the space variable.*

Proof. The sign preservation follows from the positivity of all terms in (4). Next, we could use integration by parts to obtain (we skip the details since we prove this result in more general settings in Theorem 6.5)

$$\int_0^\infty u(x,t)\mathrm{d}t = \frac{A}{k}. \tag{5}$$

Similarly, summing over x we get

$$\sum_{x=0}^{\infty} A\frac{k^x}{x!}t^x\,\mathrm{e}^{-kt} = A\,\mathrm{e}^{-kt}\sum_{x=0}^{\infty}\frac{(kt)^x}{x!} = A\,\mathrm{e}^{-kt}\,\mathrm{e}^{kt} = A. \tag{6}$$

□

If we go deeper and analyse values obtained in (5) and (6), we get the first indication of the relationship of the semidiscrete transport equation with stochastic processes.

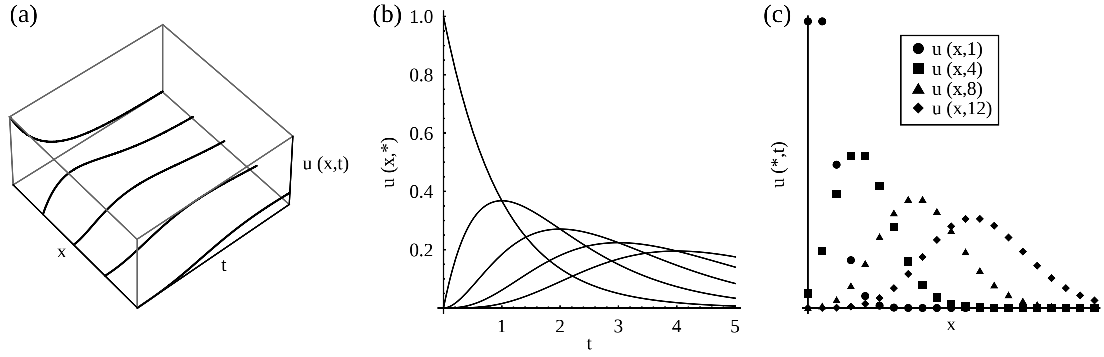

Figure 2. Solution of the transport equation with discrete space and continuous time (3) with $A = 1$ and $k = 1$.

Remark 1. If $A = k$ then time sections of solution (4) generate the probability density function of Erlang distributions (note that for $x = 0$ we get the exponential distribution and that Erlang distributions are special cases of Gamma distributions).

Similarly, if $A = 1$ the space sections of (4) form the probability mass functions of Poisson distributions (see Figure 2).

Consequently, if $A = k = 1$ the solution $u(x, t)$ describes Poisson process. All these facts are further discussed in Section 7.

We conclude this section with two natural extensions. Firstly, we mention possible generalizations to other discrete space structures.

Remark 2. If we consider problem (3) on a domain with a discrete space having the constant step $\mu_x > 0$, not necessarily $\mu_x = 1$, we obtain qualitatively equivalent problem, since

$$\begin{aligned} u_t(x,t) + k\frac{u(x,t) - u(x-\mu_x,t)}{\mu_x} &= u_t(x,t) + \frac{k}{\mu_x}\big(u(x,t) - u(x-\mu_x,t)\big) \\ &= u_t(x,t) + \hat{k}\nabla_x u(x,t). \end{aligned}$$

In contrast to the rest of this paper, the value of μ_x does not play essential role here. Therefore, for presentation purposes, we restricted our attention to $\mu_x = 1$.

Finally, we discuss more general initial condition and show that the solution is the sum of point initial conditions which justifies their use not only in this section but also in the remainder of this paper.

Corollary 4.3. *The unique solution of*

$$\begin{cases} u_t(x,t) + k\nabla_x u(x,t) = 0, & t \in \mathbb{R}_0^+, \ x \in \mathbb{Z}, \\ u(x,0) = C_x, \end{cases} \tag{7}$$

is given by

$$u(x,t) = \sum_{i=-\infty}^{x} C_i \frac{(kt)^{x-i}}{(x-i)!} \mathrm{e}^{-kt}. \tag{8}$$

Proof. One could split (7) into problems with point initial conditions, use Lemma 4.1 to solve them and then employ linearity of the equation to get (8). □

5. Discrete space and discrete time

In this section, we assume that both time and space are homogeneously discrete with steps $\mu_t > 0$ and $\mu_x > 0$, respectively. In other words, we consider a discrete domain

$$\Omega = \{(x,t) = (m\mu_x, n\mu_t), \ \text{with} \ m \in \mathbb{Z}, n \in \mathbb{N}_0\}.$$

The transport equation and the corresponding problem then have the form

$$\begin{cases} \Delta_t u(x,t) + k\nabla_x u(x,t) = 0, & (x,t) \in \Omega, \\ u(x,0) = \begin{cases} A, & x = 0, \\ 0, & x \neq 0, \end{cases} \end{cases} \tag{9}$$

where $A > 0, k > 0$. Using the definition of partial differences in (1), we can easily rewrite the equation in (9) into

$$u(x, t+\mu_t) = \left(1 - \frac{k\mu_t}{\mu_x}\right)u(x,t) + \frac{k\mu_t}{\mu_x}u(x-\mu_x, t) \tag{10}$$

and derive the unique solution.

Lemma 5.1. *Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The unique solution of* (9) *has the form:*

$$u(m\mu_x, n\mu_t) = \begin{cases} A\binom{n}{m}\left(1 - \frac{k\mu_t}{\mu_x}\right)^{n-m}\left(\frac{k\mu_t}{\mu_x}\right)^m, & n \geq m \geq 0, \\ 0, & 0 \leq n < m, \ \text{or} \ m < 0. \end{cases} \tag{11}$$

Proof. First, let us show that the solution vanishes uniquely for $u(-m\mu_x, n\mu_x) = 0$ for all $m, n \in \mathbb{N}$. Consulting (10), we observe that the value of $u(-m\mu_x, n\mu_x)$ is obtained as a linear combination of initial conditions $u(-m\mu_x, 0)$, $u(-(m+1)\mu_x, 0), \ldots,$ $u(-(m+n)\mu_x, 0)$, i.e. a linear combination of $n+1$ zeros.

We prove the rest of the statement by induction. Apparently,

$$u(0, n\mu_t) = \left(1 - \frac{k\mu_t}{\mu_x}\right)^n u(0,0) = A\left(1 - \frac{k\mu_t}{\mu_x}\right)^n.$$

Next, let us assume that $u(m\mu_x, n\mu_t)$ satisfies (11), then

$$\begin{aligned} u\big((m+1)\mu_x, n\mu_t\big) &= \left(1 - \frac{k\mu_t}{\mu_x}\right)^n u((m+1)\mu_x, 0) \\ &+ \sum_{r_{m+1}=0}^{n-1}\left(1 - \frac{k\mu_t}{\mu_x}\right)^{n-1-r_{m+1}} A\left(1 - \frac{k\mu_t}{\mu_x}\right)^{r_{m+1}-m}\left(\frac{k\mu_t}{\mu_x}\right)^{m+1}\sum_{r_m=0}^{r_{m+1}-1}\cdots\sum_{r_2=0}^{r_3-1}\sum_{r_1=0}^{r_2-1} 1 \\ &= A\left(1 - \frac{k\mu_t}{\mu_x}\right)^{n-(m+1)}\left(\frac{k\mu_t}{\mu_x}\right)^{m+1}\sum_{r_{m+1}=0}^{n-1}\cdots\sum_{r_2=0}^{r_3-1}\sum_{r_1=0}^{r_2-1} 1. \end{aligned}$$

At this stage, let us observe the properties of the falling factorials (see, e.g. [8], Section 2.1 or [12], Section 2.1) to get that

$$\sum_{r_{m+1}=0}^{n-1} \cdots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1 = \frac{n^{\underline{m+1}}}{(m+1)!} = \binom{n}{m+1},$$

which finishes the proof. □

The closed-form solution enables us to analyse sign and integral conservation.

Lemma 5.2. *If the inequality*

$$1 - \frac{k\mu_t}{\mu_x} > 0 \tag{D1}$$

holds then the solution of (9) *satisfies*

(i) $u(x,t) \geq 0$,
(ii) $\sum_{m=-\infty}^{\infty} u(m\mu_x, t)$ *is constant for all* $t = \{0, \mu_t, 2\mu_t, \dots\}$,
(iii) $\sum_{n=0}^{\infty} u(x, n\mu_t)$ *is constant for all* $x = \{0, \mu_x, 2\mu_x, \dots\}$.

Proof.

(i) The inequality follows immediately from Lemma 5.1.
(ii) If we fix t and sum up equation (10) over x we get

$$\sum_{m=-\infty}^{\infty} u(m\mu_x, t+\mu_t) = \left(1 - \frac{k\mu_t}{\mu_x}\right) \sum_{m=-\infty}^{\infty} u(m\mu_x, t) + \frac{k\mu_t}{\mu_x} \sum_{m=-\infty}^{\infty} u((m-1)\mu_x, t).$$

The assumption (D1) implies that the sum on the left-hand side is a linear combination of two sums on the right-hand side. Since these sums are equal, we get that

$$\sum_{m=-\infty}^{\infty} u(m\mu_x, t+\mu_t) = \sum_{m=-\infty}^{\infty} u(m\mu_x, t).$$

(iii) Similarly, one could sum up equation (10) over t to get for a fixed $x > 0$

$$\sum_{n=1}^{\infty} u(x, n\mu_t) = \left(1 - \frac{k\mu_t}{\mu_x}\right) \sum_{n=0}^{\infty} u(x, n\mu_t) + \frac{k\mu_t}{\mu_x} \sum_{n=0}^{\infty} u(x-\mu_x, n\mu_t).$$

Since $u(x,0) = 0$ for $x > 0$ we have that

$$\sum_{n=0}^{\infty} u(x, n\mu_t) = \sum_{n=0}^{\infty} u(x-\mu_x, n\mu_t).$$

□

Once again, we could study the solutions' relationship to probability distributions.

Theorem 5.3. *Let* $u(x,t)$ *be a solution of* (9). *Then the space and time sections* $\mu_x u(x, \cdot)$ *and* $\mu_t u(\cdot, t)$ *form probability mass functions if and only if the assumptions* (D1)

$$\frac{A\mu_x}{k} = 1 \tag{D2}$$

and

$$A\mu_x = 1 \tag{D3}$$

hold.

Proof. Lemma 5.2 yields that the solutions are non-negative and conserve sums. It suffices to include step lengths μ_x and μ_t and identify conditions under which $\mu_x \sum_x u(x,0) = 1$ and $\mu_t \sum_t u(0,t) = 1$. Given the initial condition, the former sum is equal to $A\mu_x$. Hence the assumption (D3). Finally, since $u(-\mu_x, t) = 0$, equation (10) implies that $u(0, n\mu_t) = A\left(1 - (k\mu_t/\mu_x)\right)^n$. Consequently,

$$1 = A\mu_t \sum_{n=0}^{\infty} \left(1 - \frac{k\mu_t}{\mu_x}\right)^n = \frac{A\mu_x}{k}.$$

□

COROLLARY 5.4. *Let $u(x,t)$ be a solution of* (9). *Then the space and time sections $\mu_x u(x,\cdot)$ and $\mu_t u(\cdot,t)$ form probability mass functions if and only if $k = 1$, $\mu_t < \mu_x$ and $A = 1/\mu_x$.*

Proof. (D2) and (D3) hold if and only if $k = 1$. Consequently, (D1) could be satisfied if and only if $\mu_t < \mu_x$. □

Closer examination again reveals that the sections form probability mass functions of discrete probability distributions related to Bernoulli counting processes (see Figure 3).

Remark 3. Let us consider solution (11). If we put $A = k = \mu_x = 1$ and $\mu_t = p$, we get

$$u(n, m \cdot p) = \binom{n}{m} (1-p)^{n-m} p^m, \quad n \geq m,$$

which forms, for each fixed $n \in \mathbb{N}_0$, a probability mass function of the binomial distribution. Similarly, for each fixed $m \in \mathbb{N}_0$, $p = \mu_t$-multiple forms a probability mass function of a version of the negative binomial distribution (the value $p \cdot u(n, m \cdot p)$ describes a probability that for m failures we need n trials). Consequently, the solution of (9) describes a counting Bernoulli stochastic process (see [2]).

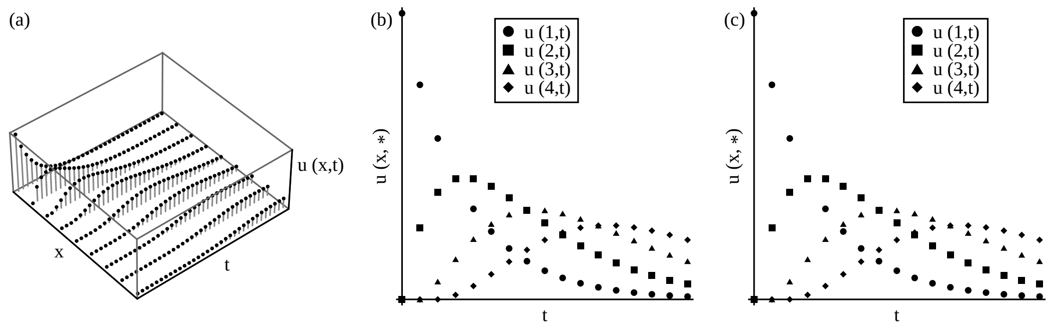

Figure 3. Solution of the transport equation with discrete space and discrete time (9) with $A = 1$, $k = 1$, $\mu_t = 0.25$ and $\mu_x = 1$.

6. Discrete space and general time

Let us extend the results from the last two sections by considering more general time structures. Let $\mathbb{T}$ be a time scale such that $\min \mathbb{T} = 0$ and $\sup \mathbb{T} = +\infty$. In this paragraph we consider domains

$$\Omega = \{(x,t) : x \in \mu_x \mathbb{Z},\ \ t \in \mathbb{T}\},$$

and the problem:

$$\begin{cases} u^{\Delta_t}(x,t) + k\nabla_x u(x,t) = 0, & (x,t) \in \Omega, \\ u(x,0) = \begin{cases} A, & x = 0, \\ 0, & x \neq 0, \end{cases} \end{cases} \tag{12}$$

where $A > 0$, $k > 0$ and $\nabla_x u(x,t)$ is the backward difference defined in (1) and u^{Δ_t} is the delta derivative in time variable. Since the space is discrete, we could again rewrite equation (12) into

$$u^{\Delta_t}(x,t) = -\frac{k}{\mu_x}\big(u(x,t) - u(x-\mu_x,t)\big). \tag{13}$$

In order to conserve the sign of solutions, we assume that

$$1 - \frac{k\mu_t(t)}{\mu_x} > 0, \tag{TS1}$$

i.e. the condition which is similar to the positive regressivity in the time scale theory (e.g. [4], Section 2.2) or the so-called CFL condition in the discretization of the transport equation (e.g. [13], Section 4.4).

Let u be a solution of (12). One could use [14] (Proposition 5.2) to show that $u(x,t) = 0$ for all $x < 0$ is the unique solution there. Since $u(-\mu_x, t) = 0$, we could see that $u^{\Delta_t}(0,t) = -(k/\mu_x)u(0,t)$. Given the initial condition and assumption (TS1), we get $u(0,t) = A\,\mathrm{e}_{-k/\mu_x}(t;0)$, where $\mathrm{e}_{-k/\mu_x}(t;0)$ is a time scale exponential function (see [4], Section 2).

Lemma 6.1. *The solution of* (12) *satisfies*

(i) $\lim_{t\to\infty} u(0,t) = 0$,
(ii) $\int_0^\infty u(0,t)\Delta t = A(\mu_x/k)$.

Proof.

(i) Follows directly from assumption (TS1) and the properties of the exponential functions [4] (Section 2.2),
(ii)

$$\int_0^\infty u(0,t)\Delta t = A\int_0^\infty \mathrm{e}_{-\frac{k}{\mu_x}}(t;0)\Delta t = \lim_{t\to\infty} -A\frac{\mu_x}{k}\left(\mathrm{e}_{-(k/\mu_x)}(t;0) - 1\right) = A\frac{\mu_x}{k}.$$

□

Unique solutions of $u(m\mu_x, t)$ could be found using the variation of constants (see, e.g. [4], Theorem 2.77). However, these computations depend critically on a particular time scale and cannot be performed in general. For example, one could compute that the second branch of the solution has the form

$$u(\mu_x, t) = A\frac{k}{\mu_x}\mathrm{e}_{-\frac{k}{\mu_x}}(t;0)\int_0^t \frac{\Delta\tau}{1-(k\mu_t(\tau)/\mu_x)}.$$

This implies that we cannot derive closed-form solutions as in previous sections. Formally, these solutions can be expressed as Taylor-like series with generalized polynomials whose form depends on particular time scale (see [14] and [4] (Section 1.6)). We determine these solutions in special cases (see Lemmata 4.1, 5.1 and 7.1). Therefore, we are forced to use another means to show the properties of solutions we are interested in.

LEMMA 6.2. *Let $x \in \mu_x\mathbb{N}$. If* (TS1) *is satisfied and $u(x-\mu_x, t) \geq 0$ and for all $t \in \mathbb{T}$ and $u(x-\mu_x, t) > 0$ at least for one $t \in \mathbb{T}$, then $u(x,t) \geq 0$ for all $t \in \mathbb{T}$.*

Proof. First, note that $u(x,0) = 0$ for all $x > 0$. Consequently, (13) implies that $u^{\Delta_t}(x,t) > 0$ at the beginning of the support of $u(x-\mu_x, t)$ and $u(x,t)$ is strictly increasing there.

- If t is right-scattered then we can rewrite equation (13) into

$$u(x, t+\mu_t) = \left(1-\frac{k\mu_t}{\mu_x}\right)u(x,t) + \frac{k\mu_t}{\mu_x}u(x-\mu_x, t).$$

  If $u(x,t) \geq 0$, then this is the weighted average of two non-negative values and thus non-negative as well.
- If t is right-dense then equation (16) has the form

$$u_t(x,t) = -\frac{k}{\mu_x}u(x,t) + \frac{k}{\mu_x}u(x-\mu_x, t).$$

  Since both $u(x-\mu_x, t) \geq 0$ and $u(x,t) \geq 0$, we have that $u_t(x,t) \geq -(k/\mu_x)u(x,t)$ and thus $u(x,t)$ cannot become negative.

Following the induction principle (e.g. [4], Theorem 1.7), we could see that $u(x,t) \geq 0$ for all $t \in \mathbb{T}$. □

Lemma 6.2 serves as the inductive step in the proof of the sign conservation.

THEOREM 6.3. *If* (TS1) *holds then $u(x,t) \geq 0$ for all $(x,t) \in \Omega$.*

Proof. We prove the statement by mathematical induction. Firstly, $u(0,t) = A\,\mathrm{e}_{-k/\mu_x}(t;0) > 0$. Secondly, if $u(x,t) \geq 0$ then Lemma 6.2 implies that $u(x+\mu_x, t) \geq 0$ which finishes the proof. □

The following auxiliary lemma shows that the variation of constant formula which generates further branches of solutions conserve zero-limits at infinity.

LEMMA 6.4. *Let us consider a time scale $\mathbb{T}$, a constant K such that $1-\mu K > 0$ and a function $f : \mathbb{T} \to [0,\infty)$ such that the integral $\int_0^\infty f(t)\Delta t$ is finite. If we define $g : \mathbb{T} \to [0,\infty)$ by*

$$g(t) = \int_0^t \mathrm{e}_{-K}(t, \sigma(\tau))f(\tau)\Delta\tau,$$

then $\lim_{t\to\infty} g(t) = 0$.

Proof. Since $\int_0^\infty f(t)\Delta(t)$ is finite we know that for each $\varepsilon > 0$ there exists $T > 0$ such that for all $t \in \mathbb{T}$, $t > T$ the inequality

$$\int_t^\infty f(\tau)\Delta\tau < \frac{\varepsilon}{2}, \tag{14}$$

holds. Similarly, properties of time scale exponential function imply that for each $\varepsilon > 0$ and $T > 0$ there exists $R > T$ such that for all $t \in \mathbb{T}$, $t > R$ the following inequality is satisfied

$$\int_0^T \mathrm{e}_{-K}(t;\sigma(\tau))\Delta\tau < \frac{\varepsilon}{2F}, \tag{15}$$

with $F = \max_{t\in\mathbb{T}} f(t)$. Consequently, inequalities (14) and (15) imply that for each for each $\varepsilon > 0$ there exists $T > 0$ and $R > T$ such that for all $t > R$

$$\begin{aligned}
g(t) &= \int_0^t \mathrm{e}_{-K}(t,\sigma(\tau))f(\tau)\Delta\tau \\
&= \int_0^T \mathrm{e}_{-K}(t;\sigma(\tau))f(\tau)\Delta\tau + \int_T^t \mathrm{e}_{-K}(t;\sigma(\tau))f(\tau)\Delta\tau \\
&\leq F\int_0^T \mathrm{e}_{-K}(t;\sigma(\tau))\Delta\tau + \int_T^t f(\tau)\Delta\tau \\
&< F\frac{\varepsilon}{2F} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

which implies that $\lim_{t\to\infty} g(t) = 0$. □

Consequently, we are able to show that the integrals are constant for each fixed $x \geq 0$.

THEOREM 6.5. *If* (TS1) *holds and* $u(x,t)$ *is a solution of* (12)*, then*

$$\int_0^\infty u(x,t)\Delta t = \int_0^\infty u(0,t)\Delta t = A\frac{\mu_x}{k}$$

for all $x \in \mu_x\mathbb{N}_0$.

Proof. We proceed by mathematical induction.

- For $x = 0$ the convergence of the integral to $A(\mu_x/k)$ follows from Lemma 6.1(ii).
- Let us fix $x \in \mu_x\mathbb{N}$ and assume that the statement holds for a function $u(x-\mu_x,t)$. If we integrate (13) we get

$$\int_0^\infty u^{\Delta_t}(x,\tau)\Delta\tau = -\frac{k}{\mu_x}\left(\int_0^\infty u(x,\tau)\Delta\tau - \int_0^\infty u(x-\mu_x,\tau)\Delta\tau\right). \tag{16}$$

Let us concentrate on the left-hand side term. The variation of constants formula [4] (Theorem 2.77) implies that

$$u(x,t) = \int_0^t \mathrm{e}_{-(k/\mu_x)}(t,\sigma(\tau))u(x-\mu_x,\tau)\Delta\tau.$$

Consequently, Lemma 6.4 implies that $\lim_{t\to\infty} u(x,t) = 0$. Using the initial condition

$u(x, 0) = 0$, we could rewrite the left-hand side of (16) into

$$\int_0^\infty u^{\Delta_t}(x, \tau)\Delta\tau = \lim_{t\to\infty} u(x, t) - u(x, 0) = 0.$$

This implies that (16) could be rewritten into

$$0 = -\frac{k}{\mu_x}\left(\int_0^\infty u(x, \tau)\Delta\tau - \int_0^\infty u(x - \mu_x, \tau)\Delta\tau\right),$$

or equivalently into

$$\int_0^\infty u(x, \tau)\Delta\tau = \int_0^\infty u(x - \mu_x, \tau)\Delta\tau,$$

which finishes the proof. □

Finally, we show that the integrals (sums in this case) remains constant in time as well.

THEOREM 6.6. *If* (TS1) *holds and* $u(x, t)$ *is a solution of* (12)*, then*

$$\int_0^\infty u(x, t)\Delta x = \mu_x \sum_{m=0}^{\infty} u(m\mu_x, t) = A\mu_x$$

for all $t \in \mathbb{T}$.

Proof. Let $u(x, t)$ be a solution of (12). We define a function $S : \mathbb{T} \to \mathbb{R}$ by

$$S(t) := \int_0^\infty u(x, t)\Delta x = \mu_x \sum_{m=0}^{\infty} u(m\mu_x, t),$$

and show that $S^{\Delta_t}(t) = 0$ for all $t \in \mathbb{T}$.

We can rewrite equation (12) into

$$u^{\Delta_t}(x, t) = -\frac{k}{\mu_x}u(x, t) + \frac{k}{\mu_x}u(x - \mu_x, t).$$

Consequently,

$$S^{\Delta_t}(t) = \mu_x \sum_{m=0}^{\infty} u^{\Delta_t}(m\mu_x, t) \tag{17}$$

$$= -k\sum_{m=0}^{\infty} u(m\mu_x, t) + k\sum_{m=0}^{\infty} u((m-1)\mu_x, t) \tag{18}$$

$$= 0. \tag{19}$$

We have to justify the first equality (17), i.e. the interchangeability of the delta derivative and summation at each $t_0 \in \mathbb{T}$. If t_0 is right-scattered, the non-negativity of the solution implies

$$S^{\Delta_t}(t_0) = \frac{\mu_x \sum_{m=0}^{\infty} u(m\mu_x, t_0 + \mu_t(t_0)) - \mu_x \sum_{m=0}^{\infty} u(m\mu_x, t_0)}{\mu_t(t_0)}$$
$$= \mu_x \sum_{m=0}^{\infty} \frac{u(m\mu_x, t_0 + \mu_t(t_0)) - u(m\mu_x, t_0)}{\mu_t(t_0)} = \mu_x \sum_{m=0}^{\infty} u^{\Delta_t}(m\mu_x, t_0).$$

If t_0 is right-dense and there is a continuous interval $[t_0, s]$, $s > t_0$, we show that the sum $\sum_{m=0}^{\infty} u^{\Delta_t}(m\mu_x, t)$ converge uniformly on $[t_0, s]$. First, let us note that (18) yields that this is implied by the uniform convergence of $\sum_{m=0}^{\infty} u(m\mu_x, t)$. One could use Corollary 4.3 to get ($\kappa = k/\mu_x$):

$$\sum_{m=0}^{\infty} u(m\mu_x, t) = \sum_{m=0}^{\infty} \left(\mathrm{e}^{-\kappa(t-t_0)} \sum_{i=0}^{m} C_i \frac{(\kappa(t-t_0))^{m-i}}{(m-i)!} \right)$$
$$= \mathrm{e}^{-\kappa(t-t_0)} \sum_{m=0}^{\infty} \frac{(\kappa(t-t_0))^m}{m!} \cdot \sum_{i=0}^{\infty} C_i.$$

If $\sum_{i=0}^{\infty} C_i$ is finite (i.e. $S(t_0)$ is finite), then this sum converge uniformly on an arbitrary closed interval. Finally, if t_0 is right-dense and there is no continuous interval $[t_0, s]$, $s > t_0$, we consider a function $v(x, t)$ with $v(m\mu_x, t_0) = u(m\mu_x, t_0)$ for all m such that v is a solution on a domain with a continuous interval $[t_0, s]$, $s > t_0$. Obviously, equation (15) implies that $v_t(m\mu_x, t_0) = u^{\Delta_t}(m\mu_x, t_0)$ for all m. Moreover for each $\delta > 0$, there is $\theta > 0$ such that for all $t \in [t_0, t_0 + \theta]_{\mathbb{T}}$:

$$(1-\delta) \sum_{m=0}^{\infty} v(m\mu_x, t) \le \sum_{m=0}^{\infty} u(m\mu_x, t) \le (1+\delta) \sum_{m=0}^{\infty} v(m\mu_x, t).$$

Consequently,

$$0 = \sum_{m=0}^{\infty} u^{\Delta_t}(m\mu_x, t_0) = \sum_{m=0}^{\infty} v_t(m\mu_x, t_0)$$
$$= \left(\sum_{m=0}^{\infty} v(m\mu_x, t_0) \right)_t = \left(\sum_{m=0}^{\infty} u(m\mu_x, t_0) \right)^{\Delta_t}.$$

Taking into account the fact that $u(x, 0)$ is given by the initial condition in (12), we see that $S(0) = A\mu_x$. Consequently, (17)–(19) imply that $S(t) = A\mu_x$. □

We could now study the relationship with probability distributions and we begin by generalizing probability density and mass functions.

Definition 6.7. *We say that a function* $f : \mathbb{T} \to \mathbb{R}_0^+$ *is a* dynamic probability density function *if*

$$\int_{-\infty}^{\infty} f(t) \Delta t = 1.$$

Note that if $\mathbb{T} = \mathbb{R}$ then f is a probability density function. If $\mathbb{T} = \mu_t\mathbb{Z}$ then $\mu_t f$ is a probability mass functions (see Theorem 5.3).

Combining Theorems 6.5 and 6.6, we get the necessary and sufficient condition for sections to generate probability distributions.

LEMMA 6.8. *Let* $u(x, t)$ *be a solution of* (12).

(1) $u(\cdot, t)$ *is a dynamic probability density function for all* $x \in \mu_x\mathbb{N}_0$ *if and only if* $A\mu_x/k = 1$ *and* $\mu_t(t) < \mu_x$ *for all* $t \in \mathbb{T}$.
(2) $u(x, \cdot)$ *is a dynamic probability density function for all* $t \in \mathbb{T}$ *if and only if* $A\mu_x = 1$ *and* (TS1) *holds.*

Proof. The proof is a direct application of Theorems 6.5 and 6.6. □

Finally, we provide the necessary and sufficient condition for both sections.

THEOREM 6.9. *Let* $u(x, t)$ *be a solution of* (12). *Then both* $u(x, \cdot)$ *and* $u(\cdot, t)$ *are dynamic probability density functions for all* $t \in \mathbb{T}$ *and* $x \in \mu_x\mathbb{N}_0$ *if and only if* $k = 1$, $A\mu_x = 1$ *and* $\mu_t(t) < \mu_x$ *for each* $t \in \mathbb{T}$.

Proof. The proof follows from Lemma 6.8. □

7. Applications

As suggested in Remarks 1 and 3, the time and space sections of solutions of the transport equation on various domains generate important probability distributions (*cf.* Table 1).

In other words, the solutions correspond to the so-called counting stochastic processes describing number of occurrences of certain random events (arrival of customers in a queue, device failures, phone calls, scored goals, etc.) (e.g. [15] (Chapters 4 and 5), [10]). They have following properties:

(1) Probability of number of events (occurrences) at time t is given by $u(\cdot, t)$ (Poisson distribution, binomial distribution).
(2) Probability distribution of the time of the first occurrence is given by $u(0, t)$ (exponential or geometric distribution).
(3) Probability distributions that at least x events have happened until time t are given by $u(x - 1, \cdot)$ (Erlang or negative binomial distributions).
(4) Probability distribution of the waiting time until the next occurrence is given by $u(0, t)$ (exponential or geometric distribution).

Our analysis in Section 6, summarized in Theorem 6.9, suggests that properties (1)–(3) are conserved on general domains $\mathbb{Z} \times \mathbb{T}$. Properties (2) and (3) are conserved in

Table 1. Correspondence of time and space sections with probability distributions.

	$u(\cdot, t)$	$u(0, \cdot)$	$u(x, \cdot),\ x \geq 0$
$\mathbb{Z} \times \mathbb{R}$	Poisson dist.	Exponential dist.	Erlang (Gamma) dist.
$\mathbb{Z} \times p\mathbb{Z}$	Binomial dist.	Geometric dist.	Negative binomial dist.

the sense of Definition 6.7 (see Examples 7.2 and 7.3 below). Property (4) does not apply because of the underlying inhomogeneous time structure.

The convergence relationship between the distributions from Table 1 is well known [10]. Our analysis strengthens this relationship since the convergence is based on the solution of the same partial equation with changing underlying structures.

We conclude this section by suggesting two applications which emphasize the time scale choice. First, let us consider Bernoulli trials with non-constant probability of successes. For example, Ref. [7] shows that the probability that a goal is scored in each minute of the association football match is not constant but increases throughout the game, especially in the last minutes of each half-time. Let us derive an explicit solution on arbitrary heterogeneous discrete structure.

LEMMA 7.1. *Let us consider a heterogeneous discrete time scale* $\mathbb{T} = \{0, \mu_1, \mu_1 + \mu_2, \dots, \sum_{i=1}^{n} \mu_i, \dots\}$. *Then the solution of* (12) *has the form*

$$u\left(m\mu_x, \sum_{i=1}^{n} \mu_i\right) = A \sum_{\pi \in P_m^{n-m}} \prod_{i=1}^{n} K_i^{\pi_i} L_i^{1-\pi_i}, \tag{20}$$

where $K_i = 1 - k(\mu_i/\mu_x)$, $L_i = k(\mu_i/\mu_x)$ *and* P_r^q *denote a set of all permutation vectors containing* q *ones and* r *zeros.*

Proof. We base our proof on the relationship

$$u\left(m\mu_x, \sum_{i=1}^{n} \mu_i\right) = \left(1 - \frac{k\mu_n}{\mu_x}\right) u(x,t) + \frac{k\mu_n}{\mu_x} u(x - \mu_x, t)$$

and proceed by induction. First, the initial condition implies that the statement holds for $n = 0$. Next, let us assume that the statement holds for $n \in \mathbb{N}_0$, i.e. (20) is satisfied. Then we have $u\left(m\mu_x, \sum_{i=1}^{n+1} \mu_n\right) = 0$ for $m \notin 0, 1, \dots, n+1$. Furthermore, for $m = 0$

$$u\left(0, \sum_{i=1}^{n+1} \mu_i\right) = K_{n+1}(AK_1K_2 \cdots K_n) + 0 = AK_1K_2 \cdots K_nK_{n+1}.$$

Next, for $m \in \big(1, 2, \dots, n\big)$:

$$\begin{aligned} u\left(m\mu_x, \sum_{i=1}^{n+1} \mu_i\right) &= K_{n+1} A \sum_{\pi \in P_m^{n-m}} \prod_{i=1}^{n} K_i^{\pi_i} L_i^{1-\pi_i} + L_{n+1} A \sum_{\pi \in P_{m-1}^{n-m+1}} \prod_{i=1}^{n} K_i^{\pi_i} L_i^{1-\pi_i} \\ &= A \sum_{\pi \in P_m^{n+1-m}} \prod_{i=1}^{n+1} K_i^{\pi_i} L_i^{1-\pi_i}. \end{aligned}$$

Finally, for $m = n + 1$ we have

$$u\left((n+1)\mu_x, \sum_{i=1}^{n+1} \mu_i\right) = 0 + L_{n+1}(AL_1L_2 \cdots L_n) = AL_1L_2 \cdots L_nL_{n+1}.$$

□

We could immediately apply this result to obtain generalizations of standard Bernoulli processes.

Example 7.2. (Heterogeneous Bernoulli process) Let us consider a repeated sequence of trials and assume that the probability of success p_i in ith trial is non-constant, in contrast to standard Bernoulli process discussed in Section 5. If we construct a discrete time scale

$$\mathbb{T} = \left\{ 0, p_1, p_1 + p_2, \ldots, \sum_{i=1}^{n} p_i, \ldots \right\},$$

then the solution $u(x, t)$ of (12) generates the probability distributions discussed above. Let us choose, for example, $A = \mu_x = k = 1$. Then, $u(\cdot, \sum_{i=1}^{n-1} p_i)$ is the probability mass function describing number of successes in the first n trials. Moreover, $p_i u(x, \cdot)$ is the probability mass function of the number of trials needed to get $x + 1$ successes.

To illustrate, let us choose

$$\mathbb{T} = \left\{ 0, \frac{1}{2}, \frac{1}{2} + \frac{1}{3}, \ldots, \sum_{i=1}^{n} \frac{1}{i+1}, \ldots \right\}$$

to study a process in which the probability of successful trial decreases harmonically. We could use Lemma 7.1 to determine that

$$u\left(m, \sum_{i=1}^{n} p_i \right) = \sum_{\pi \in P_m^{n-m}} \prod_{i=1}^{n} \left(\frac{1}{i+1} \right)^{\pi_i} \left(\frac{i}{i+1} \right)^{1-\pi_i}, \quad 0 \le m \le n.$$

For example, the probability mass function for the first successful trial appearing in kth trial, i.e. $p_i u(0, \cdot)$, has the form

$$f(k) = \frac{1}{k(k+1)}, \quad k \in \mathbb{N}.$$

Finally, we consider a mixed time scale, which, coupled with the transport equation, generates mixed processes and distributions.

Example 7.3. (Stop–start Bernoulli–Poisson process) Let us assume that a device is regularly used throughout a constant period and then switched off for another one. Let us assume that the probability of failure when the device is in use is determined by a continuous process, whereas the probability of failure in the rest mode is given by a discrete process. This leads to mixed probability distributions which could be generated, for example, by

$$\mathbb{T} = \bigcup_{i=0}^{\infty} \left[i, i + \frac{1}{2} \right].$$

Again $u(x, \cdot)$ describes the mixed probability distribution of $x + 1$ failures, in the sense of Definition 6.7. Similarly, $u(\cdot, t)$ is the probability mass function describing the number of failures at time t. Note that the probability of failure in the rest mode is given by the length of the discrete gap (*cf.* Definition 6.7). As in the previous example, we are not able to find

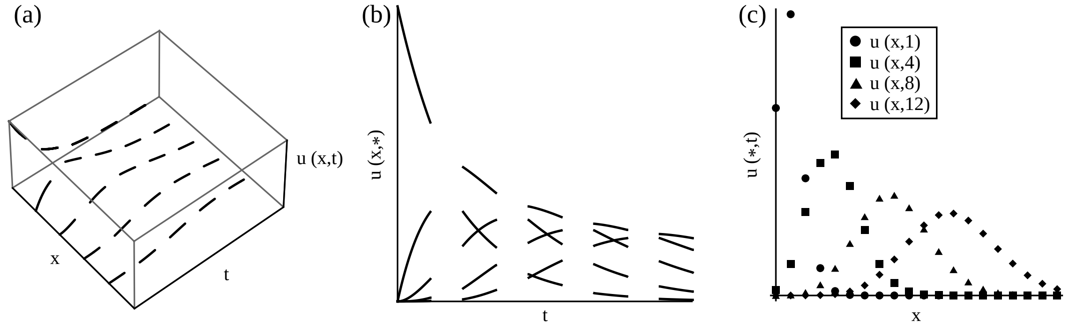

Figure 4. Solution of the transport equation with discrete space and general time (12) with $A = 1$, $\mu_x = 1$, $k = 1$ and $\mathbb{T} = \cup_{i=0}^{\infty}[i, i + (1/2)]$.

the closed-form solutions but one could tediously solve the separate equations to get that:

$$
\begin{aligned}
u(0,t) &= \frac{1}{2^n}\mathrm{e}^{(n/2)-t},\\
u(1,t) &= \frac{2t+n}{2^{n+1}}\mathrm{e}^{(n/2)-t},\\
u(2,t) &= \frac{4t^2+4nt+(n^2-4n)}{2!\cdot 2^{n+2}}\mathrm{e}^{(n/2)-t},\\
u(3,t) &= \frac{8t^3+12nt^2+6(n^2-4n)t+(n^3-12n^2+16n)}{3!\cdot 2^{n+3}}\mathrm{e}^{(n/2)-t},\\
&\dots\\
u(x,t) &= \frac{\text{polynomial of order } x}{x!\cdot 2^{n+x}}\mathrm{e}^{(n/2)-t},
\end{aligned}
$$

for $n \in \mathbb{N}_0$ (nth continuous part) and $t \in \left[n, n + (1/2)\right]$. See Figure 4 for illustration.

8. Conclusion and future directions

There are a number of open questions related to the analysis presented in this paper. In Section 6, we were unable to provide a general closed-form solution of problem (12). With the connection to probability distributions, is it possible to provide one for further special choices of $\mathbb{T}$ (see e.g. Examples 7.2 and 7.3)?

In the classical case, the solution is propagated along characteristics. Obviously, our analysis in Sections 4 and 5 implies that this is not the case on semidiscrete domains. However, one could show that at least the maxima are propagated along characteristics on discrete–continuous or discrete–discrete domains (computing directly or using modes of probability distributions). Having no closed-form solutions on time scales, could we prove this property for an arbitrary time scale? This question is closely related to modes of the corresponding probability distributions and the question could be therefore formulated in more general way. Can we, at least in special cases, determine the descriptive statistics related to the generated probability distributions?

From the theoretical point of view, there is also a natural extension to consider a transport equation with continuous space and general time, or general space and time. The applicability of these settings is limited by the fact that such problems do not conserve sign in general (*cf.* assumption $\mu_t(t) < \mu_x$ in Theorem 6.9).

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Note

1. We assume that $k > 0$ so that the solution is bounded and does not vanish. Moreover, we use the nabla difference instead of delta difference. The single reason is the simpler form of the solution (4). If we used the delta difference, we would consider $k < 0$ and the solution would propagate to the quadrant with $t > 0$ and $x < 0$. This applies also to the problems which we study in the following sections.

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APPENDIX B

Maximium and minimum principles for nonlinear transport equations on discrete-space domains

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MAXIMUM AND MINIMUM PRINCIPLES FOR NONLINEAR TRANSPORT EQUATIONS ON DISCRETE-SPACE DOMAINS

JONÁŠ VOLEK

Abstract. We consider nonlinear scalar transport equations on the domain with discrete space and continuous time. As a motivation we derive a conservation law on these domains. In the main part of the paper we prove maximum and minimum principles that are later applied to obtain an a priori bound which is applied in the proof of existence of solution and its uniqueness. Further, we study several consequences of these principles such as boundedness of solutions, sign preservation, uniform stability and comparison theorem which deals with lower and upper solutions.

1. Introduction

The transport equation is one of the simplest nonlinear partial differential equations. Its importance follows from the fact that it describes traveling waves and that it forms the basis for study of hyperbolic equations of second order. The reader can see, e.g., [11] for details about transport PDE.

We study transport equations on the domain with discrete space and continuous time. This is a combination of difference and differential equations. As an application of these models we can mention semidiscrete numerical methods of Rothe or Galerkin (see [10, 16]). We consider nonlinear equations that arise from conservation laws. Linear equations that combine continuous, discrete and time-scale variables are studied in [20]. In that paper authors present some interesting relations between equations of this type and stochastic processes of Poisson–Bernoulli type.

In recent years so called dynamical systems on lattices have been studied extensively. In [6, 7, 12] authors deal with these related problems and focus on PDEs of reaction–diffusion type on finite space lattices. Their results can be helpful, e.g., in the modelling of binary alloys (see [7]).

Moreover, in the last few years the analysis of equations on infinite lattices has attracted some researchers. We can refer to [2, 3, 4, 21] for the introduction to these problems. These papers are concerned mainly with existence of traveling waves in discrete reaction–diffusion equations and their properties. The reader is invited to

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see [21] where the main ideas and principles of this field are presented. Problems in [2, 3, 4, 21] are solved often by topological methods using fixed point theorems, degree theory, comparison principles and lower and upper solutions.

Our analysis can contribute to this mathematical area. Our problem can be understood as an equation on infinite lattice. With the help of maximum and minimum principles we derive new comparison theorem that deals with ordering of lower and upper solutions.

In general, we study simpler problems than reaction-diffusion equations but on the other hand, our work can be interesting for another reason as well. It can be useful just from the point of view of maximum and minimum principles. These principles are strong tools in the theory of differential equations. They have many applications and important consequences. We can mention, e.g., a priori bounds that can be applied in proofs of existence and uniqueness of solution, oscillation results. For the review about these topics in ODEs and PDEs see [14] or more recent book [15]. In discrete problems these principles have rich behavior. The reader is invited to see papers [13, 17, 18, 19] or survey book about partial difference equations [5] for further details. Consequently, we want to explore if the transport equation where we combine continuous and discrete approach has some fruitful properties as in these works.

The structure of our paper is as follows. First, we motivate our study, derive a conservation law in discrete space and formulate our main problem in Section 2. In Section 3 we prove maximum and minimum principles for the nonlinear equation by the so-called *stairs method*. Then we deal with existence and uniqueness of solution in Section 4 and with other consequences in Section 5. In Section 6, we study a related nonlinear problem. At the end of the paper, in Section 7, we present some open problems and directions of future research.

We denote the intervals $[0, +\infty)$ and $(0, +\infty)$ by $\mathbb{R}_0^+$ and $\mathbb{R}^+$ respectively. Partial derivative of $u(x, t)$ w.r.t. t is denoted by $u_t(x, t)$ and partial difference w.r.t. x by

$$\nabla_x u(x, t) = u(x, t) - u(x - 1, t).$$

2. Conservation law and nonlinear transport equation

As a motivation we derive the conservation law in discrete space. It leads to partial equations on discrete-space domain. Corresponding continuous conservation laws are presented, e.g., in [11].

We consider one dimensional discrete space. We simulate it by integers. Further, we suppose the density $u = u(x, t)$ which changes continuously in time and which is distributed in discrete space. The magnitude u can express, e.g., the concentration of mass or population, energy etc.

We denote by φ the flux of u. The flux $\varphi(i, t)$, $i \in \mathbb{Z}$, $t \in \mathbb{R}_0^+$, quantifies the amount of u that passes between positions $x = i$ and $x = i + 1$ in time t. Further, $f = f(x, t)$ is the source function.

Therefore, consider an arbitrary space segment between $x = i$ and $x = j$ when $i < j$. The time change of total amount in that space segment between $x = i$ and $x = j$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{x=i}^{j} u(x, t) = \varphi(i - 1, t) - \varphi(j, t) + \sum_{x=i}^{j} f(x, t). \tag{2.1}$$

We call (2.1) the conservation law in global form. Let us modify (2.1) as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x=i}^{j} u(x,t) = -\big[&\varphi(j,t)-\varphi(j-1,t)+\varphi(j-1,t)-\dots-\varphi(i,t)+\varphi(i,t)\\ &-\varphi(i-1,t)\big]+\sum_{x=i}^{j} f(x,t),\end{aligned}$$

and finally, we obtain

$$\sum_{x=i}^{j}\big[u_t(x,t)+\nabla_x\varphi(x,t)-f(x,t)\big]=0.$$

The space segment is arbitrary and thus, the following conservation law in local form has to hold necessarily

$$u_t(x,t)+\nabla_x\varphi(x,t)=f(x,t). \tag{2.2}$$

We study the case of

$$\varphi(x,t)=F(x,t,u(x,t)) \quad \text{when} \quad F:\mathbb{Z}\times\mathbb{R}_0^+\times\mathbb{R}\to\mathbb{R}.$$

This leads to the nonlinear transport equation with discrete space. Therefore, we deal with the following initial-boundary value problem (I-BVP):

$$\begin{gathered} u_t(x,t)+\nabla_x F(x,t,u(x,t))=f(x,t), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\ t\in\mathbb{R}^+,\\ u(x,0)=\phi(x), \quad \phi:\mathbb{Z}\to\mathbb{R},\\ u(a,t)=\xi(t), \quad \xi\in C(\mathbb{R}_0^+)\cap C^1(\mathbb{R}^+), \end{gathered}\tag{2.3}$$

where $F:\mathbb{Z}\times\mathbb{R}_0^+\times\mathbb{R}\to\mathbb{R}$ and $f:\mathbb{Z}\times\mathbb{R}_0^+\to\mathbb{R}$. We prove maximum and minimum principles for lower and upper solutions.

Definition 2.1. The function $v(x,t)$ is called a *lower solution* of (2.3) if

$$\begin{gathered} v_t(x,t)+\nabla_x F(x,t,v(x,t))\le f(x,t), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\ t\in\mathbb{R}^+,\\ v(x,0)\le\phi(x), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\\ v(a,t)\le\xi(t), \quad t\in\mathbb{R}_0^+. \end{gathered}$$

The function $w(x,t)$ is an *upper solution* of (2.3) if

$$\begin{gathered} w_t(x,t)+\nabla_x F(x,t,w(x,t))\ge f(x,t), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\ t\in\mathbb{R}^+,\\ w(x,0)\ge\phi(x), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\\ w(a,t)\ge\xi(t), \quad t\in\mathbb{R}_0^+. \end{gathered}$$

3. Maximum and minimum principles

In this section we derive main tools of our study, the maximum and minimum principles. Let us mention that if we consider problem (2.3) with more general difference

$$\nabla_x^{(\mu)}u(x,t)=\frac{u(x,t)-u(x-\mu,t)}{\mu}$$

with arbitrary step $\mu>0$ we can prove following results in the similar way. Hence, for the sake of simplicity we suppose only difference with unitary step $\nabla_x u(x,t)$. Next technical lemma helps us in the proof of maximum principle.

Lemma 3.1. *Let $F : \mathbb{Z} \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ satisfy*

(A1) *$F(\chi, \tau, \omega)$ is increasing in χ, i.e., for all $\chi_1 < \chi_2$ there is*
$$F(\chi_1, \tau, \omega) \le F(\chi_2, \tau, \omega),$$
(A2) *$F(\chi, \tau, \omega)$ is strictly increasing in ω, i.e., for all $\omega_1 < \omega_2$ there is*
$$F(\chi, \tau, \omega_1) < F(\chi, \tau, \omega_2).$$

Then the following holds:
$$\text{if } F(\chi_1, \tau, \omega_1) \le F(\chi_2, \tau, \omega_2) \text{ then } \chi_1 \le \chi_2 \text{ or } \omega_1 \le \omega_2, \tag{3.1}$$
$$\text{if } F(\chi_1, \tau, \omega_1) < F(\chi_2, \tau, \omega_2) \text{ then } \chi_1 < \chi_2 \text{ or } \omega_1 < \omega_2. \tag{3.2}$$

Proof. We show only (3.1). The proof of (3.2) is similar. Let us suppose by contradiction that $\chi_1 > \chi_2$ and $\omega_1 > \omega_2$. Then we have
$$F(\chi_2, \tau, \omega_2) \overset{\text{(A1)}}{\le} F(\chi_1, \tau, \omega_2) \overset{\text{(A2)}}{<} F(\chi_1, \tau, \omega_1),$$
a contradiction with the assumption of $F(\chi_1, \tau, \omega_1) \le F(\chi_2, \tau, \omega_2)$. □

Theorem 3.2 (Maximum principle)**.** *Assume that $F(\chi, \tau, \omega)$ satisfies* (A1) *and* (A2) *and $f(\chi, \tau) \le 0$ for all $\chi \in \mathbb{Z}, \chi > a$, $\tau \in \mathbb{R}^+$. Let $u(x, t)$ be a lower solution of* (2.3)*. Then*
$$u(x, t) \le \sup_{\substack{x \in \mathbb{Z}, x \ge a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\}$$
holds for all $x \in \mathbb{Z}$, $x \ge a$, and for all $t \in \mathbb{R}_0^+$.

Proof. We prove the statement by the so-called *stairs method.* The idea of our proof is shown on Figure 1. First, we denote
$$M := \sup_{\substack{x \in \mathbb{Z}, x \ge a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\}.$$
Assume by contradiction that there exist $x_0 \in \mathbb{Z}$, $x_0 > a$, and $t_0 \in \mathbb{R}^+$ such that
$$u(x_0, t_0) > M. \tag{3.3}$$
Now from assumptions (A2), (3.3) and from the fact that $u(x, t)$ is a lower solution we obtain
$$u_t(x_0, t_0) \le F(x_0 - 1, t_0, u(x_0 - 1, t_0)) - F(x_0, t_0, u(x_0, t_0)), \tag{3.4}$$
$$u_t(x_0, t_0) < F(x_0 - 1, t_0, u(x_0 - 1, t_0)) - F(x_0, t_0, M). \tag{3.5}$$
Now there are two possibilities.

(1) If $F(x_0 - 1, t_0, u(x_0 - 1, t_0)) > F(x_0, t_0, M)$ then from (3.2) in Lemma 3.1 we get $u(x_0 - 1, t_0) > M$. Hence, in this case we define
$$x_1 = x_0 - 1 \quad \text{and} \quad t_1 = t_0.$$

(2) The second possibility is that $F(x_0 - 1, t_0, u(x_0 - 1, t_0)) \le F(x_0, t_0, M)$ holds. From (3.5) there is $u_t(x_0, t_0) < 0$. Therefore, the function $u(x_0, t)$ is strictly decreasing in $t = t_0$ and we can define
$$\overline{t_0} = \inf\{\tau = [0, t_0] : u(x_0, t) \text{ is strictly decreasing on the interval } (\tau, t_0)\}.$$

If $\overline{t_0} = 0$ then we have a contradiction with the definition of M via the initial condition $\phi(x)$. If $\overline{t_0} > 0$ then there is necessarily $u_t(x_0, \overline{t_0}) = 0$ and from (3.4) we obtain

$$F(x_0, \overline{t_0}, u(x_0, \overline{t_0})) \le F(x_0 - 1, \overline{t_0}, u(x_0 - 1, \overline{t_0})).$$

Then (3.1) in Lemma 3.1 implies $u(x_0, \overline{t_0}) \le u(x_0 - 1, \overline{t_0})$ which gives

$$M < u(x_0, t_0) < u(x_0, \overline{t_0}) \le u(x_0 - 1, \overline{t_0}).$$

Consequently, in this case we define

$$x_1 = x_0 - 1 \quad \text{and} \quad t_1 = \overline{t_0}.$$

Finally, we have $u(x_1, t_1) > M$. If we continue iteratively then after at most $x_0 - a$ steps we get a contradiction with definition of M. □

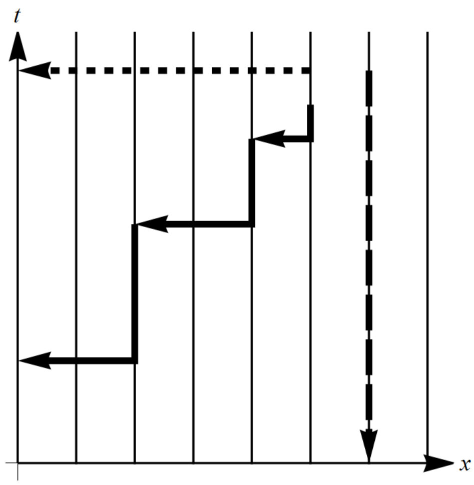

FIGURE 1. The idea of the stairs method. The dotted line shows the situation when only possibility (1) occurs which yields a contradiction via the boundary condition $\xi(t)$. The bold line shows the combination of possibilities (1) and (2) and a contradiction via the boundary condition $\xi(t)$ again. The dashed line shows the situation when we get a contradiction via the initial condition $\phi(x)$ in possibility (2).

Next we have the minimum principle which can be proved by a stairs method similarly to the one in Theorem 3.2.

Theorem 3.3 (Minimum principle). *Assume that $F(\chi, \tau, \omega)$ satisfies* (A2) *and*

(A3) *$F(\chi, \tau, \omega)$ is decreasing in χ,*

and $f(\chi, \tau) \ge 0$ for all $\chi \in \mathbb{Z}, \chi > a$, $\tau \in \mathbb{R}^+$. Let $u(x,t)$ be an upper solution of (2.3). *Then*

$$\inf_{\substack{x \in \mathbb{Z},\, x \ge a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\} \le u(x,t)$$

holds for all $x \in \mathbb{Z}$, $x \ge a$, and for all $t \in \mathbb{R}_0^+$.

4. Existence and uniqueness of solution

In this section we use maximum and minimum principles as a priori bounds to prove the existence and uniqueness of solution of (2.3). The proof is based on induction and further, we use the following lemma about global solution of IVP for ordinary differential equation.

Lemma 4.1 ([9, Corollary 8.64]). *Consider the following IVP for ordinary differential equation*

$$\begin{gathered} u'(t) = g(t, u(t)), \quad g : I \times \mathbb{R}^n \to \mathbb{R}^n, \\ u(t_0) = u_0, \quad u_0 \in \mathbb{R}^n, \end{gathered} \tag{4.1}$$

when $I \subset \mathbb{R}$ is an interval. Assume that $h : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous and there is a $v_0 \in \mathbb{R}_0^+$ such that

$$\int_{v_0}^{+\infty} \frac{\mathrm{d}s}{h(s)} = +\infty.$$

Let the function $g : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let

$$\|g(\tau, \omega)\| \le h(\|\omega\|)$$

hold for all $(\tau, \omega) \in [t_0, +\infty) \times \mathbb{R}^n$. Then for all $u_0 \in \mathbb{R}^n$ with $\|u_0\| \le v_0$ all solutions of (4.1) exist on $[t_0, +\infty)$.

Theorem 4.2 (Existence and uniqueness). *Suppose that:*

- (A4) *$\phi(x)$, $\xi(t)$ are bounded; i.e., there exist $K > 0$ such that for all $x \in \mathbb{Z}, x \ge a$, and for all $t \in \mathbb{R}_0^+$ $|\phi(x)| \le K$ and $|\xi(t)| \le K$ hold,*
- (A5) *$f(\chi, \tau) = 0$ identically,*

the function $F = F(\tau, \omega)$ is independent of χ, satisfies (A2) *and*

- (A6) *$F(\tau, \omega)$ is continuous w.r.t. τ on $\mathbb{R}_0^+$,*
- (A7) *$F(\tau, \omega)$ is locally Lipschitz continuous w.r.t. ω on $\mathbb{R}_0^+ \times \mathbb{R}$, i.e., for all $\tau_0 \in \mathbb{R}_0^+$ and for all $\omega_0 \in \mathbb{R}$ there exists a rectangle*
$$\mathcal{R}(\tau_0, \omega_0) = \left\{(\tau, \omega) \in \mathbb{R}_0^+ \times \mathbb{R} : \quad 0 \le \tau - \tau_0 \le a, |\omega - \omega_0| \le b\right\}$$
and $L = L(\tau_0, \omega_0) > 0$ such that for all $(\tau, \omega_1), (\tau, \omega_2) \in \mathcal{R}(\tau_0, \omega_0)$ there is
$$|F(\tau, \omega_1) - F(\tau, \omega_2)| \le L|\omega_1 - \omega_2|,$$
- (A8) *$F(\tau, \omega)$ is sublinear w.r.t. ω, i.e., there exist $A, B > 0$ such that for all $\tau \in \mathbb{R}_0^+$ and for all $\omega \in \mathbb{R}$ there is*
$$|F(\tau, \omega)| \le A|\omega| + B.$$

Then (2.3) possesses a unique solution $u(x, t)$ which is defined for all $x \in \mathbb{Z}$, $x \ge a$, and $t \in \mathbb{R}_0^+$.

Proof. We prove the statement by induction on $x \in \mathbb{Z}$, $x \ge a$.

(1) For $x = a$ we put $u(a, t) = \xi(t)$.

(2) Let us have a solution $u(\overline{x}, t)$ which is unique and defined for all $\overline{x} \in \mathbb{Z}$, $a \le \overline{x} < x$, on $\mathbb{R}_0^+$. Then for fixed x we get from (2.3) the following IVP for ordinary differential equation

$$\begin{gathered} u_t(x, t) = F(x-1, t, u(x-1, t)) - F(x, t, u(x, t)), \\ u(x, 0) = \phi(x), \quad \phi(x) \in \mathbb{R}, \end{gathered} \tag{4.2}$$

where $F(x-1, t, u(x-1, t))$ is a given function of t from the induction hypothesis.

• Assumptions (A6), (A7) and Picard-Lindelöf's theorem (see [9, Theorem 8.13]) imply the existence and uniqueness of a local solution $u(x,t)$ of (4.2) on some small interval $[0,\delta]$, $\delta > 0$.

• We can make the estimate

$$\begin{aligned}
&|F(x-1,t,u(x-1,t)) - F(x,t,u(x,t))| \\
&\le |F(x-1,t,u(x-1,t))| + |F(x,t,u(x,t))| \\
&\overset{\text{(A8)}}{\le} A|u(x-1,t)| + A|u(x,t)| + 2B \\
&\overset{\text{Th. 3.2+Th. 3.3+(A4)}}{\le} A|u(x,t)| + AK + 2B.
\end{aligned}$$

If we define $g(t,u) = F(x-1,t,u(x-1,t)) - F(x,t,u)$, $h(s) = As + AK + 2B$ and $v_0 = |\phi(x)|$ then assumptions of Lemma 4.1 are satisfied. Therefore, the local solution $u(x,t)$ can be extended to the whole $\mathbb{R}_0^+$.

• Finally, we have to check if there is no other solution from some time $t_0 > 0$ which disjoins from $u(x,t)$ in t_0. Hence, suppose by contradiction that there is a $t_0 > 0$ such that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of (4.2) with $u_1(x,t) = u_2(x,t)$ on $[0,t_0]$ and $u_1(x,t) \ne u_2(x,t)$ on $(t_0, t_0+\epsilon)$, $\epsilon > 0$. Let us denote $u_{t_0} = u_1(x,t_0)$ and investigate the solvability of the IVP

$$\begin{gathered}
u_t(x,t) = F(x-1,t,u(x-1,t)) - F(x,t,u(x,t)), \quad t > t_0, \\
u(x,t_0) = u_{t_0}.
\end{gathered} \tag{4.3}$$

The right-hand side of equation in (4.3) is unique by induction hypotheses. Functions $u_1(x,t)$, $u_2(x,t)$ solve (4.3) on $[t_0, t_0+\epsilon)$. But assumptions of Picard–Lindelf's theorem are also satisfied for (4.3) thanks to (A6), (A7) and consequently, there cannot be two distinct solutions. This is a contradiction which finishes the proof. $\square$

Remark 4.3. If we omit the assumption (A7) of local Lipschitz continuity of $F(\tau,\omega)$ in Theorem 4.2 then the uniqueness is not guaranteed and we get only the existence result by the same procedure with the help of Cauchy–Peano's theorem (see [9, Theorem 8.27]) instead of Picard–Lindelöf's theorem.

We present the following example for an illustration what functions $F(\chi,\tau,\omega)$ can be considered in Theorem 4.2.

Example 4.4. Assumptions of Theorem 4.2 are satisfied, e.g., for following functions $F(\tau,\omega)$:

- $F(\tau,\omega) = k(\tau)\omega$ when $k(\tau) > 0$ (linear equation),
- $F(\tau,\omega) = k(\tau)\arctan\omega$ when $k(\tau) > 0$.

For the following function F we have only existence guaranteed (cf. Remark 4.3):

- $F(\tau,\omega) = \begin{cases} -\sqrt[3]{-\omega}, & \text{for } \omega < 0, \\ \sqrt[3]{\omega}, & \text{for } \omega \ge 0. \end{cases}$

5. Consequences of maximum and minimum principles

In this section we study well-known consequences of maximum and minimum principles. Corresponding results for classical differential equations can be found in [14]. The next two corollaries follow immediately from Theorems 3.2 and 3.3.

Corollary 5.1 (Boundedness of solutions). *Let $F = F(\tau, \omega)$ satisfy Assumption* (A2)*, $f(\chi, \tau) = 0$ identically, $\phi(x)$ and $\xi(t)$ be bounded and $u(x,t)$ be a solution of* (2.3)*. Then $u(x,t)$ is bounded.*

Corollary 5.2 (Sign preservation). *Let $F = F(\chi, \tau, \omega)$ satisfy* (A2) *and* (A3)*, $f(\chi, \tau)$ be nonnegative, $\phi(x)$ and $\xi(t)$ be nonnegative and $u(x,t)$ be a solution of* (2.3)*. Then $u(x,t)$ is nonnegative.*

Last application of maximum and minimum principles from Theorems 3.2 and 3.3 is the uniform stability of solutions of the linear problem and its consequences. Thus, let us consider the linear problem

$$\begin{gathered} u_t(x,t) + \nabla_x [k(t)u(x,t)] = 0, \quad x \in \mathbb{Z}, \quad x > a \in \mathbb{Z},\ t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \\ u(a,t) = \xi(t), \quad \xi \in C(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+), \end{gathered} \tag{5.1}$$

where $k(t) > 0$.

Corollary 5.3 (Uniform stability). *Let $u_1(x,t)$ be a solution of* (5.1) *with initial-boundary conditions $\phi_1(x)$ and $\xi_1(x)$. Let $u_2(x,t)$ be a solution of* (5.1) *with initial-boundary conditions $\phi_2(x)$ and $\xi_2(x)$. Then*

$$\sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} |u_1(x,t) - u_2(x,t)| \leq \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\} \tag{5.2}$$

holds.

Proof. Define function $v(x,t) = u_1(x,t) - u_2(x,t)$. Then $v(x,t)$ solves I-BVP (5.1) with the initial-boundary conditions $\phi_1(x) - \phi_2(x)$ and $\xi_1(t) - \xi_2(t)$. Assumptions of the maximum principle in Theorem 3.2 are satisfied and hence, we obtain

$$\begin{aligned} u_1(x,t) - u_2(x,t) = v(x,t) &\leq \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi_1(x) - \phi_2(x), \xi_1(t) - \xi_2(t)\} \\ &\leq \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\}. \end{aligned} \tag{5.3}$$

Similarly, assumptions of the minimum principle in Theorem 3.3 are satisfied and therefore, there is

$$\begin{aligned} u_1(x,t) - u_2(x,t) = v(x,t) &\geq \inf_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi_1(x) - \phi_2(x), \xi_1(t) - \xi_2(t)\} \\ &\geq - \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\}. \end{aligned} \tag{5.4}$$

Finally, inequalities in (5.3) and (5.4) yield (5.2). ∎

Corollary 5.3 directly implies the following claim.

Corollary 5.4. *Let $\{u_n\}_{n=1}^{+\infty}$ be a sequence of solutions $u_n(x,t)$ of* (5.1) *with the initial-boundary conditions $\phi_n(x)$ and $\xi_n(t)$ such that*

$$\phi_n(x) \rightrightarrows \phi(x) \text{ for } x \in \mathbb{Z}, x \geq a, \quad \text{and} \quad \xi_n(t) \rightrightarrows \xi(t) \text{ for } t \in \mathbb{R}_0^+.$$

Assume that $u(x,t)$ is a solution of (5.1) *with the initial-boundary conditions $\phi(x)$ and $\xi(t)$. Then*

$$u_n(x,t) \rightrightarrows u(x,t) \quad \text{for} \quad x \in \mathbb{Z}, x \geq a, \quad \text{and} \quad t \in \mathbb{R}_0^+.$$

6. Similar problem with space difference inside nonlinearity

In this section we analyze a similar problem as (2.3). We consider the following I-BVP where the nonlinear function F depends on difference of $u(x,t)$:

$$\begin{gathered} u_t(x,t) + F(x,t,\nabla_x u(x,t)) = f(x,t), \quad x \in \mathbb{Z},\ x > a \in \mathbb{Z},\ t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \\ u(a,t) = \xi(t), \quad \xi \in C(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+). \end{gathered} \tag{6.1}$$

Remark 6.1. We define lower and upper solutions of (6.1) similarly as in Definition 2.1.

The following two theorems are the maximum and minimum principles for (6.1). We let proofs to the reader because we can prove them by stairs method again.

Theorem 6.2 (Maximum principle)**.** *Assume that $F(\chi,\tau,\omega)$ satisfies*

(A9) *for all $\chi \in \mathbb{Z}, \chi > a$, and for all $\tau \in \mathbb{R}^+$, there is*

$$F(\chi,\tau,\omega) \begin{cases} > 0, & \text{for } \omega > 0, \\ < 0, & \text{for } \omega < 0, \\ = 0, & \text{for } \omega = 0, \end{cases}$$

and $f(\chi,\tau) \leq 0$ for all $\chi \in \mathbb{Z}, \chi > a$, $\tau \in \mathbb{R}^+$. Let $u(x,t)$ be a lower solution of (6.1)*. Then*

$$u(x,t) \leq \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\}$$

holds for all $\chi \in \mathbb{Z}$, $\chi \geq a$, and for all $\tau \in \mathbb{R}^+$.

Theorem 6.3 (Minimum principle)**.** *Assume that $F(\chi,\tau,\omega)$ satisfies* (A9) *and $f(\chi,\tau) \geq 0$ for all $x \in \mathbb{Z}$, $x > a$, and for all $t \in \mathbb{R}_0^+$. Let $u(x,t)$ be an upper solution of* (6.1)*. Then*

$$\inf_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\} \leq u(x,t)$$

holds for all $x \in \mathbb{Z}$, $x \geq a$, and for all $t \in \mathbb{R}_0^+$.

Now, we introduce analogue results for (6.1) as in Sections 4 and 5. We omit proofs again because they are also similar as for (2.3).

Theorem 6.4 (Existence and uniqueness)**.** *Suppose that* (A4), (A5) *hold, function $F(\chi,\tau,\omega)$ satisfies* (A6)–(A9)*. Then* (6.1) *possesses a unique solution $u(x,t)$ which is defined for all $x \in \mathbb{Z}$, $x \geq a$, and $t \in \mathbb{R}_0^+$.*

Corollary 6.5 (Boundedness of solutions). *Let $F(\chi,\tau,\omega)$ satisfy Assumption* (A9), *$f(\chi,\tau)=0$ identically, $\phi(x)$ and $\xi(t)$ be bounded and $u(x,t)$ be a solution of* (6.1). *Then $u(x,t)$ is bounded.*

Corollary 6.6 (Sign preservation). *Let $F(\chi,\tau,\omega)$ satisfy* (A9), *$f(\chi,\tau)$ be nonnegative, $\phi(x)$ and $\xi(t)$ be nonnegative and $u(x,t)$ be a solution of* (6.1). *Then $u(x,t)$ is nonnegative.*

Finally, in contrast to previous sections about the problem (2.3), we are able to prove following assertions about nonlinear problem (6.1).

Corollary 6.7 (Uniform stability). *Consider a function $F(\chi,\tau,\omega)$ for which the partial derivative $F_\omega(\chi,\tau,\omega)$ is a continuous and positive function. Let $u_1(x,t)$ be a solution of* (6.1) *with initial-boundary conditions $\phi_1(x)$ and $\xi_1(t)$. Let $u_2(x,t)$ be a solution of* (6.1) *with initial-boundary conditions $\phi_2(x)$ and $\xi_2(t)$. Then*

$$\sup_{\substack{x\in\mathbb{Z},x\geq a\\ t\in\mathbb{R}_0^+}} |u_1(x,t)-u_2(x,t)| \leq \sup_{\substack{x\in\mathbb{Z},x\geq a\\ t\in\mathbb{R}_0^+}} \{|\phi_1(x)-\phi_2(x)|,|\xi_1(t)-\xi_2(t)|\}$$

holds.

Proof. We prove the statement with the help of maximum and minimum principles from Theorems 6.2 and 6.3. Thanks to the assumption that $u_1(x,t)$ and $u_2(x,t)$ are solutions we get the equality

$$(u_1)_t(x,t)+F(x,t,\nabla_x u_1(x,t))-(u_2)_t(x,t)-F(x,t,\nabla_x u_2(x,t))=0.$$

Applying the mean value theorem we can rewrite it to the form

$$(u_1)_t(x,t)-(u_2)_t(x,t)+F_\omega(x,t,\theta(x,t))\nabla_x(u_1(x,t)-u_2(x,t))=0,$$

where $\theta(x,t)=\alpha\nabla_x u_1(x,t)+(1-\alpha)\nabla_x u_2(x,t)$, $\alpha\in[0,1]$. Let us define an auxiliary function $v(x,t)=u_1(x,t)-u_2(x,t)$. Consequently, $v(x,t)$ solves

$$\begin{gathered}
v_t(x,t)+F_\omega(x,t,\theta(x,t))\nabla_x v(x,t)=0,\\
v(x,0)=\phi_1(x)-\phi_2(x),\\
v(a,t)=\xi_1(t)-\xi_2(t),
\end{gathered}$$

when the assumptions of Theorems 6.2 and 6.3 are satisfied. Thus, from Theorem 6.2 we obtain

$$\begin{aligned}
u_1(x,t)-u_2(x,t)=v(x,t) &\leq \sup_{\substack{x\in\mathbb{Z},x\geq a\\ t\in\mathbb{R}_0^+}} \{\phi_1(x)-\phi_2(x),\xi_1(t)-\xi_2(t)\}\\
&\leq \sup_{\substack{x\in\mathbb{Z},x\geq a\\ t\in\mathbb{R}_0^+}} \{|\phi_1(x)-\phi_2(x)|,|\xi_1(t)-\xi_2(t)|\}.
\end{aligned}$$

Similarly, from Theorem 6.3, there is

$$\begin{aligned} u_1(x,t) - u_2(x,t) = v(x,t) &\geq \inf_{\substack{x\in\mathbb{Z}, x\geq a\\ t\in\mathbb{R}_0^+}} \{\phi_1(x) - \phi_2(x), \xi_1(t) - \xi_2(t)\} \\ &\geq - \sup_{\substack{x\in\mathbb{Z}, x\geq a\\ t\in\mathbb{R}_0^+}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\} \end{aligned}$$

which completes the proof. □

Corollary 6.8. *Consider a function $F(\chi,\tau,\omega)$ for which the partial derivative $F_\omega(\chi,\tau,\omega)$ is a continuous and positive function. Let $\{u_n\}_{n=1}^{+\infty}$ be a sequence of solutions $u_n(x,t)$ of* (6.1) *with the initial-boundary conditions $\phi_n(x)$ and $\xi_n(t)$ for that*

$$\phi_n(x) \rightrightarrows \phi(x) \text{ for } x\in\mathbb{Z}, x\geq a, \quad \text{and} \quad \xi_n(t) \rightrightarrows \xi(t) \text{ for } t\in\mathbb{R}_0^+.$$

Assume that $u(x,t)$ is a solution of (6.1) *with the initial-boundary conditions $\phi(x)$ and $\xi(t)$. Then*

$$u_n(x,t) \rightrightarrows u(x,t) \quad \text{for } x\in\mathbb{Z},\ x\geq a,\ t\in\mathbb{R}_0^+.$$

Corollary 6.9 (Comparison theorem)**.** *Consider a function $F(\chi,\tau,\omega)$ for which the partial derivative $F_\omega(\chi,\tau,\omega)$ is continuous and positive function. Suppose, there exists a solution $u(x,t)$ of* (6.1)*. Moreover, let $v(x,t)$ be a lower solution and $w(x,t)$ be an upper solution of* (6.1)*. Then*

$$v(x,t) \leq u(x,t) \leq w(x,t)$$

is necessarily satisfied for all $x\in\mathbb{Z}$, $x\geq a$, and for all $t\in\mathbb{R}_0^+$.

Proof. We define two auxiliary functions $\overline{v}(x,t) = u(x,t) - v(x,t)$ and $\overline{w}(x,t) = w(x,t) - u(x,t)$ and investigate their sign.

(1) First, we study the function $\overline{v}(x,t)$. Because $v(x,t)$ is a lower solution we get

$$0 \leq u_t(x,t) + F(x,t,\nabla_x u(x,t)) - v_t(x,t) - F(x,t,\nabla_x v(x,t)).$$

Thanks to assumptions on F we can use the mean value theorem and we can continue with our estimate,

$$\begin{aligned} 0 &\leq u_t(x,t) + F(x,t,\nabla_x u(x,t)) - v_t(x,t) - F(x,t,\nabla_x v(x,t)) \\ &= [u(x,t) - v(x,t)]_t + F_\omega(x,t,\theta(x,t))\,[\nabla_x u(x,t) - \nabla_x v(x,t)] \\ &= \overline{v}_t(x,t) + F_\omega(x,t,\theta(x,t))\nabla_x \overline{v}(x,t). \end{aligned}$$

for some $\theta(x,t) = \alpha\nabla_x u(x,t) + (1-\alpha)\nabla_x v(x,t)$, $\alpha\in[0,1]$. For initial and boundary conditions we have

$$\begin{aligned} \overline{v}(x,0) &= u(x,0) - v(x,0) \geq 0, \\ \overline{v}(a,t) &= u(a,t) - v(a,t) \geq 0. \end{aligned}$$

Thus, assumptions of Theorem 6.3 are satisfied for $\overline{v}(x,t)$ which implies

$$\overline{v}(x,t) \geq 0, \quad \text{i.e.,} \quad v(x,t) \leq u(x,t).$$

(2) For the function $\overline{w}(x,t)$ it is similar. By the same procedure we get

$$\overline{w}(x,t) \geq 0, \quad \text{i.e.,} \quad u(x,t) \leq w(x,t).$$

□

Remark 6.10. *If we would like to prove the similar assertions for* (2.3) *by the same procedure then proofs would fail after using the mean value theorem. In that case, the backward difference operator* ∇_x *would be applied on the partial derivative* $F_\omega(x,t,\theta(x,t))$*. Hence, we would not be able to satisfy assumptions of Theorems 3.2 and 3.3 because we would not know the behavior of the function* $\theta(x,t)$*.*

7. Concluding remarks

In this paper we present some maximum and minimum principles for transport equations with discrete space and continuous time and derive several applications. But there are still many open questions left.

First, we can try to find another maximum principles with distinct or weaker assumptions or we can try to derive another properties of solutions of (2.3) and (6.1). Next, we should say that, although, we consider nonlinear function F as a function $F(\chi,\tau,\omega)$ in our problems, in many cases we have to assume that F is not a function of χ. Therefore, we can try to improve it and find better conditions.

We study only initial–boundary value problems as well. We can ask what will change if we consider an initial value problem on the whole $\mathbb{Z}$. One can show that in that case we cannot prove maximum or minimum principles in the same way by *stairs method* as Theorem 3.2. Moreover, we cannot use mathematical induction to prove the existence of solution of IVP because we have not where to start.

Further, we could try to generalize our results for more general time and space structures as in [17, 18, 19] (in these papers dynamic equations on time-scales are studied, for more information about time-scale calculus see [1, 8]).

In this paper we analyze equations with one space variable and hence, we can state the question what happens if we consider more space variables as on finite-dimensional lattice dynamical systems in [6, 7, 12].

Another natural generalization is to study evolutionary equations of higher order, e.g., diffusion or wave-type equations on discrete-space domains as in [2, 3, 4, 21].

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APPENDIX C

Maximum principles for discrete and semidiscrete reaction-difussion equation

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Research Article

Maximum Principles for Discrete and Semidiscrete Reaction-Diffusion Equation

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We study reaction-diffusion equations with a general reaction function f on one-dimensional lattices with continuous or discrete time u_x' (or $\Delta_t u_x$) $= k(u_{x-1} - 2u_x + u_{x+1}) + f(u_x)$, $x \in \mathbb{Z}$. We prove weak and strong maximum and minimum principles for corresponding initial-boundary value problems. Whereas the maximum principles in the semidiscrete case (continuous time) exhibit similar features to those of fully continuous reaction-diffusion model, in the discrete case the weak maximum principle holds for a smaller class of functions and the strong maximum principle is valid in a weaker sense. We describe in detail how the validity of maximum principles depends on the nonlinearity and the time step. We illustrate our results on the Nagumo equation with the bistable nonlinearity.

1. Introduction

Reaction-diffusion equation $\partial_t u = k\partial_{xx} u + f(u)$ (sometimes called FKPP equation, which abbreviates Fisher, Kolmogorov, Petrovsky, Piskounov) serves as a nonlinear model to describe a class of (biological, chemical, economic, and so forth) phenomena in which two factors are combined. Firstly, the diffusion process causes the concentration of a substance (animals, wealth, and so forth) to spread in space. Secondly, a local reaction leads to dynamics based on the concentration values.

For the sake of applications and correctness of numerical procedures it makes sense to consider partially or fully discretized reaction-diffusion equation. In certain situations (e.g., spatially structured environment) it is natural to study reaction-diffusion equations with discretized space variable and continuous time (we refer to it as a semidiscrete problem and use $u_x(t) = u(x,t)$):

$$u_x'(t) = k\left(u_{x-1}(t) - 2u_x(t) + u_{x+1}(t)\right) + f\left(u_x(t)\right), \quad x \in \mathbb{Z},\ t \in [0,+\infty), \tag{1}$$

or, for example, if nonoverlapping populations are considered, with both time and space variables being discrete (a discrete problem, $u_{x,t} := u(x,t)$):

$$\frac{u_{x,t+h} - u_{x,t}}{h} = k\left(u_{x-1,t} - 2u_{x,t} + u_{x+1,t}\right) + f\left(u_{x,t}\right), \quad x \in \mathbb{Z},\ t \in \{0, h, 2h, \dots\}. \tag{2}$$

Examples of such phenomena are chemical reactions related to crystal formation, see Cahn [1], or myelinated nerve axons, see Bell and Cosner [2] and Keener [3]. Existence and nonexistence of travelling waves in those models have been recently studied in Chow [4], Chow et al. [5], and Zinner [6] mostly with the cubic (or bistable, double-well) nonlinearities of the form $f(u) = \lambda u(u-a)(1-u)$, with $\lambda > 0$ and $a \in (0,1)$ (this special case of FKPP equation is being referred to as Nagumo equation). In contrast, various reaction functions have been proposed in models without spatial interaction, for example, Xu et al. [7].

Motivated by these facts, we allow for a general form of the reaction function f in this paper (i.e., we do not restrict ourselves to cubic nonlinearities). We prove a priori estimates for discrete reaction-diffusion equation (2) and then use Euler method to show their validity for semidiscrete reaction-diffusion equation (1). Whereas the maximum principles in the semidiscrete case exhibit similar features to those of continuous reaction-diffusion model (i.e., they hold under similar assumptions), in the discrete case the weak maximum principle holds for a smaller class of functions and the strong maximum principle is only valid in a weaker sense involving the domain of dependence. Finally, we use the maximum principles to get the global existence of solutions of the initial-boundary problem for the semidiscrete case (1). All our results are illustrated in detail in Nagumo equations with a symmetric bistable nonlinearity; that is, we consider problems (1)-(2) with $f(u) = \lambda u(1 - u^2)$.

Our motivation is twofold. First, maximum principles could be used to obtain comparison principles (Protter and Weinberger [8]), which in turn could serve as a valuable tool in the study of traveling waves, for example, Bell and Cosner [2]. Moreover, similarly as in the case of (non)existence of traveling wave solutions for Nagumo equations, it has been shown that discrete and semidiscrete structures influence the validity of maximum principles in a significant way. Even the simplest one-dimensional linear problems require additional assumptions on the step size; see Mawhin et al. [9] and Stehlík and Thompson [10]. In the case of partial difference and semidiscrete equations, the strong influence of the underlying structure on maximum principles has been described in the linear case for transport equation in Stehlík and Volek [11] and for diffusion-type equations in Slavík and Stehlík [12] and Friesl et al. [13] (interestingly, the proofs of maximum principles in this case are based on product integration; see Slavík [14]). Finally, simple maximum principles for nonlinear transport equations on semidiscrete domains have been presented in Volek [15].

In the classical case, maximum principles for diffusion (and parabolic) equations go back to Picone [16] and Levi [17]. Strong maximum principles were later established by Nirenberg [18] and a survey of various versions and applications could be found in a classical monograph Protter and Weinberger [8].

This paper is segmented in the following way. In Section 2, we briefly summarize results for the classical reaction-diffusion equation. Next, we prove weak and strong maximum principles for the discrete case (2) (Sections 3 and 4). In the case of the initial-boundary value problem for the semidiscrete equation (1) we provide local existence results (Section 5) and maximum principles (Section 6) which we consequently apply to get global existence of solutions in Section 7. Our results are then applied to the Nagumo equation with a symmetric bistable nonlinearity, that is, problems (1)-(2) with $f(u) = \lambda u(1 - u^2)$, in Section 8.

2. Reaction-Diffusion Partial Differential Equation

In order to motivate and compare our results for the reaction-diffusion equations on discrete-space domains with the classical reaction-diffusion equation we briefly summarize few basic results for the following initial-boundary problem:

$$\begin{aligned} &\partial_t u(x,t) = k\partial_{xx} u(x,t) + f(x,t,u(x,t)), \\ &\qquad\qquad x \in (a,b),\ t \in \mathbb{R}^+,\ k > 0, \\ &u(x,0) = \varphi(x), \quad x \in [a,b], \\ &u(a,t) = \xi_a(t), \quad t \in \mathbb{R}_0^+, \\ &u(b,t) = \xi_b(t), \quad t \in \mathbb{R}_0^+, \end{aligned} \tag{3}$$

where $f : (a,b) \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a reaction function and $\varphi : [a,b] \to \mathbb{R}$, $\xi_a, \xi_b : \mathbb{R}_0^+ \to \mathbb{R}$ are initial-boundary conditions satisfying $\varphi(a) = \xi_a(0)$ and $\varphi(b) = \xi_b(0)$.

The following existence and uniqueness result for (3) can be found, for example, in [19, page 298].

Theorem 1. *Let $T > 0$ be arbitrary and let f be uniformly Hölder continuous in x and t and Lipschitz in u for $(x,t) \in (a,b)\times(0,T]$. Then for all Hölder continuous initial-boundary conditions φ, ξ_a, ξ_b problem (3) has a unique bounded solution which is defined on $[a,b] \times [0,T]$.*

We define the following two numbers for the brevity:

$$\begin{aligned} M_T &:= \max_{x\in[a,b],t\in[0,T]} \{\varphi(x), \xi_a(t), \xi_b(t)\}, \\ m_T &:= \min_{x\in[a,b],t\in[0,T]} \{\varphi(x), \xi_a(t), \xi_b(t)\}. \end{aligned} \tag{4}$$

For the linear diffusion equation (i.e., (3) with $f(x,t,u) \equiv 0$) the maximum principle is proved, for example, in [8, Chapter 3.1]. For the nonlinear problem (3) (i.e., $f(x,t,u) \neq 0$) the following weak maximum principle holds (see [19, Theorem 1]).

Theorem 2. *Let $T > 0$ be arbitrary and let f be uniformly Hölder continuous in x and t and Lipschitz in u for $(x,t) \in (a,b) \times (0,T]$ and assume that*

$$\begin{aligned} &f(x,t,M_T) \le 0 \le f(x,t,m_T) \\ &\qquad\qquad \forall (x,t) \in (a,b) \times (0,T]. \end{aligned} \tag{5}$$

Let u be a continuous solution of (3) with Hölder continuous initial-boundary conditions φ, ξ_a, ξ_b. Then

$$m_T \le u(x,t) \le M_T \tag{6}$$

holds for all $(x,t) \in [a,b] \times [0,T]$.

Moreover, the strong maximum principle also holds (see [19, Theorem 2]).

Theorem 3. *Let the assumptions of Theorem 2 be satisfied and let u be a solution of (3) on $[a,b]\times[0,T]$. If $u(x_0,t_0) = M_T$ (or $u(x_0,t_0) = m_T$) for some $(x_0,t_0) \in (a,b) \times (0,T]$ then*

$$\begin{aligned} &u(x,t) = M_T \\ &(\text{or } u(x,t) = m_T) \\ &\qquad \forall (x,t) \in [a,b] \times [0,t_0]. \end{aligned} \tag{7}$$

3. Discrete Reaction-Diffusion Equation: Weak Maximum Principles

Let us consider the initial-boundary value problem for the discrete reaction-diffusion equation (which could be obtained, e.g., by Euler discretization of (3)):

$$\begin{gathered}\frac{u(x,t+h)-u(x,t)}{h}=k\Delta^2_{xx}u(x-1,t)\\ +f(x,t,u(x,t)),\\ x\in(a,b)_{\mathbb{Z}},\ t\in h\mathbb{N}_0,\ h>0,\ k>0,\\ u(x,0)=\varphi(x),\quad x\in(a,b)_{\mathbb{Z}},\\ u(a,t)=\xi_a(t),\quad t\in h\mathbb{N}_0,\\ u(b,t)=\xi_b(t),\quad t\in h\mathbb{N}_0,\end{gathered}\tag{8}$$

where $f:(a,b)_{\mathbb{Z}}\times h\mathbb{N}_0\times\mathbb{R}\to\mathbb{R}$ is a reaction function, $\varphi:(a,b)_{\mathbb{Z}}\to\mathbb{R},\xi_a,\xi_b:h\mathbb{N}_0\to\mathbb{R}$ are initial-boundary conditions, $h\mathbb{N}_0:=\{hn,n\in\mathbb{N}_0\}$, $(a,b)_{\mathbb{Z}}:=(a,b)\cap\mathbb{Z}$, and $\Delta^2_{xx}u(x-1,t):=u(x-1,t)-2u(x,t)+u(x+1,t)$ (for brevity, we assume the space discretization step $h_x=1$, but all our results are easily extendable to an arbitrary step $h_x>0$ if we use the diffusion constant $\tilde{k}=k/h_x^2$ instead of k; we discuss this in detail in a specific example at the end of Section 8).

Straightforwardly, problem (8) has a unique solution which is defined in $[a,b]_{\mathbb{Z}}\times h\mathbb{N}_0$, since $u(x,t+h)$ is uniquely given by

$$u(x,t+h)=\begin{cases}u(x,t)+h\left(k\Delta^2_{xx}u(x-1,t)+f(x,t,u(x,t))\right), & x\in(a,b)_{\mathbb{Z}},\\ \xi_a(t+h), & x=a,\\ \xi_b(t+h), & x=b.\end{cases}\tag{9}$$

For $T\in h\mathbb{N}_0$, we define the following two numbers:

$$M_T:=\max_{x\in(a,b)_{\mathbb{Z}},t\in[0,T]_{h\mathbb{N}_0}}\{\varphi(x),\xi_a(t),\xi_b(t)\},\tag{10}$$

$$m_T:=\min_{x\in(a,b)_{\mathbb{Z}},t\in[0,T]_{h\mathbb{N}_0}}\{\varphi(x),\xi_a(t),\xi_b(t)\}.\tag{11}$$

For brevity of the following assertions we formulate the assumption in the reaction function f:

(D) Let $T\in h\mathbb{N}_0$ and let f satisfy

$$\frac{2hk-1}{h}(u-m_T)\le f(x,t,u)\le\frac{2hk-1}{h}(u-M_T),\tag{12}$$

for all $x\in(a,b)_{\mathbb{Z}}$, $t\in[0,T]_{h\mathbb{N}_0}$ and $u\in[m_T,M_T]$.

Remark 4. The inequalities (12) imply that for all fixed x and t the graph of function $f(x,t,\cdot)$ does not intersect the forbidden area depicted in Figure 1.

Remark 5. Let us notice that for $h\to 0+$ the slope $(2hk-1)/h$ goes to $-\infty$; that is, the forbidden area from Remark 4 is smaller in the sense of inclusion and it is easier to satisfy assumption (D) if we decrease the time discretization step h. We illustrate this fact in Figure 1.

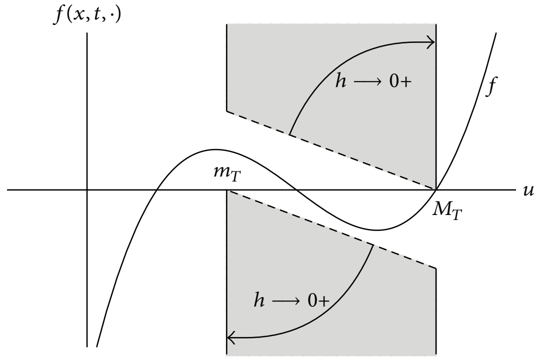


Figure 1: The forbidden area for the function $f(x,t,\cdot)$ in assumption (D). The change of this area if $h\to 0+$. The slope of the dashed line is given by $2k-1/h$; see assumption (D).

Proposition 6. *Assume that $m_T<M_T$. If $h>1/2k$ then (D) does not hold for any function f.*

Note that the inequality $h\le 1/2k$ is the necessary condition for the validity of maximum principles even in the linear case; see, for example, [13, Theorem 2.4].

Proof. If $h>1/2k$ (i.e., $2hk-1>0$) then from (12) there should be

$$\begin{aligned}0&<\frac{2hk-1}{h}(u-m_T)\le f(x,t,u)\\&\le\frac{2hk-1}{h}(u-M_T)<0\quad\text{for }u\in(m_T,M_T),\end{aligned}\tag{13}$$

a contradiction. □

Remark 7. Notice that if $m_T=M_T$ then (D) implies that $f(x,t,m_T)=f(x,t,M_T)=0$ for all $x\in(a,b)_{\mathbb{Z}}$ and $t\in[0,T]_{h\mathbb{N}_0}$. This situation corresponds to the case of the constant initial-boundary conditions $\varphi(x)\equiv M_T$ and $\xi_a(t)\equiv\xi_b(t)\equiv M_T$. From $f(x,t,M_T)=0$ and from (9) there is

$$u(x,t)=\varphi(x)=M_T\quad\text{for }t\in[0,T]_{h\mathbb{N}_0}.\tag{14}$$

Now we state an auxiliary lemma which is crucial in the proof of the maximum principle.

Lemma 8. *Let $T\in h\mathbb{N}_0$, let function f satisfy (D), and let u be the unique solution of (8). Then for all $x\in[a,b]_{\mathbb{Z}}$ and for all $t\in[0,T)_{h\mathbb{N}_0}$*

$$\begin{aligned}&m_T\le u(x,t)\le M_T\\&\qquad\textit{implies that } m_T\le u(x,t+h)\le M_T.\end{aligned}\tag{15}$$

Proof. For the sake of brevity, we only show that $u(x,t+h)\le M_T$. The inequality $m_T\le u(x,t+h)$ can be proved in the same way.

Let $t\in h\mathbb{N}_0$, $t<T$, be arbitrary. Then $u(a,t+h)=\xi_a(t+h)\le M_T$ and $u(b,t+h)=\xi_b(t+h)\le M_T$ trivially from

the definition of M_T (10) (recall that $t < T$, i.e., $t + h \leq T$). If $x \in (a, b)_{\mathbb{Z}}$ then we can estimate

$$\begin{aligned} u(x, t+h) &= u(x,t) + h\big(ku(x-1,t) - 2ku(x,t) \\ &\quad + ku(x+1,t) + f(x,t,u(x,t))\big) \leq 2hkM_T + (1 \\ &\quad - 2hk)\, u(x,t) + hf(x,t,u(x,t)). \end{aligned} \tag{16}$$

Thanks to the assumptions (12) and $m_T \leq u(x,t) \leq M_T$ we get

$$hf(x,t,u(x,t)) \leq (2hk-1)\big(u(x,t) - M_T\big). \tag{17}$$

Therefore,

$$\begin{aligned} u(x,t+h) &\leq 2hkM_T + (1-2hk)\, u(x,t) \\ &\quad + hf(x,t,u(x,t)) \\ &\leq 2hkM_T + (1-2hk)\, u(x,t) \\ &\quad + (2hk-1)\big(u(x,t) - M_T\big) = M_T. \end{aligned} \tag{18}$$

□

The weak maximum principle follows immediately.

Theorem 9. *Let $T \in h\mathbb{N}_0$ be arbitrary, let function f satisfy (D), and let u be the unique solution of (8). Then*

$$m_T \leq u(x,t) \leq M_T \tag{19}$$

holds for all $x \in [a,b]_{\mathbb{Z}}$ and $t \in [0,T]_{h\mathbb{N}_0}$.

Proof. From (10) to (11) we get $m_T \leq u(x,0) \leq M_T$ for all $x \in [a,b]_{\mathbb{Z}}$. Immediately, Lemma 8 yields that (19) holds for all $x \in [a,b]_{\mathbb{Z}}$ and $t \in [0,T]_{h\mathbb{N}_0}$. □

Remark 10. If the reaction function f does not satisfy the inequalities (12) we can find a counterexample that the maximum principle does not hold in general. For example, let us consider (8) with $a = -1$, $b = 1$, $t \in \mathbb{N}_0$, and $\varphi(x) \equiv 0$, $\xi_a(t) \equiv \xi_b(t) \equiv 0$. Let us assume that, for example, the latter inequality in (12) does not hold; that is,

$$f(0,t,0) > \frac{2hk-1}{h}\big(0 - M_T\big) = 0, \tag{20}$$

for some $t \in \mathbb{N}_0$. Assuming without loss of generality that $t = 0$, then the maximum principle is straightforwardly violated since

$$\begin{aligned} u(0,1) &= ku(-1,0) + (1-2k)\, u(0,0) + ku(1,0) \\ &\quad + f(0,0,0) = f(0,0,0) > 0 = M_T. \end{aligned} \tag{21}$$

In certain cases, the function f could fail to satisfy (D) but could still provide a priori bounds for solutions of (8) if the following inequalities hold.

(D') Let $T \in h\mathbb{N}_0$ and let there exist $S \geq M_T$ and $R \leq m_T$ such that

$$\frac{2hk-1}{h}(u-R) \leq f(x,t,u) \leq \frac{2hk-1}{h}(u-s), \tag{22}$$

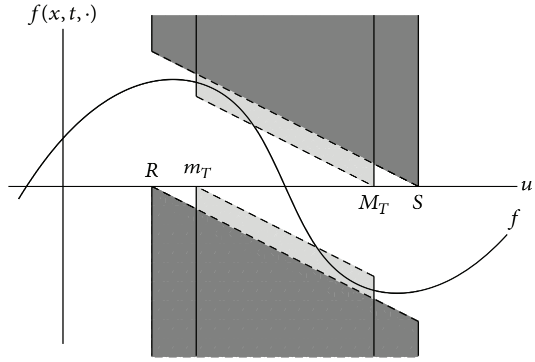


Figure 2: The example of the function f that does not satisfy (D) but satisfies (D') for some constants R, S. Such a function consequently provides a priori bounds for solutions of (8) in the sense of Theorem 11.

for all $x \in (a,b)_{\mathbb{Z}}$ and $t \in h\mathbb{N}_0$ such that $0 \leq t \leq T$ and $u \in [R,S]$.

In that case, we obtain a general version of the weak maximum principle (for the illustration of (D') see Figure 2).

Theorem 11. *Let $T \in h\mathbb{N}_0$ be arbitrary, let function f satisfy (D'), and let u be the unique solution of (8). Then*

$$R \leq u(x,t) \leq S \tag{23}$$

holds for all $x \in (a,b)_{\mathbb{Z}}$ and $t \in h\mathbb{N}_0$ such that $0 \leq t \leq T$.

Proof. For $t = 0$ we have

$$R \leq m_T \leq u(x,0) \leq M_T \leq S, \quad \forall x \in (a,b)_{\mathbb{Z}}. \tag{24}$$

Now we can proceed analogously as in the proofs of Lemma 8 and Theorem 9 where we use (D') instead of (D). We omit the details. □

Example 12. The set of nonlinear reaction functions f that could be considered in Theorem 9 or 11 includes, for example, (for the detailed analysis with $f(x,t,u) = \lambda u(1-u^2)$ see Section 8)

(i) $f(x,t,u) = -|u|^{p-1}u$ with $p > 1$,
(ii) the logistic function $f(x,t,u) = u(1-u)$,
(iii) the bistable nonlinearity $f(x,t,u) = \lambda u(u-a)(1-u)$, $a \in (0,1)$,
(iv) $f(x,t,u) = \lambda u(1-u^p)$ where $p \in \mathbb{N}$,
(v) $f(x,t,u) = -|x| \arctan(t^2 u)$.

We state the following two claims that are direct corollaries of Theorem 9.

Corollary 13. *Assume that ξ_a, ξ_b are bounded. Let f satisfy (D) for all $T > 0$. Then the unique solution u of (8) is bounded.*

Corollary 14. *Assume that φ, ξ_a, ξ_b are nonnegative. Let f satisfy (D) for all $T > 0$. Then the unique solution u of (8) is nonnegative.*

4. Discrete Reaction-Diffusion Equation: Strong Maximum Principle

As in the case of classical reaction-diffusion equation (3) (Theorem 3) we naturally turn our attention to strong maximum principles. Straightforwardly, the strong maximum principle does not hold in the discrete case in the sense of Theorem 3.

Example 15. Let us consider problem (8) with $x \in [-2,2]_{\mathbb{Z}}$, $t \in \mathbb{N}_0$, $f(x,t,u) \equiv 0$, and $k = 1/2$ (note that $h = 1/2k$) and let

$$\begin{aligned} \varphi(x) &\equiv M > 0, \quad x \in \{-1,0,1\}, \\ \xi_{-2}(t) &\equiv \xi_2(t) \equiv 0, \quad t \in \mathbb{N}_0. \end{aligned} \tag{25}$$

Then from (9) we get

$$\begin{aligned} u(0,1) = {} & u(0,0) + ku(-1,0) - 2ku(0,0) + ku(1,0) \\ & + f(0,0,0) = M. \end{aligned} \tag{26}$$

Analogously, we can deduce that

$$\begin{aligned} u(-2,1) &= u(2,1) = 0, \\ u(-1,1) &= u(1,1) = \frac{M}{2}. \end{aligned} \tag{27}$$

Consequently, the strong maximum principle does not hold.

Nonetheless, given the fact that the values of $u(x,t)$ are given by (9), we can easily construct the domain of dependence of (x_0,t_0):

$$\begin{aligned} \mathscr{D}(x_0,t_0) := \Big\{ & (x,t) \in [a,b]_{\mathbb{Z}} \times h\mathbb{N}_0 : t \le t_0,\ x = x_0 \\ & \pm j,\ j = 0,1,\ldots,\frac{t_0 - t}{h} \Big\} \end{aligned} \tag{28}$$

and the domain of influence of (x_0,t_0):

$$\begin{aligned} \mathscr{I}(x_0,t_0) := \Big\{ & (x,t) \in [a,b]_{\mathbb{Z}} \times h\mathbb{N}_0 : t \ge t_0,\ x = x_0 \\ & \pm j,\ j = 0,1,\ldots,\frac{t - t_0}{h} \Big\}. \end{aligned} \tag{29}$$

Considering the following:

(D'') Let $T \in h\mathbb{N}_0$ and let f satisfy for all $x \in (a,b)_{\mathbb{Z}}$, $t \in [0,T]_{h\mathbb{N}_0}$:

(a) $f(x,t,u) < ((2hk-1)/h)(u - M_T)$ when $u \in [m_T, M_T)$,

(b) $f(x,t,u) > ((2hk-1)/h)(u - m_T)$ when $u \in (m_T, M_T]$,

(c) $f(x,t,M_T) \le 0$ and $f(x,t,m_T) \ge 0$,

the weaker version of the strong maximum principle follows immediately.

Theorem 16. *Assume that the function f satisfies (D'') for all $T \in h\mathbb{N}_0$. Let u be the unique solution of (8) and $(x_0,t_0) \in [a,b]_{\mathbb{Z}} \times h\mathbb{N}_0$.*

(1) *If $u(x_0,t_0) = M_T$ (or $u(x_0,t_0) = m_T$), then $u(x,t) = M_T$ (or $u(x,t) = m_T$) on $\mathscr{D}(x_0,t_0)$.*

(2) *If $u(x_0,t_0) < M_T$ (or $u(x_0,t_0) > m_T$), then $u(x,t) < M_T$ (or $u(x,t) > m_T$) on $\mathscr{I}(x_0,t_0)$.*

Proof. Let us only focus on the former statement of the theorem; the latter could be proved in very similar way. We show that if the function f satisfies (D'') and $u(x_0,t_0) = M_T$ for some $x_0 \in (a,b)_{\mathbb{Z}}$, $t_0 \in h\mathbb{N}_0$, $0 < t_0 \le T$, then $u(x_0-1,t_0-h) = u(x_0,t_0-h) = u(x_0+1,t_0-h) = M_T$. The rest follows by induction.

Assume by contradiction first that $u(x_0-1,t_0-h) < M_T$ (the case $u(x_0+1,t_0-h) < M_T$ follows easily). Using this assumption, (9), and Theorem 9 we can estimate

$$\begin{aligned} u(x_0,t_0) = {} & hk\left(u(x_0-1,t_0-h) + u(x_0+1,t_0-h)\right) \\ & + (1-2hk)\,u(x_0,t_0-h) \\ & + hf\left(x_0,t_0-h,u(x_0,t_0-h)\right) \\ < {} & 2hkM_T + (1-2hk)\,u(x_0,t_0-h) \\ & + hf\left(x_0,t_0-h,u(x_0,t_0-h)\right) + M_T \\ & - M_T \\ = {} & (1-2hk)\left(u(x_0,t_0-h) - M_T\right) \\ & + hf\left(x_0,t_0-h,u(x_0,t_0-h)\right) + M_T. \end{aligned} \tag{30}$$

Thus, (D'') yields

$$\begin{aligned} & (1-2hk)\left(u(x_0,t_0-h) - M_T\right) \\ & \quad + hf\left(x_0,t_0-h,u(x_0,t_0-h)\right) \le 0. \end{aligned} \tag{31}$$

Consequently, there has to be

$$u(x_0,t_0) < M_T, \tag{32}$$

a contradiction.

If $u(x_0,t_0-h) < M_T$, then by the similar procedure as above we obtain

$$\begin{aligned} u(x_0,t_0) \le {} & (1-2hk)\left(u(x_0,t_0-h) - M_T\right) \\ & + hf\left(x_0,t_0-h,u(x_0,t_0-h)\right) + M_T. \end{aligned} \tag{33}$$

Since $u(x_0,t_0-h) \in [m_T, M_T)$ in this case, (D'') implies that

$$\begin{aligned} & (1-2hk)\left(u(x_0,t_0-h) - M_T\right) \\ & \quad + hf\left(x_0,t_0-h,u(x_0,t_0-h)\right) < 0. \end{aligned} \tag{34}$$

Hence,

$$u(x_0,t_0) < M_T, \tag{35}$$

a contradiction. □

In the case of nonconstant time discretization $h = h(t)$ we can follow similar techniques and consider (D) (eventually, (D') or (D'')) with $h_{\max}$ (or $h_{\sup}$ for $T \to \infty$); see Remark 5.

5. Semidiscrete Reaction-Diffusion Equation: Local Existence

In this section we study the local existence of the following initial-boundary value problem on semidiscrete domains:

$$\begin{aligned} u_t(x,t) &= k\Delta_{xx}^2 u(x-1,t) + f(x,t,u(x,t)), \\ &\qquad x \in (a,b)_{\mathbb{Z}},\ t \in \mathbb{R}_0^+,\ k > 0, \\ u(x,0) &= \varphi(x), \quad x \in (a,b)_{\mathbb{Z}}, \\ u(a,t) &= \xi_a(t), \quad t \in \mathbb{R}_0^+, \\ u(b,t) &= \xi_b(t), \quad t \in \mathbb{R}_0^+, \end{aligned} \tag{36}$$

where $u_t = \partial_t u$ denotes the time derivative, $f : (a,b)_{\mathbb{Z}} \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a reaction function, $\varphi : (a,b)_{\mathbb{Z}} \to \mathbb{R}$, $\xi_a, \xi_b \in C^1(\mathbb{R}_0^+)$ are initial-boundary conditions, and $\mathbb{R}_0^+ := [0,+\infty)$.

Given the fact that (36) can be interpreted as a vector ODE, we can rewrite it as

$$\begin{aligned} \mathbf{u}'(t) &= \mathbf{g}(t,\mathbf{u}(t)), \quad t \in \mathbb{R}_0^+, \\ \mathbf{u}(0) &= \tilde{\mathbf{u}}, \end{aligned} \tag{37}$$

where $\mathbf{u} : \mathbb{R}_0^+ \to \mathbb{R}^N$, $\mathbf{g} : \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and $\tilde{\mathbf{u}} \in \mathbb{R}^N$.

Naturally, we use the well-known result of Picard and Lindelöf to get the local existence for the initial value problem (37) (see [20, Theorem 8.13]).

Theorem 17. *Assume that* $\mathbf{g}$ *is continuous on the rectangle*

$$Q = \left\{(t,\mathbf{u}) \in \mathbb{R}_0^+ \times \mathbb{R}^N : 0 \le t \le \alpha,\ \|\mathbf{u} - \tilde{\mathbf{u}}\| \le \beta\right\} \tag{38}$$

and satisfies the Lipschitz condition on Q; that is, there exists $L > 0$ *such that, for all* $(t_1,\mathbf{u}_1), (t_2,\mathbf{u}_2) \in Q$

$$\|\mathbf{g}(t_1,\mathbf{u}_1) - \mathbf{g}(t_2,\mathbf{u}_2)\| \le L\|\mathbf{u}_1 - \mathbf{u}_2\| \tag{39}$$

holds. Then there exists $\eta > 0$ *such that (37) has a unique solution* $\mathbf{u}$ *defined on* $[0,\eta]$.

We apply Theorem 17 to get the local existence for the semidiscrete reaction-diffusion equation (36). We use the following two assumptions:

(C_{cont}) Let $f(x,t,u)$ be continuous in $(t,u) \in \mathbb{R}_0^+ \times \mathbb{R}$ for all $x \in (a,b)_{\mathbb{Z}}$.

(C_{lip}) Let $f(x,t,u)$ be locally Lipschitz with respect to u on $(a,b)_{\mathbb{Z}} \times \mathbb{R}_0^+ \times \mathbb{R}$; that is, for all $x_0 \in (a,b)_{\mathbb{Z}}$, $t_0 \in \mathbb{R}_0^+$, and $u_0 \in \mathbb{R}$ there exist $\alpha, \beta > 0$ and

$$\begin{aligned} Q(x_0,t_0,u_0) = \{&(x_0,t,u) \in (a,b)_{\mathbb{Z}} \times \mathbb{R}_0^+ \\ &\times \mathbb{R} : |t - t_0| \le \alpha,\ |u - u_0| \le \beta\} \end{aligned} \tag{40}$$

and $L > 0$ such that for all $(x_0,t_1,u_1), (x_0,t_2,u_2) \in Q$ there is

$$|f(x_0,t_1,u_1) - f(x_0,t_2,u_2)| \le L|u_1 - u_2|. \tag{41}$$

Theorem 18. *Let* f *satisfy* (C_{cont}) *and* (C_{lip}). *Then there exists* $\eta > 0$ *such that (36) has a unique solution defined on* $[a,b]_{\mathbb{Z}} \times [0,\eta]$.

Proof. Since the space variable x is from a finite set $(a,b)_{\mathbb{Z}}$ problem (36) corresponds to the following vector ODE:

$$\begin{aligned} \underbrace{\begin{pmatrix} u(a+1,t) \\ u(a+2,t) \\ \vdots \\ u(b-2,t) \\ u(b-1,t) \end{pmatrix}'}_{\mathbf{u}'(t)} &= k \underbrace{\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & & -2 & 1 \\ 0 & \cdots & & 1 & -2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} u(a+1,t) \\ u(a+2,t) \\ \vdots \\ u(b-2,t) \\ u(b-1,t) \end{pmatrix}}_{\mathbf{u}(t)} \\ &+ k \underbrace{\begin{pmatrix} \xi_a(t) \\ 0 \\ \vdots \\ 0 \\ \xi_b(t) \end{pmatrix}}_{\boldsymbol{\xi}(t)} + \underbrace{\begin{pmatrix} f(a+1,t,u(a+1,t)) \\ f(a+2,t,u(a+2,t)) \\ \vdots \\ f(b-2,t,u(b-2,t)) \\ f(b-1,t,u(b-1,t)) \end{pmatrix}}_{\mathbf{f}(t,\mathbf{u}(t))}, \end{aligned} \tag{42}$$

coupled with the initial condition

$$\underbrace{\begin{pmatrix} u(a+1,0) \\ u(a+2,0) \\ \vdots \\ u(b-1,0) \end{pmatrix}}_{\mathbf{u}(0)} = \underbrace{\begin{pmatrix} \varphi(a+1) \\ \varphi(a+2) \\ \vdots \\ \varphi(b-1) \end{pmatrix}}_{\boldsymbol{\varphi}}. \tag{43}$$

Thus, problem (36) can be rewritten in the vector form as follows:

$$\begin{aligned} \mathbf{u}'(t) &= k\mathbf{A}\mathbf{u}(t) + k\boldsymbol{\xi}(t) + \mathbf{f}(t,\mathbf{u}(t)), \quad t \in \mathbb{R}_0^+, \\ \mathbf{u}(0) &= \boldsymbol{\varphi}. \end{aligned} \tag{44}$$

Assumptions (C_{cont}) and (C_{lip}) yield that the nonlinear function $\mathbf{f}$ is continuous and satisfies Lipschitz condition with respect to $\mathbf{u}$ on some rectangle Q. Since the term $\mathbf{Au}$ is linear and therefore Lipschitz with respect to $\mathbf{u}$ and $\boldsymbol{\xi} \in C^1(\mathbb{R}_0^+)$ the assumptions of Theorem 17 are satisfied. Consequently, there exists $\eta > 0$ such that (44) has a unique solution on $[0, \eta]$. □

Remark 19. If we assume only (C_{cont}) then we can apply the Peano theorem [20, Theorem 8.27] instead of Theorem 17 to get the local existence of solutions of (36) which need not be unique.

6. Semidiscrete Reaction-Diffusion Equation: Maximum Principles

Having the local existence and uniqueness we focus on the maximum principles for (36). In the following analysis we approximate the solution of (36) by the solutions of the discrete problem (8) which arises from (36) by the explicit (*Euler*) discretization of the time variable.

First, we define the Euler polygon (see [21, I.7]).

Definition 20. Let $h > 0$ be a discretization step. Consider the initial value problem (37) on the interval $[0, T]$ where $T = nh$, $n \in \mathbb{N}$. Define the subdivision of interval $[0, T]$ as the set of points $t_i = ih$, $i = 0, 1, \ldots, n$, and for $i = 0, 1, \ldots, n-1$ define

$$\begin{aligned} \mathbf{y}_{i+1} &= \mathbf{y}_i + h\mathbf{g}(t_i, \mathbf{y}_i), \\ \mathbf{y}_0 &= \mathbf{u}(0) = \tilde{\mathbf{u}}. \end{aligned} \tag{45}$$

Then the continuous function $\mathbf{y}_{(h)} : [0, T] \to \mathbb{R}^N$ defined by

$$\mathbf{y}_{(h)}(t) = \mathbf{y}_i + (t - t_i)\,\mathbf{g}(t_i, \mathbf{y}_i), \quad t_i \le t \le t_{i+1}, \tag{46}$$

is called *Euler polygon*.

The following statement sums up the convergence of Euler method (see [21, I.7, Theorem 7.3 and I.9, page 54]).

Theorem 21. *Let $T > 0$ and let $\mathbf{g}$ be continuous, satisfying Lipschitz condition on*

$$Q = \left\{(t, \mathbf{u}) \in \mathbb{R}_0^+ \times \mathbb{R}^N : 0 \le t \le T,\ \|\mathbf{u} - \tilde{\mathbf{u}}\| \le \beta\right\}, \tag{47}$$

and let $\|\mathbf{g}\|$ be bounded by a constant $A > 0$ on Q. If $T \le \beta/A$ then the following hold:

(a) *for $h \to 0+$ the Euler polygons $\mathbf{y}_{(h)}(t)$ converge uniformly to a continuous function $\vartheta(t)$ on $[0, T]$,*

(b) *$\vartheta \in C^1(0, T)$ and it is the unique solution of (37) on $[0, T]$.*

We define the bounds of initial-boundary conditions similarly as in the discrete problem:

$$\begin{aligned} M_T &:= \max_{x \in (a,b)_{\mathbb{Z}}, t \in [0,T]} \{\varphi(x), \xi_a(t), \xi_b(t)\}, \\ m_T &:= \min_{x \in (a,b)_{\mathbb{Z}}, t \in [0,T]} \{\varphi(x), \xi_a(t), \xi_b(t)\}. \end{aligned} \tag{48}$$

Before we state the weak maximum principle we describe the connection between the discretization of (36) and the assumption (D).

Lemma 22. *Let $T \in \mathbb{R}_0^+$ and let f satisfy (C_{cont}), (C_{lip}), and*

(C_{sign}) $f(x, t, M_T) \le 0 \le f(x, t, m_T)$ for all $x \in (a, b)_{\mathbb{Z}}$, $t \in [0, T]$.

Then there exists $H > 0$ such that for all $h \in (0, H)$

$$\frac{2hk - 1}{h}(u - m_T) \le f(x, t, u) \le \frac{2hk - 1}{h}(u - M_T) \tag{49}$$

holds for all $x \in (a, b)_{\mathbb{Z}}$, $t \in [0, T]$, and $u \in [m_T, M_T]$.

Proof. We prove the latter inequality in (49) by contradiction. The former inequality can be proved in the same way. Let us assume that for all $H > 0$ there exist $h \in (0, H)$, $x_h \in (a, b)_{\mathbb{Z}}$, $t_h \in [0, T]$, and $u_h \in [m_T, M_T]$ such that

$$f(x_h, t_h, u_h) > \frac{2hk - 1}{h}(u_h - M_T). \tag{50}$$

Therefore, there exist sequences $\{h_m\}_{m=1}^{\infty}$ and $\{(x_m, t_m, u_m)\}_{m=1}^{\infty}$ (we denote $x_m := x_{h_m}$, $t_m := t_{h_m}$, $u_m := u_{h_m}$) such that

$$\begin{aligned} h_m &\longrightarrow 0+, \\ f(x_m, t_m, u_m) &> \frac{2h_m k - 1}{h_m}(u_m - M_T). \end{aligned} \tag{51}$$

First, we observe that if $u_m = M_T$ for some $m \in \mathbb{N}$ we get a contradiction with (C_{sign}). Thus, we can assume that $u_m < M_T$ for all $m \in \mathbb{N}$.

Now we have to distinguish between two cases.

(i) If there does not exist any subsequence $\{u_{m_l}\}_{l=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ such that $u_{m_l} \to M_T$ then the right-hand side of inequality in (51) goes to infinity. Hence, from (51) $f(x_m, t_m, u_m)$ also goes to infinity. This yields a contradiction with (C_{cont}), which implies boundedness of the function f on $(a, b)_{\mathbb{Z}} \times [0, T] \times [m_T, M_T]$.

(ii) Let there exist a subsequence $\{u_{m_l}\}_{l=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ such that $u_{m_l} \to M_T$. We show that we get a contradiction with (C_{lip}) in this case. Since the interval $(a, b)_{\mathbb{Z}}$ is bounded there exists a convergent subsequence $\{x_{m_l}\}_{l=1}^{\infty} \subset \{x_m\}_{m=1}^{\infty}$ such that $x_{m_l} \to \tilde{x}$. Analogically, since $[0, T]$ is bounded there also exists a convergent subsequence $\{t_{m_l}\}_{l=1}^{\infty} \subset \{t_m\}_{m=1}^{\infty}$ such that $t_{m_l} \to \tilde{t}$. Let $\alpha, \beta > 0$, and $L > 0$ be arbitrary. Then we can find $\hat{l} \in \mathbb{N}$ sufficiently large such that

$$\begin{aligned} \frac{1 - 2h_{m_{\hat{l}}} k}{h_{m_{\hat{l}}}} &\ge L, \\ M_T - u_{m_{\hat{l}}} &\le \beta, \\ x_{m_{\hat{l}}} &= \tilde{x}, \\ \left|t_{m_{\hat{l}}} - \tilde{t}\right| &< \alpha. \end{aligned} \tag{52}$$

If we put $\widehat{x} := x_{m_{\widehat{l}}}$, $\widehat{t} := t_{m_{\widehat{l}}}$, $\widehat{u} := u_{m_{\widehat{l}}}$, and $\widetilde{u} := M_T$ and $Q(\widetilde{x}, \widetilde{t}, \widetilde{u})$ is the rectangle from assumption (C_{lip}) with given $\alpha > 0$ and $\beta > 0$ then $(\widehat{x}, \widehat{t}, \widehat{u}), (\widetilde{x}, \widetilde{t}, \widetilde{u}) \in Q(\widetilde{x}, \widetilde{t}, \widetilde{u})$. Now from (51), (52), and (C_{sign}) we can estimate

$$\begin{aligned} L\,|\widetilde{u} - \widehat{u}| &\leq \frac{1 - 2h_{m_{\widehat{l}}}k}{h_{m_{\widehat{l}}}} (M_T - \widehat{u}) < f(\widehat{x}, \widehat{t}, \widehat{u}) \\ &\leq f(\widehat{x}, \widehat{t}, \widehat{u}) - f(\widetilde{x}, \widetilde{t}, M_T) \\ &= f(\widehat{x}, \widehat{t}, \widehat{u}) - f(\widetilde{x}, \widetilde{t}, \widetilde{u}) \\ &\leq \left| f(\widehat{x}, \widehat{t}, \widehat{u}) - f(\widetilde{x}, \widetilde{t}, \widetilde{u}) \right|, \end{aligned} \tag{53}$$

a contradiction with (C_{lip}). □

Remark 23. The assumption (C_{sign}) defines the forbidden area for a reaction function $f(x, t, \cdot)$ in the same way as (D). However, this area is reduced to a pair of half-lines. Let us notice that it is the limit case of forbidden areas for the discrete case if $h \to 0+$ (see Remark 5 and Figure 1). Moreover, it is equivalent to the assumption for classical PDEs; see (5).

Theorem 24. *Let $T > 0$ be arbitrary, let f satisfy (C_{cont}), (C_{lip}), and (C_{sign}), and let u be a solution of (36) defined on $[a,b]_{\mathbb{Z}} \times [0,T]$. Then*

$$m_T \leq u(x,t) \leq M_T \tag{54}$$

holds for all $x \in (a,b)_{\mathbb{Z}}$, $t \in [0,T]$.

Proof. We prove that for all $x \in (a,b)_{\mathbb{Z}}$, $t \in [0,T]$ there is $u(x,t) \leq M_T$. The first inequality in (54) can be proved similarly. Let us assume by contradiction that there exist $x_c \in (a,b)_{\mathbb{Z}}$ and $t_c \in (0,T]$ such that

$$u(x_c, t_c) > M_T. \tag{55}$$

From the continuity of the solution u there exist $x_0 \in (a,b)_{\mathbb{Z}}$ and $t_0 \in [0, t_c)$ such that

(a) $u(x,t) \leq M_T$ for all $x \in (a,b)_{\mathbb{Z}}$ and $t \in [0, t_0]$,

(b) $u(x_0, t_0) = M_T$,

(c) there exists $\delta > 0$ such that

$$u(x_0, t) > M_T \quad \text{on } (t_0, t_0 + \delta). \tag{56}$$

Let us analyze the new initial-boundary value problem (36) with the initial condition $u(x, t_0)$ at time t_0. Let us understand this problem as the initial value problem for the vector ODE (44) with the initial condition at time t_0.

From (C_{cont}), (C_{lip}) we know that $\mathbf{f}(t, \mathbf{u})$ is continuous and Lipschitz on some rectangle Q. From (C_{cont}) we also get that $\mathbf{f}$ is bounded by some constant $A > 0$ on Q. Therefore, Theorem 21 implies that for sufficiently small discretization steps $h > 0$ and for sufficiently small interval $[t_0, t_0 + \varepsilon]$ the Euler polygons $\mathbf{y}_{(h)}(t)$ converge uniformly to the unique solution $\mathbf{u}(t)$ on $[t_0, t_0 + \varepsilon]$.

Notice that the node points of Euler polygons $\mathbf{y}_{(h)}(t)$ are the solutions of (8). From (C_{cont}), (C_{lip}), and (C_{sign}) and from Lemma 22 the assumption (D) is satisfied (recall that h is sufficiently small) and therefore, from Theorem 9

$$y_{(h)}(x,t) \leq M_T \quad \text{on } [t_0, t_0 + \varepsilon], \ \forall x \in (a,b)_{\mathbb{Z}}. \tag{57}$$

But if $\mathbf{y}_{(h)}(t)$ converge uniformly to $\mathbf{u}(t)$ and $y_{(h)}(x,t) \leq M_T$ on $[t_0, t_0 + \min\{\delta, \varepsilon\}]$ for all $x \in (a,b)_{\mathbb{Z}}$ then there has to be

$$\begin{aligned} &u(x,t) \leq M_T \\ &\qquad \text{on } [t_0, t_0 + \min\{\delta, \varepsilon\}], \ \forall x \in (a,b)_{\mathbb{Z}}, \end{aligned} \tag{58}$$

a contradiction with (56). □

If the assumption (C_{sign}) is not satisfied but the nonlinear function f satisfies the following:

(C'_{sign}) Let $T > 0$ be arbitrary and let there exist $S \geq M_T$ and $R \leq m_T$ such that

$$f(x,t,S) \leq 0 \leq f(x,t,R) \tag{59}$$

for all $x \in (a,b)_{\mathbb{Z}}$, $t \in [0,T]$,

then we can state the following generalized weak maximum principle.

Theorem 25. *Let $T > 0$ be arbitrary, let f satisfy (C_{cont}), (C_{lip}), and (C'_{sign}), and let u be a solution of (36) defined on $[a,b]_{\mathbb{Z}} \times [0,T]$. Then*

$$R \leq u(x,t) \leq S \tag{60}$$

holds for all $x \in (a,b)_{\mathbb{Z}}$, $t \in [0,T]$.

Proof. The statement can be proved in the similar way as Lemma 22 and Theorem 24. □

As in the previous sections we want to establish the strong maximum principle. First, we recall the well-known Grönwall's inequality (see, e.g., [20, Corollary 8.62]).

Lemma 26. *Let $\beta, u : [r,s] \to \mathbb{R}$ be continuous functions and let u be differentiable on (r,s). If*

$$u'(t) \leq \beta(t)\, u(t) \quad \textit{for } t \in (r,s) \tag{61}$$

then

$$u(t) \leq u(r)\, e^{\int_r^t \beta(\tau) d\tau} \quad \textit{for } t \in [r,s]. \tag{62}$$

Further, we need the following auxiliary lemma.

Lemma 27. *Let $T > 0$ be arbitrary, let f satisfy (C_{cont}), (C_{lip}), and (C_{sign}), and let u be a solution of (36) defined on $[a,b]_{\mathbb{Z}} \times [0,T]$. If $u(x_0, t_0) = M_T$ (or $u(x_0, t_0) = m_T$) for some $x_0 \in (a,b)_{\mathbb{Z}}$ and $t_0 \in (0,T]$ then*

$$\begin{aligned} &u(x_0, t) = M_T \\ &(\textit{or } u(x_0, t) = m_T) \\ &\qquad \forall t \in [0, t_0]. \end{aligned} \tag{63}$$

Proof. We prove the statement with M_T. The case with m_T is similar. First, the weak maximum principle (Theorem 24) holds; that is, $u(x,t) \le M_T$ for all $x \in [a,b]_{\mathbb{Z}}$ and $t \in [0,T]$. Suppose by contradiction that there exists $t_c \in (0,t_0]$ such that

$$\begin{aligned} &u(x_0,t_c) = M_T, \\ &u(x_0,t) < M_T \qquad \text{on } [t_c-\varepsilon,t_c). \end{aligned} \tag{64}$$

Without loss of generality let $\varepsilon > 0$ be sufficiently small so that the function $f(x_0,t,u)$ is uniformly Lipschitz in u on $[t_c-\varepsilon,t_c)$ with Lipschitz constant $L > 0$ (follows from (C_{lip})). With the help of these facts and also from (C_{sign}) we can estimate

$$\begin{aligned} u_t(x_0,t) &= k(u(x_0-1,t) - 2u(x_0,t) + u(x_0+1,t)) \\ &\quad + f(x_0,t,u(x_0,t)) \\ &\le k(2M_T - 2u(x_0,t)) + f(x_0,t,u(x_0,t)) \\ &\quad - f(x_0,t,M_T) + \underbrace{f(x_0,t,M_T)}_{\le 0} \\ &\le k(2M_T - 2u(x_0,t)) \\ &\quad + \left|f(x_0,t,u(x_0,t)) - f(x_0,t,M_T)\right| \\ &\le k(2M_T - 2u(x_0,t)) + L\left|u(x_0,t) - M_T\right| \\ &= k(2M_T - 2u(x_0,t)) + L(M_T - u(x_0,t)) \\ &= -(2k+L)(u(x_0,t) - M_T), \end{aligned} \tag{65}$$

for all $t \in (t_c-\varepsilon,t_c)$. If we denote $\alpha := -(2k+L)$ then the function $u(x_0,t)$ satisfies $u_t(x_0,t) \le \alpha(u(x_0,t)-M_T)$ on $(t_c-\varepsilon,t_c)$. If we substitute $v(t) := u(x_0,t) - M_T$ then the function v satisfies the following differential inequality:

$$v'(t) \le \alpha v(t). \tag{66}$$

Therefore, Grönwall's inequality (Lemma 26) implies that

$$v(t) \le v(t_c-\varepsilon)\, e^{\alpha(t-t_c+\varepsilon)} \tag{67}$$

and hence

$$\begin{aligned} u(x_0,t) &\le M_T - (M_T - u(x_0,t_c-\varepsilon))\, e^{\alpha(t-t_c+\varepsilon)} \\ &< M_T \end{aligned} \tag{68}$$

on $[t_c-\varepsilon,t_c]$, a contradiction. □

The strong maximum principle for (36) follows immediately.

Theorem 28. *Let $T > 0$ be arbitrary, let f satisfy (C_{cont}), (C_{lip}), and (C_{sign}), and let u be a solution of (36) defined on $[a,b]_{\mathbb{Z}} \times [0,T]$. If $u(x_0,t_0) = M_T$ (or $u(x_0,t_0) = m_T$) for some $x_0 \in (a,b)_{\mathbb{Z}}$ and $t_0 \in (0,T]$ then*

$$\begin{aligned} &u(x,t) = M_T \\ &(\text{or } u(x,t) = m_T) \\ &\qquad \forall x \in [a,b]_{\mathbb{Z}},\ t \in [0,t_0]. \end{aligned} \tag{69}$$

Proof. Lemma 27 yields that $u(x_0,t) = M_T$ for all $t \in [0,t_0]$. If $u(x_0-1,t_c) < M_T$ (or $u(x_0+1,t_c) < M_T$) at some $t_c \in [0,t_0]$ then applying (C_{sign}) the following has to be satisfied:

$$\begin{aligned} u_t(x_0,t_c) &= k(u(x_0-1,t_c) - 2u(x_0,t_c) + u(x_0+1,t_c)) \\ &\quad + f(x_0,t_c,u(x_0,t_c)) \\ &< k(2M_T - 2M_T) + f(x_0,t_c,M_T) \le 0, \end{aligned} \tag{70}$$

a contradiction with the fact that the function $u(x_0,t)$ is constant and equal to M_T on $[0,t_0]$. Therefore, functions $u(x_0-1,t)$ and $u(x_0+1,t)$ are also constant and equal to M_T on $[0,t_0]$. Then we can continue inductively in x to the boundary points $x = a$ or $x = b$. The case with m_T is similar. □

7. Semidiscrete Reaction-Diffusion Equation: Global Existence

In this section we combine the local existence and uniqueness and the maximum principle to obtain the global existence of solution of (36).

Once again, we use known results from the theory of ordinary differential equations. First, we define the maximal interval of existence (see [20, Definition 8.31]).

Definition 29. Let $\mathbf{g}$ be continuous and let $\mathbf{u}$ be a solution of (37) defined on $[0,\eta)$. Then one says $[0,\eta)$ is a *maximal interval of existence* for $\mathbf{u}$ if there does not exist an $\eta_1 > \eta$ and a solution $\mathbf{w}$ defined on $[0,\eta_1)$ such that $\mathbf{u}(t) = \mathbf{w}(t)$ for $t \in [0,\eta)$.

In the following we apply the extendability theorem (see [20, Theorem 8.33]).

Theorem 30. *Let $\mathbf{g}$ be continuous and let $\mathbf{u}$ be a solution of (37) defined on $[0,\omega)$. Then u can be extended to a maximal interval of existence $[0,\eta)$, $0 < \eta \le \infty$. Furthermore, there is either*

$$\begin{aligned} &\eta = \infty \\ &\text{or } \|\mathbf{u}(t)\| \xrightarrow{t\to\eta-} \infty. \end{aligned} \tag{71}$$

This theorem enables us to conclude with the global existence for (36).

Theorem 31. *Let f satisfy (C_{cont}), (C_{lip}), and (C_{sign}) for all $T > 0$. Then (36) has a unique solution defined on $[a,b]_{\mathbb{Z}}\times\mathbb{R}_0^+$ which satisfies*

$$\inf_{x\in(a,b)_{\mathbb{Z}},t\in\mathbb{R}_0^+}\{\varphi(x),\xi_a(t),\xi_b(t)\}\le u(x,t)\le\sup_{x\in(a,b)_{\mathbb{Z}},t\in\mathbb{R}_0^+}\{\varphi(x),\xi_a(t),\xi_b(t)\},\tag{72}$$

for all $(x,t)\in[a,b]_{\mathbb{Z}}\times\mathbb{R}_0^+$.

Proof. Let us understand problem (36) as the initial value problem for the vector ODE (44). From Theorem 18 there exists a uniquely determined local solution $\mathbf{u}(t)$. From Theorem 30 the solution can be extended to a maximal interval of existence $[0,\eta)$ which is open from right and either

$$\eta=\infty \quad \text{or} \quad \|\mathbf{u}(t)\|\xrightarrow{t\to\eta-}\infty.\tag{73}$$

If $\|\mathbf{u}(t)\|\to\infty$ for $t\to\eta-$ then for all $K>0$ there have to exist $x_K\in(a,b)_{\mathbb{Z}}$ and $t_K\in[0,\eta)$ such that

$$|u(x_K,t_K)|>K.\tag{74}$$

If we put $K:=\max\{|m_\eta|,|M_\eta|\}<\infty$ (ξ_a, ξ_b are C^1 functions on $\mathbb{R}_0^+$ and therefore bounded on $[0,\eta]$) then

$$|u(x_K,t_K)|>\max\{|m_\eta|,|M_\eta|\},\tag{75}$$

a contradiction with the maximum principle in Theorem 24 (which holds thanks to (C_{cont}), (C_{lip}), and (C_{sign})).

Therefore, there has to be $\eta=\infty$; that is, the solution $\mathbf{u}(t)$ is defined on the interval $[0,\infty)$ and from (C_{lip}) it has to be unique. □

Remark 32. All nonlinear functions f listed in Example 12 can be considered in Theorems 24 and 31. However, it is worth noting that we have additional assumptions on the nonlinearity in the semidiscrete case (conditions (C_{cont}) and (C_{lip})). Thus, for non-Lipschitz functions (e.g.,

$$f(x,t,u)=\begin{cases}-|u|^{p-1}u, & u\neq0,\\ 0, & u=0\end{cases}\tag{76}$$

with $p\in[0,1)$), we only get the maximum principles in discrete case. On the other hand, in discrete case, the validity of maximum principle depends strongly on the interaction between the discretization step h and the nonlinearity f (see (12); we illustrate this dependence in detail in Section 8).

Let us finish with the two corollaries that are immediate consequences of Theorems 24 and 31.

Corollary 33. *Assume that ξ_a, ξ_b are bounded. Let f satisfy (C_{cont}), (C_{lip}), and (C_{sign}) for all $T>0$. Then the unique solution u of (36) is bounded.*

Corollary 34. *Assume that φ, ξ_a, ξ_b are nonnegative. Let f satisfy (C_{cont}), (C_{lip}), and (C_{sign}) for all $T>0$. Then the unique solution u of (36) is nonnegative.*

8. Application: Discrete and Semidiscrete Nagumo Equation

In this section we apply the results of this paper to the most common nonlinearity occurring in the connection with the reaction-diffusion equation, the bistable/double-well nonlinearity. For simplicity, we consider only the symmetric case and use interval $[-1,1]$ so that our arguments for positive values can be directly reproduced for the negative ones; that is, we study

$$f(x,t,u)=\lambda u\left(1-u^2\right),\quad \lambda\in\mathbb{R}.\tag{77}$$

Throughout this section we assume that the initial-boundary conditions φ, ξ_a, ξ_b are such that $m_T=-1$ and $M_T=1$ (or possibly $m_T\ge-1$ and $M_T\le1$) for all $T>0$.

Starting with the semidiscrete case (36), we observe that f is continuous and locally Lipschitz continuous and satisfies $f(x,t,-1)=0=f(x,t,1)$. Consequently, for any given $\lambda\in\mathbb{R}$ we can apply Theorems 24 and 31 to get that there exists a unique solution of the semidiscrete problem (36) such that $u(x,t)\in[-1,1]$ for all $x\in(a,b)_{\mathbb{Z}}$ and $t\in\mathbb{R}_0^+$.

We encounter a more interesting situation if we consider the problem for the discrete Nagumo equation

$$\begin{aligned}\frac{u(x,t+h)-u(x,t)}{h}&=k\Delta_{xx}^2u(x-1,t)+\lambda u(x,t)\left(1-u^2(x,t)\right),\\ u(x,0)&=\varphi(x),\quad x\in(a,b)_{\mathbb{Z}},\\ u(a,t)&=\xi_a(t),\quad t\in h\mathbb{N}_0,\\ u(b,t)&=\xi_b(t),\quad t\in h\mathbb{N}_0,\end{aligned}\tag{78}$$

with $\lambda\in\mathbb{R}$, $x\in(a,b)_{\mathbb{Z}}$, $t\in h\mathbb{N}_0$, $h>0$, $k>0$.

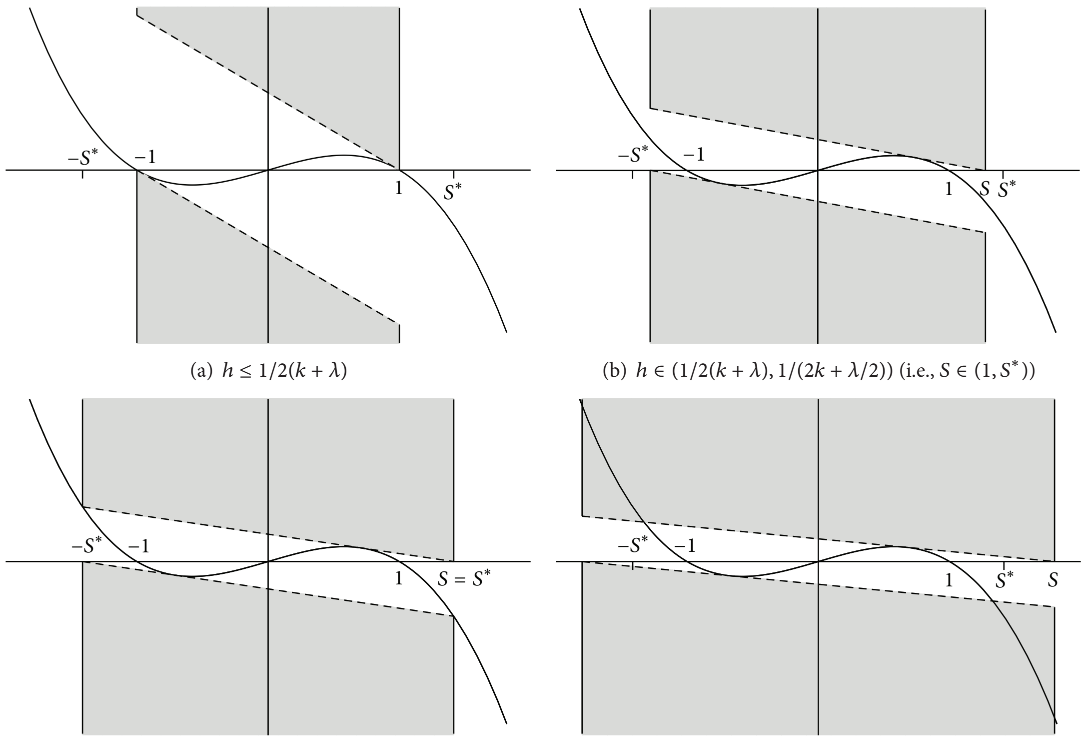
Let us assume first that $\lambda>0$. We observe that $f'(1)=-2\lambda$. Hence the application of Theorem 9 is restricted to cases for which the slope of the dashed line in the forbidden area (see Figure 1) given by $2k-1/h$ (see the assumption (D)) satisfies

$$2k-\frac{1}{h}\le-2\lambda,\quad\text{or equivalently}\quad h\le\frac{1}{2(k+\lambda)}.\tag{79}$$

Consequently, if $h\le1/2(k+\lambda)$, we can apply Theorem 9 to get that $u(x,t)\in[-1,1]$ for all $x\in(a,b)_{\mathbb{Z}}$, $t\in h\mathbb{N}_0$ (see Figure 3(a)).

Once $h>1/2(k+\lambda)$, Theorem 9 is no longer available and we proceed to Theorem 11. We split our argument into two steps:

(i) Since $f(u)$ is strictly concave on $[0,1]$ and attains a local maximum at $u=1/\sqrt{3}$, we can for each $h>0$



(a) $h \leq 1/2(k+\lambda)$

(b) $h \in (1/2(k+\lambda), 1/(2k+\lambda/2))$ (i.e., $S \in (1, S^*)$)

(c) $h = 1/(2k+\lambda/2)$ (i.e., $S = S^*$)

(d) $h > 1/(2k+\lambda/2)$ (i.e., $S > S^*$)

Figure 3: Validity of maximum principles for the discrete Nagumo equation (78) with $m_T = -1$ and $M_T = 1$. If h is small enough we can apply Theorem 9; see subplot (a). For $h \in (1/2(k+\lambda), 1/(2k+\lambda/2)]$ we can get weaker bounds (83) by the application of Theorem 11, subplots (b) and (c). If h is greater than $1/(2k+\lambda/2)$ our results cannot be applied; see subplot (d).

such that $2k - 1/h > -2\lambda$ (or equivalently $h > 1/2(k+\lambda)$) find a point $T \in (1/\sqrt{3}, 1)$ such that the slope of a tangent line of f at T is $2k - 1/h$. One could compute that

$$T = \sqrt{\frac{1}{3} - \frac{2kh-1}{3\lambda h}}. \quad (80)$$

(ii) Let us denote by S the x-intercept of the tangent line of f at T. Since the slope of this tangent line is $2k-1/h$, we can deduce that

$$S = \frac{-2T^3}{1-3T^2} = \frac{2\lambda h\sqrt{(1/3-(2kh-1)/3\lambda h)^3}}{1-2kh}. \quad (81)$$

Given the shape of $f(u)$ for $u < 0$ (decreasing convexly for $u < -1/\sqrt{3}$), Theorem 11 could be applied once $f(-S)$ lies below or on the tangent line (cf. Figures 2 and 3). We observe the following interesting fact: choosing $S = S^* := \sqrt{2}$ the tangent line of f at T and the function f intersect at the point $[-\sqrt{2}, \lambda\sqrt{2}]$ for each $\lambda > 0$ and $k > 0$ (see Figure 3(c)). Consequently, we can apply Theorem 11 whenever $S \leq \sqrt{2}$, which is equivalent to

$$h \leq \frac{1}{2k+\lambda/2}. \quad (82)$$

If we choose $S > S^*$, then we can easily observe that $f(-S)$ lies above the tangent line and therefore Theorem 11 cannot be applied (see Figure 3(d)).

Since we intentionally chose a symmetric f, we can repeat the same argument on the lower bound of solutions of (78).

If $\lambda = 0$, problem (78) reduces to the linear case and we can trivially apply (12) whenever $h \leq 1/2k$. Finally, if $\lambda < 0$, then the assumption (D) is satisfied as long as the line $(2k - 1/h)(u-1)$ does not intersect for $u < 0$ or is tangential to $f(u) = \lambda u(1-u^2)$. One can easily compute that the tangential case occurs if $2k - 1/h = \lambda/4$. Therefore, we can apply Theorem 9 whenever $h \leq 1/(2k-\lambda/4)$. If this condition is violated we cannot use Theorem 11 since $f(u) > 0$ for $u > 1$ and for each $S > 1$ the assumption (D') does not hold.

To sum up, depending on values of λ and h we obtain the following bounds for the solution of (78)

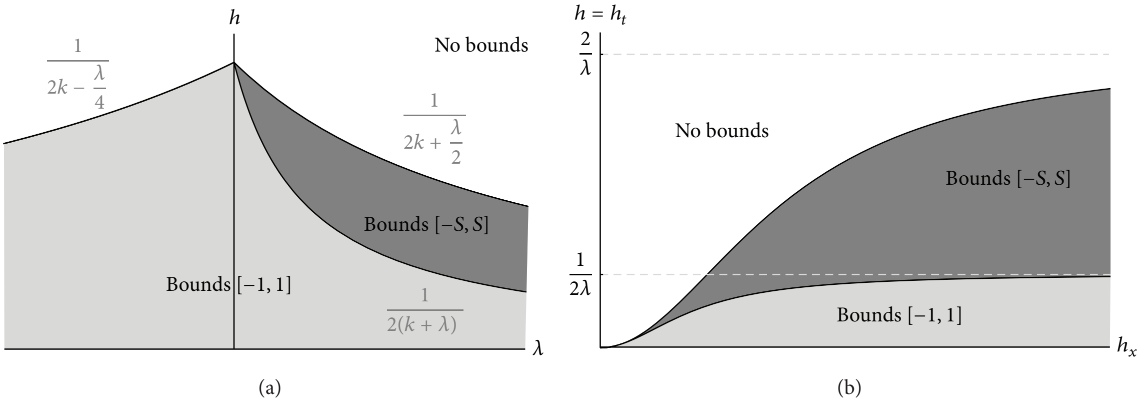


Figure 4: Bounds of solutions of the discrete Nagumo equation (78) with $m_T=-1$ and $M_T=1$ and their dependence on the values of λ and h (a) and on the values of space and time discretization steps h_x and $h=h_t$ for a fixed $\lambda>0$ (b). In the light gray area, the bounds follow from Theorem 9. In the dark gray area, the bounds are implied by Theorem 11 and S is given by (81). In the white area we have no bounds on solutions.

$$u(x,t) \in \begin{cases} [-1,1], & \text{if } \lambda\le 0,\ h\le \dfrac{1}{2k-\lambda/4}, \\ [-1,1], & \text{if } \lambda>0,\ h\le \dfrac{1}{2(k+\lambda)}, \\ \left[-\dfrac{2\lambda h\sqrt{(1/3-(2kh-1)/3\lambda h)^3}}{1-2kh}, \dfrac{2\lambda h\sqrt{(1/3-(2kh-1)/3\lambda h)^3}}{1-2kh}\right], & \text{if } \lambda>0,\ h\in\left(\dfrac{1}{2(k+\lambda)}, \dfrac{1}{2k+\lambda/2}\right]; \end{cases} \tag{83}$$

see Figure 4(a) for the illustrative summary of our results in this section.

Interestingly, if we considered general space discretization step $h_x>0$ in (78), that is,

$$\begin{aligned} \frac{u(x,t+h)-u(x,t)}{h_t} &= k\frac{u(x-h_x,t)-2u(x,t)+u(x+h_x,t)}{h_x^2} \\ &\quad +\lambda u(x,t)\left(1-u^2(x,t)\right), \end{aligned} \tag{84}$$

one could get the same bounds as in (83) by replacing k with k/h_x^2. The dependence of regions of maximum principles' validity on time and space discretization steps h_t and h_x for $\lambda>0$ is depicted in Figure 4(b) (notice that very small values of h_t are necessary for small h_x).

9. Final Remarks

In this paper, we studied a priori bounds for solutions of initial-boundary value problems related to discrete and semidiscrete diffusion. Our main motivation for the initial-boundary problems was the direct comparison with the classical results (Theorems 2 and 3). However, note that, in the discrete case, the results would be identical if we dealt with an initial problem on $\mathbb{Z}$. On the other hand, in the semidiscrete or classical case, even the solutions of linear diffusion equations are not necessarily bounded (see, e.g., [12]).

Similarly, the ideas of this paper could be easily extended to a general reaction-diffusion-type equation (possibly with nonconstant time steps $h=h(t)$)

$$\begin{aligned} \frac{u(x,t+h)-u(x,t)}{h} &= a(x,t)\,u(x-1,t) \\ &\quad +b(x,t)\,u(x,t) \\ &\quad +c(x,t)\,u(x+1,t) \\ &\quad +f(x,t,u(x,t)). \end{aligned} \tag{85}$$

For example, the weak maximum principle is then valid if $h\ge -1/b$ (replacing $h\le 1/2k$) and the following generalization of (D) holds (we assume that $b=-(a+c)<0$):

$$\begin{aligned} \frac{m_T(1-ah-ch)-(1+bh)\,u}{h} &\le f(x,t,u) \\ &\le \frac{M_T(1-ah-ch)-(1+bh)\,u}{h}. \end{aligned} \tag{86}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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APPENDIX D

Well-posedness and maximum principles for lattice reaction-diffusion equations

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Well-posedness and maximum principles for lattice reaction-diffusion equations

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Abstract

Existence, uniqueness and continuous dependence results together with maximum principles represent key tools in the analysis of lattice reaction-diffusion equations. In this paper we study these questions in full generality, by considering nonautonomous reaction functions, possibly nonsymmetric diffusion and continuous, discrete or mixed time. First, we prove the local existence and global uniqueness of bounded solutions, as well as continuous dependence of solutions on the underlying time structure and on initial conditions. Next, we obtain the weak maximum principle, which enables us to get global existence of solutions. Finally, we provide the strong maximum principle, which exhibits an interesting dependence on the time structure. Our results are illustrated by the autonomous Fisher and Nagumo lattice equations, and a nonautonomous logistic population model with a variable carrying capacity.

Keywords: reaction-diffusion equation; lattice equation; existence and uniqueness; continuous dependence; maximum principle; time scale

MSC 2010 subject classification: 34A33, 34A34, 34N05, 35A01, 35B50, 35F25, 39A14, 65M12

1 Introduction

The classical reaction-diffusion equation $\partial_t u = k\partial_{xx}u + f(u)$ is a nonlinear partial differential equation frequently used to describe the evolution of numerous natural quantities (chemical concentrations, temperatures, populations, etc.). These phenomena combine a local dynamics (via the reaction function f) and a spatial dynamics (via the diffusion). It is well known that solutions to reaction-diffusion systems can exhibit rich behavior, such as the existence of traveling waves or formation of spatial patterns [32].

Motivated by applications in biology, chemistry and kinematics [2, 10, 12, 19], various authors have considered the lattice reaction-diffusion equation [8, 9, 36, 37]

$$\partial_t u(x,t) = k(u(x+1,t) - 2u(x,t) + u(x-1,t)) + f(u(x,t)), \quad x \in \mathbb{Z}, \quad t \in [0,\infty), \tag{1.1}$$

as well as the discrete reaction-diffusion equation [6, 9, 18]

$$u(x,t+1) - u(x,t) = k(u(x+1,t) - 2u(x,t) + u(x-1,t)) + f(u(x,t)), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0. \tag{1.2}$$

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Naturally, equations (1.1) and (1.2) are also interesting from the standpoint of numerical mathematics, since they correspond to semi- or full discretization of the original reaction-diffusion equation [18].

The literature dealing with equations (1.1) and (1.2) studies mainly the dynamical properties such as the asymptotic behavior [5, 33, 34], existence of traveling wave solutions [6, 9, 10, 21, 35, 36, 37] and pattern formation [7, 8, 9], in particular for specific nonlinearities (e.g., the Fisher or Nagumo equation). A growing number of studies have dealt with those questions in nonautonomous cases [17, 24]. In this paper, we study (1.1)–(1.2) with a general time- and space-dependent nonlinearity f. Our focus lies on the existence, uniqueness, continuous dependence (both on the initial condition as well as on the underlying time structure/numerical discretization), and a priori bounds in the form of weak and strong maximum principles. Note that both continuous dependence and maximum principles are key assumptions in the proofs of existence of traveling waves [21, 35]. Our goal is to explore and describe them in full generality.

In order to consider both (1.1), (1.2) at once and motivated by convergence issues and continuous dependence of solutions on the time discretization, we use the language of the time scale calculus [4, 16]. We do not restrict ourselves to symmetric diffusion (see the following paragraph) and consider nonautonomous reaction-diffusion processes

$$u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{T}, \tag{1.3}$$

where $a, b, c \in \mathbb{R}$, $\mathbb{T} \subseteq \mathbb{R}$ is a time scale, and the symbol u^{Δ} denotes the delta derivative with respect to time. Our results are new even in the special cases $\mathbb{T} = \mathbb{R}$ (when u^{Δ} becomes the partial derivative $\partial_t u$) and $\mathbb{T} = \mathbb{Z}$ (when u^{Δ} is the partial difference $u(x,t+1) - u(x,t)$).

If $a = c$ and $b = -2a$ then (1.3) becomes the symmetric lattice reaction-diffusion equation. The asymmetric case $a \neq c$, $b = -(a+c)$ corresponds to the lattice reaction-advection-diffusion equation. Next, if $a = 0$ and $c = -b > 0$ then (1.3) reduces to the lattice reaction-transport equation. For more details and other special cases see [28, Section 1].

In Section 2, we formulate (1.3) as an abstract nonautonomous dynamic equation and prove the local existence of solutions. In comparison with the existing literature [5, 33, 34] we do not work in the Hilbert space $\ell^2(\mathbb{Z})$ or in the weighted spaces $\ell^2_\delta(\mathbb{Z})$ but in the Banach space $\ell^\infty(\mathbb{Z})$; as explained in [12], this is a much more natural choice. We also prove the uniqueness of bounded solutions. In Section 3, we use techniques from the Kurzweil-Stieltjes integration theory to show the continuous dependence of solutions on the time scale (time discretization). In the special case, this implies the convergence of solutions of (1.2) to the solution of (1.1) as the time discretization step tends to zero. Following the ideas from [31] (which deals with initial-boundary-value problems on finite subsets of $\mathbb{Z}$), we provide weak maximum and minimum principles in Section 4. These a priori bounds, as usual, depend strongly on the time structure. Combined with the local existence results they enable us to prove the global existence of bounded solutions to (1.3). We illustrate our findings on the autonomous logistic and bistable nonlinearities (Fisher and Nagumo equations) and a nonautonomous logistic population model with a variable carrying capacity. Finally, in Section 5, we conclude with the strong maximum principle. In the linear case $f \equiv 0$, the weak maximum principle was already proved in [28, Theorem 4.7], but the strong maximum principle is new even for linear equations.

2 Local existence and uniqueness of solutions

In this section, we study the local existence and global uniqueness of solutions to the initial-value problem

$$\begin{aligned} &u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \quad t \in [t_0,T]^{\kappa}_{\mathbb{T}}, \\ &u(x,t_0) = u^0_x, \quad x \in \mathbb{Z}, \end{aligned} \tag{2.1}$$

where $\{u_x^0\}_{x\in\mathbb{Z}}$ is a bounded real sequence, $a, b, c \in \mathbb{R}$, $\mathbb{T} \subseteq \mathbb{R}$ is a time scale and t_0, $T \in \mathbb{T}$. We use the notation $[\alpha, \beta]_{\mathbb{T}} = [\alpha, \beta] \cap \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$, and

$$[t_0, T]^{\kappa}_{\mathbb{T}} = \begin{cases} [t_0, T]_{\mathbb{T}} & \text{if } T \text{ is left-dense,} \\ [t_0, T)_{\mathbb{T}} & \text{if } T \text{ is left-scattered.} \end{cases}$$

We impose the following conditions on the function $f : \mathbb{R} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$:

(H_1) *f is bounded on each set $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, where $B \subset \mathbb{R}$ is bounded.*

(H_2) *f is Lipschitz-continuous in the first variable on each set $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, where $B \subset \mathbb{R}$ is bounded.*

(H_3) *For each bounded set $B \subset \mathbb{R}$ and each choice of $\varepsilon > 0$ and $t \in [t_0, T]_{\mathbb{T}}$, there exists a $\delta > 0$ such that if $s \in (t - \delta, t + \delta) \cap [t_0, T]_{\mathbb{T}}$, then $|f(u, x, t) - f(u, x, s)| < \varepsilon$ for all $u \in B$, $x \in \mathbb{Z}$.*

We begin with a local existence result. Given a function $U : \mathbb{T} \to \ell^\infty(\mathbb{Z})$, the symbol $U(t)_x$ denotes the x-th component of the sequence $U(t)$, and should not be confused with the derivative of U with respect to x (which never appears in this paper).

Theorem 2.1 (local existence)**.** *Assume that $f : \mathbb{R} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ satisfies (H_1)–(H_3). Then for each $u^0 \in \ell^\infty(\mathbb{Z})$, the initial-value problem* (2.1) *has a bounded local solution defined on $\mathbb{Z} \times [t_0, t_0 + \delta]_{\mathbb{T}}$, where $\delta > 0$ and $\delta \geq \mu(t_0)$. The solution is obtained by letting $u(x, t) = U(t)_x$, where $U : [t_0, t_0 + \delta]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ is a solution of the abstract dynamic equation*

$$U^{\Delta}(t) = \Phi(U(t), t), \quad U(t_0) = u^0, \tag{2.2}$$

with $\Phi : \ell^\infty(\mathbb{Z}) \times [t_0, T]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ being given by

$$\Phi(\{u_x\}_{x\in\mathbb{Z}}, t) = \{a u_{x+1} + b u_x + c u_{x-1} + f(u_x, x, t)\}_{x\in\mathbb{Z}}.$$

Proof. (H_1) guarantees that Φ indeed takes values in $\ell^\infty(\mathbb{Z})$. Choose an arbitrary $\rho > 0$, denote $\mathcal{B} = \{u \in \ell^\infty(\mathbb{Z}); \|u - u^0\|_\infty \leq \rho\}$, and $B = \left[\inf_{x\in\mathbb{Z}} u_x^0 - \rho, \sup_{x\in\mathbb{Z}} u_x^0 + \rho\right] \subset \mathbb{R}$. Note that if $u, v \in \mathcal{B}$, then $u_x, v_x \in B$ for all $x \in \mathbb{Z}$. If L is the Lipschitz constant for the function f on $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, we get

$$\|\Phi(u,t) - \Phi(v,t)\|_\infty \leq \|a\{u_{x+1} - v_{x+1}\}_{x\in\mathbb{Z}}\|_\infty + \|b\{u_x - v_x\}_{x\in\mathbb{Z}}\|_\infty + \|c\{u_{x-1} - v_{x-1}\}_{x\in\mathbb{Z}}\|_\infty$$
$$+ \|\{f(u_x, x, t) - f(v_x, x, t)\}_{x\in\mathbb{Z}}\|_\infty \leq (|a| + |b| + |c|)\|u - v\|_\infty + L\|u - v\|_\infty.$$

This means that Φ is Lipschitz-continuous in the first variable on $\mathcal{B} \times [t_0, T]_{\mathbb{T}}$.

Next, we observe that Φ is bounded on $\mathcal{B} \times [t_0, T]_{\mathbb{T}}$. Indeed, let M be the boundedness constant for the function $|f|$ on $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$. For each $u \in \mathcal{B}$, we have $u_x \in B$ for each $x \in \mathbb{Z}$, and consequently

$$\begin{aligned} \|\Phi(u,t)\|_\infty &\leq \|a\{u_{x+1}\}_{x\in\mathbb{Z}}\|_\infty + \|b\{u_x\}_{x\in\mathbb{Z}}\|_\infty + \|c\{u_{x-1}\}_{x\in\mathbb{Z}}\|_\infty + \|\{f(u_x, x, t)\}_{x\in\mathbb{Z}}\|_\infty \\ &\leq (|a| + |b| + |c|)\|u\|_\infty + M \leq (|a| + |b| + |c|)(\|u^0\|_\infty + \rho) + M. \end{aligned}$$

Finally, we claim that Φ is continuous on $\mathcal{B} \times [t_0, T]_{\mathbb{T}}$. To see this, consider an arbitrary $\varepsilon > 0$ and a fixed pair $(u, t) \in \mathcal{B} \times [t_0, T]_{\mathbb{T}}$. Let $\delta > 0$ be the corresponding number from (H_3). Then for all $(v, s) \in \mathcal{B} \times [t_0, T]_{\mathbb{T}}$ with $\|u - v\|_\infty < \varepsilon$ and $s \in (t - \delta, t + \delta) \cap [t_0, T]_{\mathbb{T}}$, we have

$$\begin{aligned} \|\Phi(u,t) - \Phi(v,s)\|_\infty &\leq \|\Phi(u,t) - \Phi(v,t)\|_\infty + \|\Phi(v,t) - \Phi(v,s)\|_\infty \\ &\leq (|a| + |b| + |c| + L)\|u - v\|_\infty + \|\{f(v_x, x, t) - f(v_x, x, s)\}_{x\in\mathbb{Z}}\|_\infty \\ &\leq (|a| + |b| + |c| + L + 1)\varepsilon, \end{aligned}$$

which proves that Φ is continuous at the point (u, t).

By [4, Theorem 8.16], the initial-value problem

$$U^{\Delta}(t) = \Phi(U(t), t), \quad U(t_0) = u^0,$$

has a local solution defined on $[t_0, t_0 + \delta]_{\mathbb{T}}$, where $\delta > 0$ and $\delta \geq \mu(t_0)$. Letting $u(x,t) = U(t)_x$, $x \in \mathbb{Z}$, we see that u is a solution of the initial-value problem (2.1). $\square$

Note that even in the linear case $f \equiv 0$ the solutions of (2.1) are not unique in general (see, e.g., [28, Section 3]) and the uniqueness can be expected only in the class of bounded solutions. In the next theorem, we tackle this issue for an initial-value problem which generalizes (2.1).

Theorem 2.2. *Assume that $\varphi : \ell^\infty(\mathbb{Z}) \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ satisfies the following conditions:*

1. *φ is bounded on each set $\mathcal{B} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, where $\mathcal{B} \subset \ell^\infty(\mathbb{Z})$ is bounded.*
2. *φ is Lipschitz-continuous in the first variable on each set $\mathcal{B} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, where $\mathcal{B} \subset \ell^\infty(\mathbb{Z})$ is bounded.*

Then for each $u^0 \in \ell^\infty(\mathbb{Z})$, the initial-value problem

$$u^{\Delta}(x,t) = \varphi(\{u(x,t)\}_{x\in\mathbb{Z}}, x, t), \quad u(x, t_0) = u^0_x, \quad x \in \mathbb{Z}, \quad t \in [t_0, T]^{\kappa}_{\mathbb{T}}, \tag{2.3}$$

has at most one bounded solution $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$.

Proof. Assume that u_1, u_2 are two bounded solutions that do not coincide on $\mathbb{Z} \times (t_0, T]_{\mathbb{T}}$; let

$$t = \inf\{\tau \in (t_0, T]_{\mathbb{T}};\ u_1(x,\tau) \neq u_2(x,\tau) \text{ for some } x \in \mathbb{Z}\}.$$

We claim that $u_1(x,t) = u_2(x,t)$ for every $x \in \mathbb{Z}$. If $t = t_0$, the statement is true. If $t > t_0$ and t is left-dense, then the statement follows from continuity of solutions with respect to the time variable. Finally, if $t > t_0$ and t is left-scattered, then $u_1(x, \rho(t)) = u_2(x, \rho(t))$, and the statement follows from the fact that $u_1^{\Delta}(x, \rho(t)) = u_2^{\Delta}(x, \rho(t))$.

If t is right-scattered, then $u_1(x,t) = u_2(x,t)$ and $u_1^{\Delta}(x,t) = u_2^{\Delta}(x,t)$ imply $u_1(x, \sigma(t)) = u_2(x, \sigma(t))$, a contradiction with the definition of t. Hence, t is right-dense. Since the functions $U_i(t) = \{u_i(x,t)\}_{x\in\mathbb{Z}}$, $i \in \{1,2\}$, $t \in [t_0, T]_{\mathbb{T}}$, are bounded, their values are contained in a bounded set $\mathcal{B} \subset \ell^\infty(\mathbb{Z})$. By the first assumption, there is a constant $M \geq 0$ such that $|\varphi| \leq M$ on $\mathcal{B} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$. We have

$$u_i(x,t) - u_i(x,s) = \int_s^t u_i^{\Delta}(x,\tau)\,\Delta\tau = \int_s^t \varphi(U_i(\tau), x, \tau)\,\Delta\tau, \quad i \in \{1,2\}, \quad t, s \geq t_0, \quad x \in \mathbb{Z}$$

(the last integral exists at least in the Henstock-Kurzweil's sense; see [23, Theorem 2.3]). It follows that

$$|u_i(x,t) - u_i(x,s)| \leq |t - s| M, \quad i \in \{1,2\}, \quad t, s \geq t_0, \quad x \in \mathbb{Z},$$

and therefore

$$\|U_i(t) - U_i(s)\|_\infty \leq |t - s| M, \quad i \in \{1,2\}, \quad t, s \geq t_0,$$

i.e., the functions U_1, U_2 are continuous on $[t_0, T]_{\mathbb{T}}$.

By the second assumption, φ is Lipschitz-continuous in the first variable on $\mathcal{B} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$; let L be the corresponding Lipschitz constant. Then

$$u_1(x,r) - u_2(x,r) = \int_t^r \varphi(U_1(\tau), x, \tau) - \varphi(U_2(\tau), x, \tau)\,\Delta\tau, \quad r \geq t,$$

$$\|U_1(r) - U_2(r)\|_\infty \leq \int_t^r L \|U_1(\tau) - U_2(\tau)\|_\infty\,\Delta\tau, \quad r \geq t$$

(the last integral exists since $U_1 - U_2$ is continuous). Consequently, for each $s \in [t, T]_{\mathbb{T}}$,

$$\sup_{\tau \in [t,s]} \|U_1(\tau) - U_2(\tau)\|_\infty \le (s-t)L \sup_{\tau \in [t,s]} \|U_1(\tau) - U_2(\tau)\|_\infty.$$

Since t is right-dense, there is a point $s \in [t, T]_{\mathbb{T}}$ with $s > t$ and $(s-t)L < 1$. Substituting this inequality into the previous estimate, we arrive to a contradiction. $\square$

Uniqueness of bounded solutions to the initial-value problem (2.1) is now a simple consequence of the previous theorem.

Theorem 2.3 (global uniqueness)**.** *Assume that $f : \mathbb{R} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ satisfies (H_1) and (H_2). Then for each $u^0 \in \ell^\infty(\mathbb{Z})$, the initial-value problem* (2.1) *has at most one bounded solution $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$.*

Proof. Note that (2.1) is a special case of (2.3) with the function $\varphi : \ell^\infty(\mathbb{Z}) \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ being given by

$$\varphi(\{u_x\}_{x \in \mathbb{Z}}, x, t) = a u_{x+1} + b u_x + c u_{x-1} + f(u_x, x, t).$$

Hence, it is enough to verify that the two conditions in Theorem 2.2 are satisfied.

Given an arbitrary bounded set $\mathcal{B} \subset \ell^\infty(\mathbb{Z})$, there exists a bounded set $B \subset \mathbb{R}$ such that $u \in \mathcal{B}$ implies $u_x \in B$, $x \in \mathbb{Z}$. Hence, the first condition in Theorem 2.2 is an immediate consequence of (H_1). To verify the second condition, let L be the Lipschitz constant for the function f on $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$. Then, for each pair of sequences u, $v \in \mathcal{B} \subset \ell^\infty(\mathbb{Z})$, we have

$$|\varphi(u,x,t) - \varphi(v,x,t)| \le (|a| + |b| + |c|) \cdot \|u - v\|_\infty + |f(u_x,x,t) - f(v_x,x,t)| \le (|a| + |b| + |c| + L) \cdot \|u - v\|_\infty,$$

which means that φ is Lipschitz-continuous in the first variable on $\mathcal{B} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$. $\square$

3 Continuous dependence results

This section is devoted to the study of continuous dependence of solutions to abstract dynamic equations with respect to the choice of the time scale. The results are also applicable to (2.1), whose solutions (as we know from Theorem 2.1) are obtained from solutions to a certain abstract dynamic equation.

We begin by proving a continuous dependence theorem for the so-called measure differential equations, i.e., integral equations with the Kurzweil-Stieltjes integral (also known as the Perron-Stieltjes integral) on the right-hand side. For readers who are not familiar with this concept, it is sufficient to know that the integral has the usual properties of linearity and additivity with respect to adjacent subintervals. The main advantage with respect to the Riemann-Stieltjes integral is that the class of Kurzweil-Stieltjes integrable functions is much larger. For example, if $g : [a, b] \to \mathbb{R}$ has bounded variation, then the integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists for each regulated function $f : [a, b] \to X$, where X is a Banach space (see [26, Proposition 15]).

The statement as well as the proof of the next theorem are closely related to Theorem 5.1 in [3]; for more details, see Remark 3.3.

Theorem 3.1. *Let X be a Banach space, $\mathcal{B} \subseteq X$. Consider a sequence of nondecreasing left-continuous functions $g_n : [t_0, T] \to \mathbb{R}$, $n \in \mathbb{N}_0$, such that $g_n \rightrightarrows g_0$ on $[t_0, T]$. Assume that $\Phi : \mathcal{B} \times [t_0, T] \to X$ is Lipschitz-continuous in the first variable. Let $x_n : [t_0, T] \to \mathcal{B}$, $n \in \mathbb{N}_0$, be a sequence of functions satisfying*

$$x_n(t) = x_n(t_0) + \int_{t_0}^t \Phi(x_n(s), s)\,\mathrm{d}g_n(s), \quad t \in [t_0, T], \quad n \in \mathbb{N}_0,$$

and $x_n(t_0) \to x_0(t_0)$. Suppose finally that the function $s \mapsto \Phi(x_0(s), s)$, $s \in [t_0, T]$, is regulated. Then $x_n \rightrightarrows x_0$ on $[t_0, T]$.

Proof. Since $g_n(t_0) \to g_0(t_0)$ and $g_n(T) \to g_0(T)$, the sequences $\{g_n(t_0)\}_{n=1}^{\infty}$ and $\{g_n(T)\}_{n=1}^{\infty}$ are necessarily bounded. Hence, there exists a constant $M \geq 0$ such that

$$\operatorname{var}_{t\in[t_0,T]} g_n(t) = g_n(T) - g_n(t_0) \leq M, \quad n \in \mathbb{N}.$$

The Kurzweil-Stieltjes integral $\int_{t_0}^{T} \Phi(x_0(s), s)\,\mathrm{d}(g_n - g_0)(s)$ exists, because $s \mapsto \Phi(x_0(s), s)$ is regulated and $g_n - g_0$ has bounded variation. Since $g_n - g_0 \rightrightarrows 0$, it follows from [22, Theorem 2.2] that

$$\lim_{n\to\infty} \int_{t_0}^{t} \Phi(x_0(s), s)\,\mathrm{d}(g_n - g_0)(s) = 0,$$

uniformly with respect to $t \in [t_0, T]$. Thus, for an arbitrary $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\left\| \int_{t_0}^{t} \Phi(x_0(s), s)\,\mathrm{d}(g_n - g_0)(s) \right\| \leq \varepsilon, \quad n \geq n_0, \quad t \in [t_0, T].$$

Moreover, n_0 can be chosen in such a way that $\|x_n(t_0) - x_0(t_0)\| \leq \varepsilon$ for each $n \geq n_0$.

Consequently, the following inequalities hold for each $n \geq n_0$ and $t \in [t_0, T]$:

$$\|x_n(t) - x_0(t)\| \leq \|x_n(t_0) - x_0(t_0)\| + \left\| \int_{t_0}^{t} \Phi(x_n(s), s)\,\mathrm{d}g_n(s) - \int_{t_0}^{t} \Phi(x_0(s), s)\,\mathrm{d}g_0(s) \right\|$$

$$\leq \varepsilon + \left\| \int_{t_0}^{t} (\Phi(x_n(s), s) - \Phi(x_0(s), s))\,\mathrm{d}g_n(s) \right\| + \left\| \int_{t_0}^{t} \Phi(x_0(s), s)\,\mathrm{d}(g_n - g_0)(s) \right\|$$

$$\leq 2\varepsilon + \int_{t_0}^{t} \|\Phi(x_n(s), s) - \Phi(x_0(s), s)\|\,\mathrm{d}g_n(s) \leq 2\varepsilon + L \int_{t_0}^{t} \|x_n(s) - x_0(s)\|\,\mathrm{d}g_n(s),$$

where L is the Lipschitz constant for the function Φ. Using Grönwall's inequality for the Kurzweil-Stieltjes integral (see, e.g., [25, Corollary 1.43]), we get

$$\|x_n(t) - x_0(t)\| \leq 2\varepsilon e^{L(g_n(t) - g_n(t_0))} \leq 2\varepsilon e^{LM}, \quad n \geq n_0, \quad t \in [t_0, T],$$

which completes the proof. □

We now use the relation between measure differential equations and dynamic equations to obtain a continuous dependence theorem for the latter type of equations. Since we need to compare solutions defined on different time scales (whose intersection might be empty), we introduce the following definitions.

Consider an interval $[t_0, T] \subset \mathbb{R}$ and a time scale $\mathbb{T}$ with $t_0 \in \mathbb{T}$, $\sup \mathbb{T} \geq T$. Let $g_{\mathbb{T}} : [t_0, T] \to \mathbb{R}$ be given by

$$g_{\mathbb{T}}(t) = \inf\{s \in [t_0, T]_{\mathbb{T}};\ s \geq t\}, \quad t \in [t_0, T]. \tag{3.1}$$

Each function $x : [t_0, T]_{\mathbb{T}} \to X$ can be extended to a function $x^* : [t_0, T] \to X$ by letting

$$x^*(t) = x(g_{\mathbb{T}}(t)), \quad t \in [t_0, T]. \tag{3.2}$$

Note that x^* coincides with x on $[t_0, T]_{\mathbb{T}}$, and is constant on each interval $(u, v]$ where $(u, v) \cap \mathbb{T} = \emptyset$. We will refer to x^* as the piecewise constant extension of x, see Figure 1.

We are now ready to prove a theorem dealing with continuous dependence of solutions to abstract dynamic equations with respect to the choice of the time scale and initial condition.

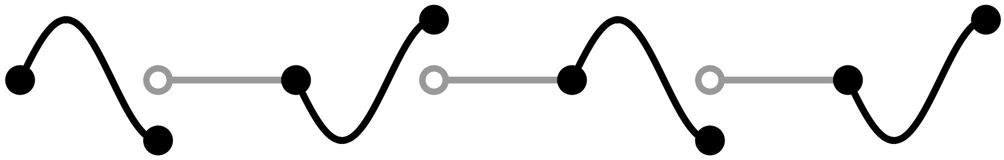

Figure 1: Piecewise constant extension x^* (gray) of a function x (black); see (3.2).

Theorem 3.2 (continuous dependence). *Let X be a Banach space, $\mathcal{B} \subseteq X$. Consider an interval $[t_0, T] \subset \mathbb{R}$ and a sequence of time scales $\{\mathbb{T}_n\}_{n=0}^{\infty}$ such that $t_0 \in \mathbb{T}_n$ and $\sup \mathbb{T}_n \geq T$ for each $n \in \mathbb{N}_0$, $T \in \mathbb{T}_0$, and $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$. Denote $\mathbb{T} = \overline{\bigcup_{n=0}^{\infty} \mathbb{T}_n}$. Suppose that $\Phi : \mathcal{B} \times [t_0, T]_{\mathbb{T}} \to X$ is continuous on its domain and Lipschitz-continuous with respect to the first variable. Let $x_n : [t_0, T]_{\mathbb{T}_n} \to \mathcal{B}$, $n \in \mathbb{N}_0$, be a sequence of functions satisfying*

$$x_n^{\Delta}(t) = \Phi(x_n(t), t), \quad t \in [t_0, T]_{\mathbb{T}_n}^{\kappa}, \quad n \in \mathbb{N}_0,$$

and $x_n(t_0) \to x_0(t_0)$. Then the sequence of piecewise constant extensions $\{x_n^*\}_{n=1}^{\infty}$ is uniformly convergent to the piecewise constant extension x_0^* on $[t_0, T]$. In particular, for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\|x_n(t) - x_0(t)\| < \varepsilon$ for all $n \geq n_0$, $t \in [t_0, T]_{\mathbb{T}_n} \cap [t_0, T]_{\mathbb{T}_0}$.

Proof. According to the assumptions, we have

$$x_n(t) = x_n(t_0) + \int_{t_0}^{t} \Phi(x_n(s), s)\, \Delta s, \quad t \in [t_0, T]_{\mathbb{T}_n}, \quad n \in \mathbb{N}_0.$$

For each $n \in \mathbb{N}_0$, let $x_n^* : [t_0, T] \to X$ be the piecewise constant extension of x_n. Using the relation between Δ-integrals and Kurzweil-Stieltjes integrals (see [27, Theorem 5] or [11, Theorem 4.5]), we conclude that x_n^* satisfy

$$x_n^*(t) = x_n^*(t_0) + \int_{t_0}^{t} \Phi(x_n^*(s), g_{\mathbb{T}_n}(s))\, \mathrm{d}g_{\mathbb{T}_n}(s), \quad t \in [t_0, T], \quad n \in \mathbb{N}_0. \tag{3.3}$$

Let $\Phi^* : \mathcal{B} \times [t_0, T] \to X$ be given by

$$\Phi^*(x, t) = \Phi(x, g_{\mathbb{T}}(t)), \quad x \in \mathcal{B}, \quad t \in [t_0, T].$$

Note that for each $s \in [t_0, T]_{\mathbb{T}_n}$, we have $\Phi(x_n^*(s), g_{\mathbb{T}_n}(s)) = \Phi(x_n^*(s), s) = \Phi(x_n^*(s), g_{\mathbb{T}}(s)) = \Phi^*(x_n^*(s), s)$. Thus, by [11, Theorem 5.1], the integral equation (3.3) is equivalent to

$$x_n^*(t) = x_n^*(t_0) + \int_{t_0}^{t} \Phi^*(x_n^*(s), s)\, \mathrm{d}g_{\mathbb{T}_n}(s), \quad t \in [t_0, T], \quad n \in \mathbb{N}_0.$$

Because x_0 is continuous on $[t_0, T]_{\mathbb{T}_0}$, its piecewise constant extension x_0^* is regulated on $[t_0, T]$ (see [27, Lemma 4]). Moreover, its one-sided limits at each point of $[t_0, T]$ are elements of $\mathcal{B}$ (note that $x_0^*([t_0, T]) = x_0([t_0, T]_{\mathbb{T}_0})$ is compact, because x_0 is continuous and $[t_0, T]_{\mathbb{T}_0}$ is compact). The function $g_{\mathbb{T}}$ is the piecewise constant extension of the identity function from $[t_0, T]_{\mathbb{T}}$ to $[t_0, T]$; hence (again by [27, Lemma 4]), $g_{\mathbb{T}}$ is regulated on $[t_0, T]$. Consequently, the function $s \mapsto (x_0^*(s), g_{\mathbb{T}}(s))$ is also regulated on $[t_0, T]$, and its one-sided limits have values in $\mathcal{B} \times [t_0, T]_{\mathbb{T}}$. Continuity of Φ on $\mathcal{B} \times [t_0, T]_{\mathbb{T}}$ implies that $s \mapsto \Phi(x_0^*(s), g_{\mathbb{T}}(s)) = \Phi^*(x_0^*(s), s)$ is regulated on $[t_0, T]$. According to Theorem 3.1, we have $x_n^* \rightrightarrows x_0^*$ on $[t_0, T]$. $\square$

Remark 3.3. The problem of continuous dependence of solutions to dynamic equations with respect to the choice of time scale has been studied by several authors; see, e.g., [1, 3, 13, 14, 15, 20]. Our approach is close to the one taken in [3] or [13]; it relies on the continuous dependence result for measure differential equations from Theorem 3.1, which is similar in spirit to Theorem 5.1 in [3]. In this context, it seems appropriate to include a few remarks:

- Although the statement of Theorem 5.1 in [3] is essentially correct, the proof provided there is based on an erroneous estimate of the form $\|\int_{t_0}^t f_n \, \mathrm{d}g_n - \int_{t_0}^t f_n \, \mathrm{d}g_0\| \leq \int_{t_0}^T M \, \mathrm{d}(g_n - g_0)$, where f_n, f_0 are certain functions whose norm is bounded by M, and g_n, g_0 are nondecreasing.
- The assumption that the Hausdorff distance between $\mathbb{T}_n$ and $\mathbb{T}_0$ tends to zero is never used in the proof of Theorem 5.1 in [3], and can be omitted. On the other hand, the assumption that the above-mentioned integral $\int_{t_0}^T f_n \, \mathrm{d}g_0$ exists is missing.
- Theorem 5.1 in [3] deals with measure *functional* differential equations; our Theorem 3.1 and its proof can be easily adapted to this type of equations.

The next result shows that each time scale can be approximated by a sequence of discrete time scales in such a way that the assumptions of Theorem 3.2 are satisfied. We introduce the following notation:

$$\overline{\mu}_{\mathbb{T}} = \max_{t \in [t_0, T)_{\mathbb{T}}} \mu(t).$$

Theorem 3.4. *If $\mathbb{T}_0 \subset \mathbb{R}$ is a time scale with $t_0, T \in \mathbb{T}_0$, there exists a sequence of discrete time scales $\{\mathbb{T}_n\}_{n=1}^\infty$ with $\mathbb{T}_n \subseteq \mathbb{T}_0$, $\min \mathbb{T}_n = t_0$, $\max \mathbb{T}_n = T$, and such that $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$.*

Moreover, if $\overline{\mu}_{\mathbb{T}_0} = 0$, then $\lim_{n \to \infty} \overline{\mu}_{\mathbb{T}_n} = 0$; otherwise, if $\overline{\mu}_{\mathbb{T}_0} > 0$, then the sequence $\{\mathbb{T}_n\}_{n=1}^\infty$ can be chosen so that $\overline{\mu}_{\mathbb{T}_n} = \overline{\mu}_{\mathbb{T}_0}$ for all $n \in \mathbb{N}$.

Proof. We start by proving that for each $\varepsilon > 0$, there exists a left-continuous nondecreasing step function $g_\varepsilon : [t_0, T] \to \mathbb{R}$ such that $g_\varepsilon(t_0) = t_0$, $g_\varepsilon(T) = T$, and $\|g_\varepsilon - g_{\mathbb{T}_0}\|_\infty \leq \varepsilon$.

Given an $\varepsilon > 0$, let $t_0 = x_0 < x_1 < \cdots < x_m = T$ be a partition of $[t_0, T]$ such that $x_i - x_{i-1} \leq \varepsilon$, $i \in \{1, \ldots, m\}$. We begin the construction of the step function $g_\varepsilon : [t_0, T] \to \mathbb{R}$ by letting $g_\varepsilon(T) = T$. Then we proceed by induction in the backward direction and define g_ε on $[x_{m-1}, x_m)$, …, $[x_0, x_1)$. At the same time, we are going to check that $\|g_{\mathbb{T}_0} - g_\varepsilon\|_\infty \leq \varepsilon$ on these subintervals, and also ensure that $g_\varepsilon(x_i) = x_i$ whenever $x_i \in \mathbb{T}_0$; this will guarantee that $g_\varepsilon(t_0) = t_0$.

Assume that g_ε is already defined at x_i, and we want to extend it to $[x_{i-1}, x_i)$. We distinguish between two possibilites:

- If $\mathbb{T}_0 \cap [x_{i-1}, x_i) = \emptyset$, then, by the definition of $g_{\mathbb{T}_0}$, we have $g_{\mathbb{T}_0}(t) = g_{\mathbb{T}_0}(x_i)$ for each $t \in [x_{i-1}, x_i)$. Let $g_\varepsilon(t) = g_\varepsilon(x_i)$, $t \in [x_{i-1}, x_i)$. Then $|g_\varepsilon(t) - g_{\mathbb{T}_0}(t)| = |g_\varepsilon(x_i) - g_{\mathbb{T}_0}(x_i)| \leq \varepsilon$, where the last inequality follows from the induction hypothesis.
- If $\mathbb{T}_0 \cap [x_{i-1}, x_i)$ is nonempty, let t_i be its supremum. Define

  $$g_\varepsilon(x_{i-1}) = \begin{cases} x_{i-1}, & \text{if } x_{i-1} \in \mathbb{T}_0, \\ t_i, & \text{if } x_{i-1} \notin \mathbb{T}_0, \end{cases} \qquad g_\varepsilon(t) = \begin{cases} t_i, & t \in (x_{i-1}, t_i], \\ g_\varepsilon(x_i), & t \in (t_i, x_i). \end{cases}$$

  Note that t_i might coincide with x_i; in this case, we necessarily have $x_i \in \mathbb{T}_0$, and therefore, by the induction hypothesis, $g_\varepsilon(x_i) = x_i$; this guarantees that g_ε is left-continuous at x_i.

  For each $t \in [x_{i-1}, t_i]$, we have $x_{i-1} \leq t \leq g_{\mathbb{T}_0}(t) \leq t_i$. Therefore, $0 \leq t_i - g_{\mathbb{T}_0}(t_i) \leq t_i - x_{i-1} \leq \varepsilon$, which in turn means that $|g_\varepsilon(t) - g_{\mathbb{T}_0}(t)| \leq \varepsilon$. For each $t \in (t_i, x_i)$, it follows from the definition of $g_{\mathbb{T}_0}$ that $g_{\mathbb{T}_0}(t) = g_{\mathbb{T}_0}(x_i)$, and therefore $|g_\varepsilon(t) - g_{\mathbb{T}_0}(t)| = |g_\varepsilon(x_i) - g_{\mathbb{T}_0}(x_i)| \leq \varepsilon$.

Observe that the function g_ε constructed in this way has the property that $g_\varepsilon(t) \geq t$, and that $g_\varepsilon(t) = t$ implies $t \in \mathbb{T}_0$.

By choosing $\varepsilon = 1/n$, $n \in \mathbb{N}$, we get a sequence of left-continuous nondecreasing step functions $\{g_{1/n}\}_{n=1}^\infty$ such that $g_{1/n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$. For each $n \in \mathbb{N}$, consider the set

$$\mathbb{T}_n = \{t \in [t_0, T];\ g_{1/n}(t) = t\}.$$

Clearly, t_0 and T are elements of $\mathbb{T}_n$, and $\mathbb{T}_n \subseteq \mathbb{T}_0$. Moreover, $\mathbb{T}_n$ is finite, since $g_{1/n}$ is a step function and therefore its graph has only finitely many intersections with the graph of the identity function. Thus, $\mathbb{T}_n$ is a discrete time scale. It follows from the definition of $\mathbb{T}_n$ that $g_{\mathbb{T}_n} = g_{1/n}$, and therefore $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$.

To prove the final part of the theorem, we distinguish between two cases:

- Assume that $\overline{\mu}_{\mathbb{T}_0} > 0$. Let $y_0 = t_0$, and construct a sequence of points $y_1 < \cdots < y_k = T$ using the recursive formula
$$y_i = \sup\,(y_{i-1}, y_{i-1} + \overline{\mu}_{\mathbb{T}_0}] \cap [t_0, T]_{\mathbb{T}_0}.$$
Since the graininess of $\mathbb{T}_0$ never exceeds $\overline{\mu}_{\mathbb{T}_0}$, the set whose supremum is being considered is never empty. Also, note that $y_{i+1} - y_{i-1} \geq \overline{\mu}_{\mathbb{T}_0}$ (otherwise, the point y_{i+1} would have been chosen directly after y_{i-1}). Thus, the recursive procedure always terminates by reaching the point $t_k = T$ for some $k \in \mathbb{N}$.

  In the construction of the function g_ε described in the beginning of this proof, we can always assume that the points $y_0, \dots, y_k$ are among $x_0, \dots, x_m$. The construction then guarantees that $g_\varepsilon(y_i) = y_i$ for each $i \in \{0, \dots, k\}$. Consequently, the points $y_0, \dots, y_k$ are contained in all of the time scales $\mathbb{T}_n$, $n \in \mathbb{N}$, and
$$\overline{\mu}_{\mathbb{T}_n} \leq \max_{1 \leq i \leq k} (y_i - y_{i-1}) \leq \overline{\mu}_{\mathbb{T}_0}.$$
On the other hand, since $\mathbb{T}_n \subseteq \mathbb{T}_0$, we have $\overline{\mu}_{\mathbb{T}_0} \leq \overline{\mu}_{\mathbb{T}_n}$, which in turn means that $\overline{\mu}_{\mathbb{T}_n} = \overline{\mu}_{\mathbb{T}_0}$.

- Assume that $\overline{\mu}_{\mathbb{T}_0} = 0$. If μ is the graininess function of an arbitrary time scale $\mathbb{T}$ with $\min \mathbb{T} = t_0$ and $\sup \mathbb{T} \geq T$, observe that $g_{\mathbb{T}}(t+) - g_{\mathbb{T}}(t) = \mu(t)$ if $t \in [t_0, T)_{\mathbb{T}}$, and $g_{\mathbb{T}}(t+) - g_{\mathbb{T}}(t) = 0$ if $t \in [t_0, T) \setminus \mathbb{T}$. Hence, we have
$$\overline{\mu}_{\mathbb{T}} = \sup_{t \in [t_0, T)_{\mathbb{T}}} \mu(t) = \sup_{t \in [t_0, T)} \left(g_{\mathbb{T}}(t+) - g_{\mathbb{T}}(t)\right).$$
Since $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$, the Moore-Osgood theorem implies that $g_{\mathbb{T}_n}(t+) - g_{\mathbb{T}_n}(t) \rightrightarrows g_{\mathbb{T}_0}(t+) - g_{\mathbb{T}_0}(t)$ on $[t_0, T)$, and therefore
$$\lim_{n \to \infty} \overline{\mu}_{\mathbb{T}_n} = \lim_{n \to \infty} \left(\sup_{t \in [t_0, T)} \left(g_{\mathbb{T}_n}(t+) - g_{\mathbb{T}_n}(t)\right) \right) = \sup_{t \in [t_0, T)} \left(g_{\mathbb{T}_0}(t+) - g_{\mathbb{T}_0}(t)\right) = \overline{\mu}_{\mathbb{T}_0} = 0. \qquad \square$$

4 Weak maximum principle and global existence

A natural task in the analysis of diffusion-type equations is to establish the maximum principles. Given an initial condition $u^0 \in \ell^\infty(\mathbb{Z})$, let

$$m = \inf_{x \in \mathbb{Z}} u_x^0, \quad M = \sup_{x \in \mathbb{Z}} u_x^0.$$

We introduce the following conditions, which will be useful for our purposes:

(H_4) *a, b, $c \in \mathbb{R}$ are such that $a, c \geq 0$, $b < 0$, and $a + b + c = 0$.*

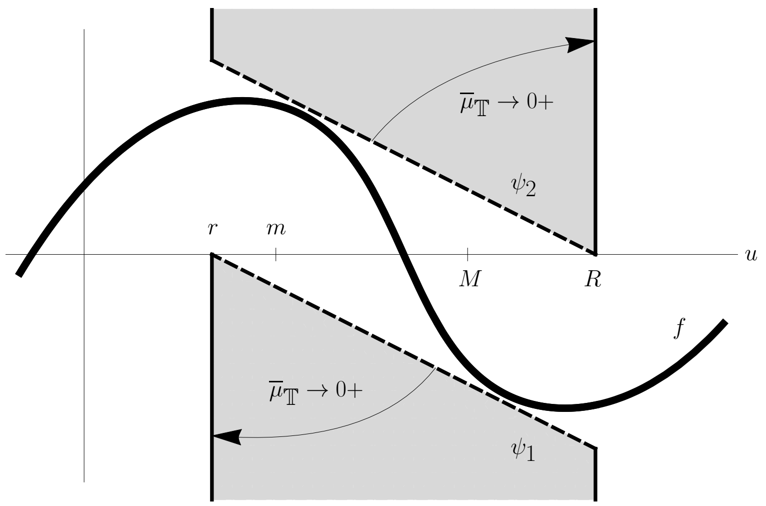

Figure 2: Illustration of (H_6). The values r, R are chosen so that the function $f(\cdot, x, t)$ does not intersect the gray forbidden areas. The slope of the boundary dashed lines is determined by the values of $\overline{\mu}_{\mathbb{T}}$.

(H_5) *$b < 0$ and $\overline{\mu}_{\mathbb{T}} \leq -1/b$.*

(H_6) *There exist $r, R \in \mathbb{R}$ such that $r \leq m \leq M \leq R$, and one of the following statements holds:*

- *$\overline{\mu}_{\mathbb{T}} = 0$ and $f(R, x, t) \leq 0 \leq f(r, x, t)$ for all $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.*
- *$\overline{\mu}_{\mathbb{T}} > 0$ and $\dfrac{1 + \overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(r - u) \leq f(u, x, t) \leq \dfrac{1 + \overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(R - u)$ for all $u \in [r, R]$, $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.*

Remark 4.1. Let us notice that:

- If (H_4)–(H_5) are not satisfied, then the maximum principle does not hold even in the linear case with $f \equiv 0$; see [28, Section 4].
- (H_6) defines forbidden areas that the function $f(\cdot, x, t)$ cannot intersect for any $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$, similarly as in [31] (see Figure 2).
- If (H_5) holds, there exists a function f satisfying (H_6); indeed, the linear functions $\psi_1(u) = \frac{1+\overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(r - u)$ and $\psi_2(u) = \frac{1+\overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(R - u)$ have identical nonpositive slopes, and the constant term of ψ_1 is less than or equal to the constant term of ψ_2. If $\overline{\mu}_{\mathbb{T}} = -1/b$ or $r = R$, then (H_6) is equivalent to $f(u, x, t) = 0$ for all $u \in [r, R]$, $x \in \mathbb{Z}$ and $t \in [t_0, T]_{\mathbb{T}}$. Finally, if $\overline{\mu}_{\mathbb{T}} > -1/b$ and $r < R$, there does not exist any function satisfying (H_6).

If (H_6) holds in the continuous case $\overline{\mu}_{\mathbb{T}} = 0$, the following lemma shows that (H_6) is also satisfied for all sufficiently fine time scales (specifically, for almost all of the discrete approximating time scales $\mathbb{T}_n$ from Theorem 3.4).

Lemma 4.2. *Assume that $\overline{\mu}_{\mathbb{T}} = 0$ and (H_2), (H_6) hold. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the following inequalities hold:*

$$\frac{1 + \varepsilon b}{\varepsilon}(r - u) \leq f(u, x, t) \leq \frac{1 + \varepsilon b}{\varepsilon}(R - u) \quad \textit{for all} \quad u \in [r, R], \quad x \in \mathbb{Z}, \quad t \in [t_0, T]. \tag{4.1}$$

Proof. Let $L \geq 0$ be the Lipschitz constant for the function f on the set $[r, R] \times \mathbb{Z} \times [t_0, T]$. Then for all $u \in [r, R]$, $x \in \mathbb{Z}$, and $t \in [t_0, T]$, we obtain

$$\begin{aligned} f(u,x,t) &\leq f(u,x,t) - f(R,x,t) \leq |f(u,x,t) - f(R,x,t)| \leq L|u - R| = L(R - u), \\ f(u,x,t) &\geq f(u,x,t) - f(r,x,t) \geq -|f(u,x,t) - f(r,x,t)| \geq -L|u - r| = L(r - u). \end{aligned}$$

Since $L(r-u) \leq f(u,x,t) \leq L(R-u)$, the two inequalities in (4.1) will be satisfied if $1/\varepsilon + b \geq L$, i.e., for all $\varepsilon \in (0, 1/(L-b)]$. □

The following lemma represents a weak maximum principle for time scales containing no right-dense points; it will be a key tool in the proof of the general weak maximum principle.

Lemma 4.3. *Assume that $[t_0, T)_{\mathbb{T}}$ does not contain any right-dense points, (H_4)–(H_6) hold, and $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ is a solution of (2.1) with $u^0 \in \ell^\infty(\mathbb{Z})$. Then*

$$r \leq u(x,t) \leq R \quad \textit{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}}. \tag{4.2}$$

Proof. We show the statement via the induction principle [4, Theorem 1.7] in the variable t. For a fixed $t \in [t_0, T]_{\mathbb{T}}$, we have to distinguish among three cases:

- For $t = t_0$, we obtain from the definition of m and M and from (H_6) that

$$r \leq m \leq u(x, t_0) \leq M \leq R \quad \text{for all} \quad x \in \mathbb{Z}.$$

- Let $t \in (t_0, T]_{\mathbb{T}}$ be left-dense and assume that $r \leq u(x,s) \leq R$ for all $s \in [t_0, t)_{\mathbb{T}}$ and $x \in \mathbb{Z}$. Then the continuity of the function $u(x, \cdot)$ on $[t_0, T]_{\mathbb{T}}$ implies

$$r \leq u(x,t) = \lim_{s \to t-} u(x,s) \leq R \quad \text{for all} \quad x \in \mathbb{Z}.$$

- Let $t \in [t_0, T)_{\mathbb{T}}$ be right-scattered, i.e., necessarily $\overline{\mu}_{\mathbb{T}} > 0$, and

$$r \leq u(x,t) \leq R \quad \text{for all} \quad x \in \mathbb{Z}. \tag{4.3}$$

  We have to show that

$$r \leq u(x, t + \mu_{\mathbb{T}}(t)) \leq R \quad \text{for all} \quad x \in \mathbb{Z}. \tag{4.4}$$

  Notice that from (H_5) and from the fact that $\overline{\mu}_{\mathbb{T}} \geq \mu_{\mathbb{T}}(t) > 0$ we get

$$0 \leq \frac{1 + \overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}} = \frac{1}{\overline{\mu}_{\mathbb{T}}} + b \leq \frac{1}{\mu_{\mathbb{T}}(t)} + b = \frac{1 + \mu_{\mathbb{T}}(t) b}{\mu_{\mathbb{T}}(t)}.$$

  Consequently, (H_6) yields

$$\frac{1 + \mu_{\mathbb{T}}(t) b}{\mu_{\mathbb{T}}(t)}(r - u) \leq f(u,x,t) \leq \frac{1 + \mu_{\mathbb{T}}(t) b}{\mu_{\mathbb{T}}(t)}(R - u) \quad \text{for all} \quad u \in [r, R], \ \ x \in \mathbb{Z}, \ \ t \in [t_0, T]_{\mathbb{T}}. \tag{4.5}$$

  Let us prove the latter inequality in (4.4). Using the equation in (2.1), we obtain the estimate

$$\begin{aligned} u(x, t + \mu_{\mathbb{T}}(t)) &= \mu_{\mathbb{T}}(t) a u(x+1, t) + (1 + \mu_{\mathbb{T}}(t) b)\, u(x,t) + \mu_{\mathbb{T}}(t) c u(x-1, t) \\ &\quad + \mu_{\mathbb{T}}(t) f(u(x,t), x, t) \\ &\overset{(H_4),(4.3)}{\leq} \mu_{\mathbb{T}}(t)(a + c) R + (1 + \mu_{\mathbb{T}}(t) b)\, u(x,t) + \mu_{\mathbb{T}}(t) f(u(x,t), x, t) \\ &\overset{(H_4)}{=} -\mu_{\mathbb{T}}(t) b R + (1 + \mu_{\mathbb{T}}(t) b)\, u(x,t) + \mu_{\mathbb{T}}(t) f(u(x,t), x, t) \\ &\overset{(4.3),(4.5)}{\leq} -\mu_{\mathbb{T}}(t) b R + (1 + \mu_{\mathbb{T}}(t) b)\, u(x,t) + (1 + \mu_{\mathbb{T}}(t) b)\,(R - u(x,t)) \\ &= R, \end{aligned}$$

  for each $x \in \mathbb{Z}$. The former inequality in (4.4) can be shown in a similar way.

We do not have to consider the case when t is right-dense, since $\mathbb{T}$ does not contain any such point. Therefore, the induction principle yields that (4.2) holds for all $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$. □

We now proceed to the general weak maximum principle for (2.1), where $\mathbb{T}$ is an arbitrary time scale (i.e., allowing right-dense points). The basic idea of the proof is to use the continuous dependence results from Theorems 3.2 and 3.4 to approximate the solution of (2.1) on any time scale by solutions of (2.1) defined on discrete time scales, for which we can apply Lemma 4.3.

Theorem 4.4 (weak maximum principle)**.** *Assume that* (H_1)–(H_6) *hold. If* $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ *is a bounded solution of* (2.1)*, then*

$$r \leq u(x,t) \leq R \quad \textit{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}}. \tag{4.6}$$

Proof. From Theorems 2.1 and 2.3 we obtain that u has to be unique and $U(t) = \{u(x,t)\}_{x \in \mathbb{Z}}$ is the unique solution of the abstract initial-value problem

$$U^{\Delta}(t) = \Phi(U(t), t), \quad U(t_0) = u^0, \tag{4.7}$$

where $\Phi : \ell^\infty(\mathbb{Z}) \times [t_0, T]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ is given by $\Phi(\{u_x\}_{x\in\mathbb{Z}}, t) = \{a u_{x+1} + b u_x + c u_{x-1} + f(u_x, x, t)\}_{x \in \mathbb{Z}}$.

According to Theorem 3.4, there exists a sequence $\{\mathbb{T}_n\}_{n=1}^\infty$ of discrete time scales such that $\mathbb{T}_n \subseteq \mathbb{T}$, $\min \mathbb{T}_n = t_0$, $\max \mathbb{T}_n = T$, $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}}$. Moreover, we have either $\overline{\mu}_{\mathbb{T}} = 0$ and $\overline{\mu}_{\mathbb{T}_n} \to 0$, or $\overline{\mu}_{\mathbb{T}_n} = \overline{\mu}_{\mathbb{T}}$ for all $n \in \mathbb{N}$. In any case, using (H_5), we get the existence of an $n_0 \in \mathbb{N}$ such that

$$\overline{\mu}_{\mathbb{T}_n} \leq -\frac{1}{b} \quad \text{for all} \quad n > n_0.$$

If $\overline{\mu}_{\mathbb{T}} = 0$, it follows from Lemma 4.2 that n_0 can be chosen in such a way that the inequalities

$$\frac{1 + \mu_{\mathbb{T}_n}(t) b}{\mu_{\mathbb{T}_n}(t)} (r - u) \leq f(u, x, t) \leq \frac{1 + \mu_{\mathbb{T}_n}(t) b}{\mu_{\mathbb{T}_n}(t)} (R - u) \quad \text{for all } u \in [r, R], \;\; x \in \mathbb{Z}, \;\; t \in [t_0, T]_{\mathbb{T}_n}$$

hold for each $n > n_0$. If $\overline{\mu}_{\mathbb{T}} > 0$, the same inequalities hold for each $n \in \mathbb{N}$ because of (H_6) and the fact that $\overline{\mu}_{\mathbb{T}_n} = \overline{\mu}_{\mathbb{T}}$.

Therefore, since $\mathbb{T}_n$ are discrete time scales, Lemma 4.3 yields that the corresponding solutions $u_n : \mathbb{Z} \times [t_0, T]_{\mathbb{T}_n} \to \mathbb{R}$ of (2.1) satisfy

$$r \leq u_n(x,t) \leq R \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}_n}, \quad n > n_0,$$

i.e., for $U_n(t) = \{u_n(x,t)\}_{x \in \mathbb{Z}}$, we have

$$r \leq \inf_{x \in \mathbb{Z}} U_n(t)_x \leq \sup_{x \in \mathbb{Z}} U_n(t)_x \leq R \quad \text{for all} \quad t \in [t_0, T]_{\mathbb{T}_n}, \quad n > n_0. \tag{4.8}$$

Since the solution U is bounded, there is an $S > 0$ such that $\|U(t)\|_\infty \leq S$ for each $t \in [t_0, T]_{\mathbb{T}}$. Let

$$\mathcal{B} = \{V \in \ell^\infty(\mathbb{Z}); \|V\|_\infty \leq \max(|r|, |R|, S)\}.$$

As in the proof of Theorem 2.1, one can show that the restriction of the mapping Φ to $\mathcal{B} \times [t_0, T]_{\mathbb{T}}$ is continuous on its domain and Lipschitz-continuous in the first variable. Therefore, if we let $\mathbb{T}_0 = \mathbb{T}$, the assumptions of Theorem 3.2 are satisfied (recall that $U_n(t) \in \mathcal{B}$ for all $t \in \mathbb{T}_n$ and $n > n_0$ from (4.8), and $U(t) \in \mathcal{B}$ for all $t \in \mathbb{T}$ immediately from the definition of $\mathcal{B}$) and hence, $U_n^* \rightrightarrows U^*$ on $[t_0, T]$.

From the definition of the piecewise constant extension U_n^* and from (4.8), it is obvious that

$$r \leq \inf_{x \in \mathbb{Z}} U_n^*(t)_x \leq \sup_{x \in \mathbb{Z}} U_n^*(t)_x \leq R \quad \text{for all} \quad t \in [t_0, T], \quad n > n_0. \tag{4.9}$$

Since $U_n^* \rightrightarrows U^*$ on $[t_0, T]$, the inequalities (4.9) imply

$$r \leq \inf_{x\in\mathbb{Z}} U^*(t)_x \leq \sup_{x\in\mathbb{Z}} U^*(t)_x \leq R \quad \text{for all} \quad t \in [t_0, T].$$

Particularly, there has to be

$$r \leq \inf_{x\in\mathbb{Z}} U(t)_x \leq \sup_{x\in\mathbb{Z}} U(t)_x \leq R \quad \text{for all} \quad t \in [t_0, T]_{\mathbb{T}},$$

which proves that (4.6) holds. □

Remark 4.5. In connection with the previous theorem, we point out the following facts:

- The classical maximum principle guarantees that $m \leq u(x,t) \leq M$, i.e., it corresponds to the case when $r = m$ and $R = M$. However, for this choice of r and R, (H_6) need not be satisfied. By choosing $r < m$ and $R > M$, we can soften (H_6), and obtain the weaker estimate $r \leq u(x,t) \leq R$.
- An examination of the proofs of Lemma 4.3 and Theorem 4.4 reveals that if we are interested only in the upper bound $u(x,t) \leq R$, it is sufficient to assume that $a + b + c \leq 0$. Symmetrically, to get the lower bound $u(x,t) \geq r$, it is enough to suppose that $a + b + c \geq 0$.

As an application of the weak maximum principle, we obtain the following global existence theorem. Since we consider a general class of nonlinearities f, the result is new even in the special case $\mathbb{T} = \mathbb{R}$.

Theorem 4.6 (global existence)**.** *If $u^0 \in \ell^\infty(\mathbb{Z})$ and (H_1)–(H_6) hold, then* (2.1) *has a unique bounded solution $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$.*

Moreover, the solution depends continuously on u^0 in the following sense: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $v^0 \in \ell^\infty(\mathbb{Z})$, $r \leq v_x^0 \leq R$ for all $x \in \mathbb{Z}$, and $\|u^0 - v^0\|_\infty < \delta$, then the unique bounded solution $v : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ of (2.1) *corresponding to the initial condition v^0 satisfies $|u(x,t) - v(x,t)| < \varepsilon$ for all $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.*

Proof. We know from Theorems 2.1 and 2.3 that bounded solutions to (2.1) are unique, and that they correspond to solutions of the initial-value problem

$$U^\Delta(t) = \Phi(U(t), t), \quad t \in [t_0, T]_{\mathbb{T}}^\kappa, \quad U(t_0) = u^0, \tag{4.10}$$

with $\Phi : \ell^\infty(\mathbb{Z}) \times [t_0, T]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ being given by $\Phi(\{u_x\}_{x\in\mathbb{Z}}, t) = \{au_{x+1} + bu_x + cu_{x-1} + f(u_x, x, t)\}_{x\in\mathbb{Z}}$. Thus, it is enough to prove that (4.10) has a solution on the whole interval $[t_0, T]_{\mathbb{T}}$.

Let $\mathcal{S}$ be the set of all $s \in [t_0, T]_{\mathbb{T}}$ such that (4.10) has a solution on $[t_0, s]_{\mathbb{T}}$, and denote $t_1 = \sup \mathcal{S}$. By Theorem 2.1, we have $t_1 > t_0$. Let us prove that $t_1 \in \mathcal{S}$. The statement is obvious if t_1 is a left-scattered maximum of $\mathcal{S}$; therefore, we can assume that t_1 is left-dense. It follows from the definition of t_1 that (4.10) has a solution U defined on $[t_0, t_1)_{\mathbb{T}}$. According to the weak maximum principle, U takes values in the bounded set $\mathcal{B} = \{u \in \ell^\infty(\mathbb{Z});\ r \leq u_x \leq R \text{ for each } x \in \mathbb{Z}\}$. As in the proof of Theorem 2.1, one can show that Φ is continuous on its domain and Lipschitz-continuous in the first variable and bounded on $\mathcal{B} \times [t_0, T]_{\mathbb{T}}$; let C be the boundedness constant for $\|\Phi\|_\infty$. Since U is a solution of (4.10), we have

$$U(t) = U(t_0) + \int_{t_0}^{t} \Phi(U(s), s)\Delta s \tag{4.11}$$

for each $t \in [t_0, t_1)_{\mathbb{T}}$. Note also that $\|U(s_1) - U(s_2)\|_\infty \leq C|s_1 - s_2|$ for all $s_1, s_2 \in [t_0, t_1)_{\mathbb{T}}$. Thus, the Cauchy condition for the existence of the limit $U(t_1-) = \lim_{s\to t_1-} U(s)$ is satisfied. If we extend U to $[t_0, t_1]_{\mathbb{T}}$ by letting $U(t_1) = U(t_1-)$, we see that (4.11) holds also for $t = t_1$. Since the mapping $s \mapsto \Phi(U(s), s)$ is continuous on $[t_0, t_1]_{\mathbb{T}}$, it follows that U is a solution of (4.10) on $[t_0, t_1]_{\mathbb{T}}$, i.e., $t_1 \in \mathcal{S}$.

If $t_1 < T$, we can use Theorem 2.1 to extend the solution U from $[t_0, t_1]_{\mathbb{T}}$ to a larger interval. However, this contradicts the fact that $t_1 = \sup \mathcal{S}$. Hence, the only possibility is $t_1 = T$, and the proof of the existence is complete.

To obtain continuous dependence of the solution on the initial condition, it is enough to show the following statement: If $u^n \in \mathcal{B}$ for $n \in \mathbb{N}$, $u^n \to u^0$ in $\ell^\infty(\mathbb{Z})$, and $U_n : [t_0, T]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ is the unique solution of the initial-value problem

$$U_n^\Delta(t) = \Phi(U_n(t), t), \quad t \in [t_0, T]_{\mathbb{T}}^\kappa, \quad U_n(t_0) = u^n,$$

then $U_n \rightrightarrows U$ on $[t_0, T]_{\mathbb{T}}$. Since we know that the solutions U_n in fact take values in $\mathcal{B}$, the statement is an immediate consequence of Theorem 3.2, where we take $\mathbb{T}_n = \mathbb{T}$ for each $n \in \mathbb{N}_0$. $\square$

Let us illustrate the application of the weak maximum principle and the global existence theorem on the following special cases of (2.1).

Example 4.7. Consider the logistic nonlinearity $f(u, x, t) = \lambda u(1 - u)$, $u \in \mathbb{R}$, $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$, where $\lambda > 0$ is a parameter. In this case, (2.1) becomes a Fisher-type reaction-diffusion equation:

$$\begin{aligned} &u^\Delta(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + \lambda u(x,t)\left(1 - u(x,t)\right), \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}}^\kappa, \\ &u(x, t_0) = u_x^0, \quad x \in \mathbb{Z}. \end{aligned} \tag{4.12}$$

Obviously, f satisfies (H_1)–(H_3). Suppose that $a, c \geq 0$, $b < 0$, $a + b + c = 0$, and $\overline{\mu}_{\mathbb{T}} \leq -1/b$; i.e., (H_4) and (H_5) hold. Consider an arbitrary nonnegative initial condition $u^0 \in \ell^\infty(\mathbb{Z})$, i.e., $m \geq 0$. We now distinguish between the cases $\overline{\mu}_{\mathbb{T}} = 0$ and $\overline{\mu}_{\mathbb{T}} > 0$:

- If $\overline{\mu}_{\mathbb{T}} = 0$, let $r = \min(m, 1)$ and $R = \max(M, 1)$. Then $f(R, x, t) \leq 0$ and $f(r, x, t) \geq 0$, i.e., (H_6) holds and there exists a unique global solution u of (4.12). Moreover, u satisfies $r \leq u(x,t) \leq R$ for all $x \in \mathbb{Z}$ and $t \in [t_0, T]_{\mathbb{T}}$. In particular, nonnegative initial conditions always lead to nonnegative solutions.

- If $\overline{\mu}_{\mathbb{T}} > 0$, Lemma 4.2 together with the analysis of the previous case guarantee that (H_6) holds with $r = \min(m, 1)$ and $R = \max(M, 1)$ whenever $\overline{\mu}_{\mathbb{T}}$ is sufficiently small. For example, if $M \leq 1$, consider the linear functions $\psi_1(u) = \frac{1+\overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(r - u)$ and $\psi_2(u) = \frac{1+\overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(R - u)$ from (H_6). We have $\psi_1(u) \leq 0 \leq f(u, x, t)$ for each for $u \in [r, R]$, i.e., the first inequality in (H_6) is satisfied. The graphs of ψ_2 and $f(\cdot, x, t)$ meet at the point $(1, 0)$. Therefore, the second inequality $f(u, x, t) \leq \psi_2(u)$ in (H_6) will be satisfied for $u \in [r, R]$ if and only if $\frac{\partial f}{\partial u}(1, x, t) \geq \psi_2'(1)$, i.e., if and only if $-\lambda \geq -(1/\overline{\mu}_{\mathbb{T}} + b)$. The last condition is equivalent to $\lambda - b \leq 1/\overline{\mu}_{\mathbb{T}}$, which holds if $\overline{\mu}_{\mathbb{T}} \leq 1/(\lambda - b)$ (note that $b < 0 < \lambda$). Under these assumptions, (H_6) holds and there exists a unique bounded global solution u of (4.12). Moreover, u satisfies $m = r \leq u(x,t) \leq R = 1$ for all $x \in \mathbb{Z}$ and $t \in [t_0, T]_{\mathbb{T}}$.

Example 4.8. Consider the so-called bistable nonlinearity $f(u, x, t) = \lambda u(1 - u^2)$, $u \in \mathbb{R}$, $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$, where $\lambda > 0$. In this case, (2.1) becomes a Nagumo-type reaction-diffusion equation:

$$\begin{aligned} &u^\Delta(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + \lambda u(x,t)\left(1 - u(x,t)^2\right), \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}}^\kappa, \\ &u(x, t_0) = u_x^0, \quad x \in \mathbb{Z}. \end{aligned} \tag{4.13}$$

Obviously, f satisfies (H_1)–(H_3). Suppose that $a, c \geq 0$, $b < 0$, $a + b + c = 0$, and $\overline{\mu}_{\mathbb{T}} \leq -1/b$; i.e., (H_4) and (H_5) hold. Consider an arbitrary initial condition $u^0 \in \ell^\infty(\mathbb{Z})$. Again, we distinguish between the cases $\overline{\mu}_{\mathbb{T}} = 0$ and $\overline{\mu}_{\mathbb{T}} > 0$:

- If $\overline{\mu}_{\mathbb{T}} = 0$, let

$$r = \begin{cases} \min(m, -1) & \text{if } m < 0, \\ \min(m, 1) & \text{if } m \geq 0, \end{cases} \qquad R = \begin{cases} \max(M, -1) & \text{if } M \leq 0, \\ \max(M, 1) & \text{if } M > 0. \end{cases}$$

Then $f(R,x,t) \le 0$ and $f(r,x,t) \ge 0$, i.e., (H_6) holds and there exists a unique bounded global solution u of (4.13). Moreover, u satisfies $r \le u(x,t) \le R$ for all $x \in \mathbb{Z}$ and $t \in [t_0,T]_{\mathbb{T}}$. In particular, nonnegative/nonpositive initial conditions always lead to nonnegative/nonpositive solutions.

- If $\overline{\mu}_{\mathbb{T}} > 0$, Lemma 4.2 together with the analysis of the previous case guarantee that (H_6) holds whenever $\overline{\mu}_{\mathbb{T}}$ is sufficiently small. For example, if $\|u^0\|_\infty \le 1$, one can follow the computations from [31, Section 8] to conclude that there exists a unique global solution u of (4.13) satisfying

$$u(x,t) \in \begin{cases} [-1,1] & \text{if } \overline{\mu}_{\mathbb{T}} \le 1/(2\lambda - b), \\ [-\tilde{R}, \tilde{R}] & \text{if } 1/(2\lambda - b) < \overline{\mu}_{\mathbb{T}} \le 2/(\lambda - 2b), \end{cases}$$

where
$$\tilde{R} = \frac{2\lambda\overline{\mu}_{\mathbb{T}}(1/3 + (1+2b\overline{\mu}_{\mathbb{T}})/3\lambda\overline{\mu}_{\mathbb{T}})^{3/2}}{1 + b\overline{\mu}_{\mathbb{T}}}.$$

We have no a priori bounds for $\overline{\mu}_{\mathbb{T}} > 2/(\lambda - 2b)$.

Example 4.9. Consider the nonautonomous nonlinearity $f(u,x,t) = \lambda u(d(x,t) - u)$, $u \in \mathbb{R}$, $x \in \mathbb{Z}$, $t \in [t_0,T]_{\mathbb{T}}$, where $\lambda > 0$ and $d : \mathbb{Z} \times [t_0,T]_{\mathbb{T}} \to \mathbb{R}$. In this case, (2.1) has the form

$$\begin{aligned} &u^\Delta(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + \lambda u(x,t)(d(x,t) - u(x,t)), \quad x \in \mathbb{Z}, \quad t \in [t_0,T]^\kappa_{\mathbb{T}}, \\ &u(x,t_0) = u^0_x, \quad x \in \mathbb{Z}. \end{aligned} \tag{4.14}$$

The equation can be interpreted as the logistic population model where the carrying capacity d depends on position and time. Assume that d has the following properties:

- d is bounded.
- For each choice of $\varepsilon > 0$ and $t \in [t_0,T]_{\mathbb{T}}$, there exists a $\delta > 0$ such that if $s \in (t-\delta, t+\delta) \cap [t_0,T]_{\mathbb{T}}$, then $|d(x,t) - d(x,s)| < \varepsilon$ for all $x \in \mathbb{Z}$.

Then the function f satisfies (H_1)–(H_3). Indeed, let D be the boundedness constant for $|d|$. If $B \subset \mathbb{R}$ is bounded, it is contained in a ball of radius ρ centered at the origin. Consequently, for all $u,v \in B$, $x \in \mathbb{Z}$, $t,s \in [t_0,T]_{\mathbb{T}}$, we get the estimates

$$\begin{aligned} |f(u,x,t)| &\le \lambda|u|(|d(x,t)| + |u|) \le \lambda\rho(D+\rho), \\ |f(u,x,t) - f(v,x,t)| &= \lambda|(u-v)(d(x,t) - u - v)| \le \lambda|u-v|(D+2\rho), \\ |f(u,x,t) - f(u,x,s)| &= \lambda|u(d(x,t) - d(x,s))| \le \lambda\rho|d(x,t) - d(x,s)|, \end{aligned}$$

which imply that (H_1)–(H_3) hold.

As an example, let us mention the model of population dynamics with a shifting habitat, which was described in [17]. The authors considered the problem (4.14) with $\mathbb{T} = \mathbb{R}$, $a = c$, $b = -2a$ (i.e., symmetric diffusion), and $d(x,t) = e(x - \gamma t)$, where $\gamma > 0$ and $e : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, and bounded. It follows that e is uniformly continuous on $\mathbb{R}$: Given an $\varepsilon > 0$, there exists a $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $|e(t_1) - e(t_2)| < \varepsilon$. Thus, we get

$$|d(x,t) - d(x,s)| = |e(x - \gamma t) - e(x - \gamma s)| < \varepsilon$$

whenever $|t - s| < \frac{\delta}{\gamma}$ and $x \in \mathbb{Z}$; this shows that d satisfies our assumptions. (We remark that some of the results presented in [17] can be found in our earlier paper [29]. In particular, the fundamental solution of the linear lattice diffusion equation was derived in [29, Example 3.1], and [17, Corollary 2.1] is a consequence of our superposition principle from [29, Theorem 2.2].)

Another simple example is obtained by letting $d(x,t) = e(t)$, where $e : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function; this choice corresponds to a population model with a periodically changing habitat. Since e is necessarily bounded and uniformly continuous on $\mathbb{R}$, it is obvious that d satisfies our assumptions.

Suppose now that $a, c \geq 0$, $b < 0$, $a + b + c = 0$, and $\overline{\mu}_{\mathbb{T}} \leq -1/b$; i.e., (H_4) and (H_5) hold. For simplicity, let us restrict ourselves to the case when d is a positive function, and let

$$d_{\min} = \inf_{(x,t)\in\mathbb{Z}\times[t_0,T]_{\mathbb{T}}} d(x,t), \qquad d_{\max} = \sup_{(x,t)\in\mathbb{Z}\times[t_0,T]_{\mathbb{T}}} d(x,t).$$

Consider an arbitrary nonnegative initial condition $u^0 \in \ell^\infty(\mathbb{Z})$, i.e., $m \geq 0$. Take $r = \min(m, d_{\min})$ and $R = \max(M, d_{\max})$. Then $f(r,x,t) \geq 0$ and $f(R,x,t) \leq 0$ for all $x \in \mathbb{Z}$ and $t \in [t_0,T]_{\mathbb{T}}$. This means that (H_6) holds if $\overline{\mu}_{\mathbb{T}} = 0$, or (by Lemma 4.2) if $\overline{\mu}_{\mathbb{T}}$ is positive and sufficiently small. In these cases, the problem (4.14) possesses a unique global solution u, and $r \leq u(x,t) \leq R$ for all $x \in \mathbb{Z}$ and $t \in [t_0,T]_{\mathbb{T}}$.

5 Strong maximum principle

In the rest of the paper we focus on the strong maximum principle for (2.1). We need the following stronger versions of (H_4)–(H_6):

$(\overline{H_4})$ *a, b, $c \in \mathbb{R}$ are such that $a, c > 0$, $b < 0$, and $a + b + c = 0$.*

$(\overline{H_5})$ *$b < 0$ and $\overline{\mu}_{\mathbb{T}} < -1/b$.*

$(\overline{H_6})$ *There exist $r, R \in \mathbb{R}$ such that $r \leq m \leq M \leq R$, and the following statements hold for all $x \in \mathbb{Z}$ and $t \in [t_0,T]_{\mathbb{T}}$:*

- *$f(R,x,t) \leq 0 \leq f(r,x,t)$.*
- *If $\overline{\mu}_{\mathbb{T}} > 0$, then $f(u,x,t) > \dfrac{1+\overline{\mu}_{\mathbb{T}}b}{\overline{\mu}_{\mathbb{T}}}(r-u)$ for all $u \in (r,R]$.*
- *If $\overline{\mu}_{\mathbb{T}} > 0$, then $f(u,x,t) < \dfrac{1+\overline{\mu}_{\mathbb{T}}b}{\overline{\mu}_{\mathbb{T}}}(R-u)$ for all $u \in [r,R)$.*

The next lemma analyzes the situation when a solution of (2.1) attains its maximum at a left-scattered point.

Lemma 5.1. *Assume that (H_1), (H_2), (H_3), $(\overline{H_4})$, $(\overline{H_5})$, $(\overline{H_6})$ hold, and $u : \mathbb{Z} \times [t_0,T]_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of (2.1). If $u(\bar{x},\bar{t}) \in \{r, R\}$ for some $\bar{x} \in \mathbb{Z}$ and a left-scattered point $\bar{t} \in (t_0,T]_{\mathbb{T}}$, then $u(x,\rho_{\mathbb{T}}(\bar{t})) = u(\bar{x},\bar{t})$ for each $x \in \{\bar{x}-1, \bar{x}, \bar{x}+1\}$.*

Proof. We consider the case when $u(\bar{x},\bar{t}) = R$; the case $u(\bar{x},\bar{t}) = r$ can be treated in a similar way. Denote $\bar{s} = \rho_{\mathbb{T}}(\bar{t})$. We have

$$u(\bar{x},\bar{t}) = \mu_{\mathbb{T}}(\bar{s})au(\bar{x}+1,\bar{s}) + (1+\mu_{\mathbb{T}}(\bar{s})b)u(\bar{x},\bar{s}) + \mu_{\mathbb{T}}(\bar{s})cu(\bar{x}-1,\bar{s}) + \mu_{\mathbb{T}}(\bar{s})f(u(\bar{x},\bar{s}),\bar{x},\bar{s}).$$

By the weak maximum principle (which holds because $(\overline{H_4})$–$(\overline{H_6})$ imply (H_4)–(H_6)), the values of u cannot exceed R. If at least one of the values $u(\bar{x}+1,\bar{s})$, $u(\bar{x}-1,\bar{s})$ is smaller than R and $u(\bar{x},\bar{s}) = R$, then

$$u(\bar{x},\bar{t}) \overset{(\overline{H_4})}{<} \mu_{\mathbb{T}}(\bar{s})(a+c)R + (1+\mu_{\mathbb{T}}(\bar{s})b)R + \mu_{\mathbb{T}}(\bar{s})f(R,\bar{x},\bar{s}) \overset{(\overline{H_4})}{=} R + \mu_{\mathbb{T}}(\bar{s})f(R,\bar{x},\bar{s}) \overset{(\overline{H_6})}{\leq} R,$$

which contradicts the fact that $u(\bar{x},\bar{t}) = R$. If $u(\bar{x},\bar{s}) < R$, then

$$\begin{aligned} u(\bar{x},\bar{t}) &\leq \mu_{\mathbb{T}}(\bar{s})(a+c)R + (1+\mu_{\mathbb{T}}(\bar{s})b)u(\bar{x},\bar{s}) + \mu_{\mathbb{T}}(\bar{s})f(u(\bar{x},\bar{s}),\bar{x},\bar{s}) \\ &\overset{(\overline{H_4}),(\overline{H_6})}{<} \mu_{\mathbb{T}}(\bar{s})(a+c)R + (1+\mu_{\mathbb{T}}(\bar{s})b)u(\bar{x},\bar{s}) + \mu_{\mathbb{T}}(\bar{s})\tfrac{1+\overline{\mu}_{\mathbb{T}}b}{\overline{\mu}_{\mathbb{T}}}(R - u(\bar{x},\bar{s})) \\ &\leq \mu_{\mathbb{T}}(\bar{s})(a+c)R + (1+\mu_{\mathbb{T}}(\bar{s})b)u(\bar{x},\bar{s}) + (1+\mu_{\mathbb{T}}(\bar{s})b)(R - u(\bar{x},\bar{s})) \overset{(\overline{H_4})}{=} R, \end{aligned} \tag{5.1}$$

which is a contradiction again. Thus, the only possibility is that

$$u(\bar{x}+1,\bar{s}) = u(\bar{x},\bar{s}) = u(\bar{x}-1,\bar{s}) = R. \qquad \square$$

We now turn our attention to the case when the maximum is attained at a left-dense point.

Lemma 5.2. *Assume that (H_1), (H_2), (H_3), $(\overline{H_4})$, $(\overline{H_5})$, $(\overline{H_6})$ hold, and $u : \mathbb{Z}\times[t_0,T]_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of (2.1). If $u(\bar{x},\bar{t}) \in \{r,R\}$ for some $\bar{x}\in\mathbb{Z}$ and a left-dense point $\bar{t}\in(t_0,T]_{\mathbb{T}}$, then $u(x,t) = u(\bar{x},\bar{t})$ for all $x\in\mathbb{Z}$ and $t\in[t_0,\bar{t}]_{\mathbb{T}}$.*

Proof. We consider the case when $u(\bar{x},\bar{t}) = R$; the case $u(\bar{x},\bar{t}) = r$ can be treated in a similar way. We begin by proving that

$$u(\bar{x},t) = R \quad \text{for all } t\in[t_0,\bar{t}]_{\mathbb{T}}. \tag{5.2}$$

Assume that there exists a $\bar{s}\in[t_0,\bar{t})_{\mathbb{T}}$ such that $u(\bar{x},\bar{s}) < R$. Let $L \geq 0$ be the Lipschitz constant for f on the set $[r,R]\times\mathbb{Z}\times[t_0,T]_{\mathbb{T}}$. Choose a partition $\bar{s} = s_0 < s_1 < \cdots < s_k = \bar{t}$ such that $s_0,\ldots,s_k\in\mathbb{T}$ and for each $i\in\{1,\ldots,k\}$, we have either $s_i - s_{i-1} < 1/(L-b)$, or $s_i = \sigma_{\mathbb{T}}(s_{i-1})$. We will use induction with respect to i to show that $u(\bar{x},s_i) < R$ for each $i\in\{0,\ldots,k\}$; this will be a contradiction with the fact that $u(\bar{x},s_k) = u(\bar{x},\bar{t}) = R$.

For $i = 0$, we know that $u(\bar{x},s_0) = u(\bar{x},\bar{s}) < R$. By the weak maximum principle (which holds because $(\overline{H_4})$–$(\overline{H_6})$ imply (H_4)–(H_6)), the values of u cannot exceed R. If $i\in\{0,\ldots,k-1\}$ is such that $s_{i+1} = \sigma_{\mathbb{T}}(s_i)$, then the induction hypothesis $u(\bar{x},s_i) < R$ and Lemma 5.1 imply that $u(\bar{x},s_{i+1}) < R$. Otherwise, we have $s_{i+1} - s_i < 1/(L-b)$. For each $t\in[s_i,s_{i+1})_{\mathbb{T}}$, we get

$$\begin{aligned} (u(\bar{x},t)-R)^{\Delta} &= au(\bar{x}+1,t) + bu(\bar{x},t) + cu(\bar{x}-1,t) + f(u(\bar{x},t),\bar{x},t) \\ &\overset{(\overline{H_4}),\text{Thm.4.4}}{\leq} (a+c)R + bu(\bar{x},t) + f(u(\bar{x},t),\bar{x},t) - f(R,\bar{x},t) + f(R,\bar{x},t) \\ &\overset{(\overline{H_4}),(\overline{H_6})}{\leq} -b(R-u(\bar{x},t)) + f(u(\bar{x},t),\bar{x},t) - f(R,\bar{x},t) \\ &\leq -b(R-u(\bar{x},t)) + |f(u(\bar{x},t),\bar{x},t) - f(R,\bar{x},t)| \\ &\leq -b(R-u(\bar{x},t)) + L\,|u(\bar{x},t)-R| \\ &\overset{\text{Thm.4.4}}{=} (b-L)\,(u(\bar{x},t)-R)\,. \end{aligned}$$

Notice that $1+\mu_{\mathbb{T}}(t)(b-L) > 0$ for all $t\in[s_i,s_{i+1})_{\mathbb{T}}$. Therefore, Grönwall's inequality [4, Theorem 6.1] yields

$$u(\bar{x},s_{i+1}) - R \leq \underbrace{(u(\bar{x},s_i)-R)}_{<0}\underbrace{\mathrm{e}_{b-L}(s_{i+1},s_i)}_{>0} < 0,$$

which completes the proof by induction and confirms that (5.2) holds.

Let us prove that $u(\bar{x}\pm1,t) = R$ for all $t\in[t_0,\bar{t}]_{\mathbb{T}}$. Assume that there exists a $t\in[t_0,\bar{t}]_{\mathbb{T}}$ such that at least one of the values $u(\bar{x}\pm1,t)$ is smaller than R. The fact that $u(\bar{x},\cdot)$ is a constant function on $[t_0,\bar{t}]_{\mathbb{T}}$ implies that $u^{\Delta}(\bar{x},t) = 0$ (note that if $t = \bar{t}$, then t is necessarily left-dense). On the other hand,

$$u^{\Delta}(\bar{x},t) = au(\bar{x}+1,t) + bu(\bar{x},t) + cu(\bar{x}-1,t) + f(u(\bar{x},t),\bar{x},t) < (a+b+c)R + f(R,\bar{x},t) \leq 0,$$

i.e., $u^{\Delta}(\bar{x},t) < 0$, a contradiction.

Once we know that $u(\bar{x}\pm1,t) = R$ for all $t\in[t_0,\bar{t}]_{\mathbb{T}}$, it follows by induction with respect to $x\in\mathbb{Z}$ that $u(x,t) = R$ for all $x\in\mathbb{Z}$ and $t\in[t_0,\bar{t}]_{\mathbb{T}}$. $\square$

With the help of previous two lemmas, we derive the strong maximum principle.

Theorem 5.3 (strong maximum principle)**.** *Assume that* (H_1), (H_2), (H_3), $(\overline{H_4})$, $(\overline{H_5})$, $(\overline{H_6})$ *hold with* $r = m \leq M = R$ *and* $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ *is a bounded solution of* (2.1)*. If* $u(\bar{x}, \bar{t}) \in \{r, R\}$ *for some* $\bar{x} \in \mathbb{Z}$ *and* $\bar{t} \in (t_0, T]_{\mathbb{T}}$*, then the following statements hold:*

(a) *If* $[t_0, \bar{t}]_{\mathbb{T}}$ *contains only isolated points, i.e.,* $t_0 = \rho^k_{\mathbb{T}}(\bar{t})$ *for some* $k \in \mathbb{N}$*, and*

$$\mathcal{D}(\bar{x}, \bar{t}) = \left\{(x, t) \in \mathbb{Z} \times [t_0, \bar{t}]_{\mathbb{T}} : \ t = \rho^j_{\mathbb{T}}(\bar{t}), \ j = 0, \ldots, k, \ \text{and } x = \bar{x} \pm i, \ i = 0, \ldots, j\right\},$$

then $u(x, t) = u(\bar{x}, \bar{t})$ *for all* $(x, t) \in \mathcal{D}(\bar{x}, \bar{t})$.

(b) *Otherwise, if* $[t_0, \bar{t}]_{\mathbb{T}}$ *contains a point which is not isolated, then* $r = R$ *and* $u(x, t) = R$ *for all* $x \in \mathbb{Z}$ *and* $t \in [t_0, T]_{\mathbb{T}}$.

Remark 5.4. In order to prevent any confusion, we emphasize that the fact whether a point is isolated or not is considered with respect to the time scale interval $[t_0, \bar{t}]_{\mathbb{T}}$, not the entire time scale $\mathbb{T}$. In other words, the statement distinguishes between the cases in which the interval $[t_0, \bar{t}]_{\mathbb{T}}$ is a finite set (part (a)) or at least countable (part (b)).

Proof. We consider the case when $u(\bar{x}, \bar{t}) = R$; the case $u(\bar{x}, \bar{t}) = r$ can be treated in a similar way. We prove the statement by analyzing two different cases:

1. Let there be a left-dense point in $[t_0, \bar{t}]_{\mathbb{T}}$. Denote

$$\mathcal{P}_{ld} = \{t \in [t_0, \bar{t}]_{\mathbb{T}} : \ t \text{ is left-dense}\} \neq \emptyset$$

and $t_{ld} = \sup \mathcal{P}_{ld}$. Given the definition of supremum and the fact that $\mathbb{T}$ is a closed set, we obtain $t_{ld} \in \mathbb{T}$. To show that t_{ld} is left-dense, let us assume by contradiction that t_{ld} is left-scattered. Thus, $t_{ld} \notin \mathcal{P}_{ld}$ and immediately from the definition of supremum we get a contradiction. From the proofs of Lemmas 5.1 and 5.2 we obtain that $u(\bar{x}, t) = R$ for all $t \in [t_0, \bar{t}]_{\mathbb{T}}$ and particularly, $u(\bar{x}, t_{ld}) = R$. Furthermore, since t_{ld} is left-dense, Lemma 5.2 yields that

$$u(x, t) = R \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0, t_{ld}]_{\mathbb{T}}. \tag{5.3}$$

There remains to prove the statement for $t \in [t_{ld}, T]_{\mathbb{T}}$. From (5.3) we get that $u(x, t_0) = u^0_x = R$ for all $x \in \mathbb{Z}$ and thus, $r = m = M = R$. Consequently, since (H_6) holds with $r = m = M = R$, Theorem 4.4 (weak maximum principle) yields that

$$R \leq u(x, t) \leq R, \quad \text{i.e.,} \quad u(x, t) = R, \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}}.$$

2. Let us assume that $[t_0, \bar{t}]_{\mathbb{T}}$ does not contain any left-dense point.

   (i) If $[t_0, \bar{t})_{\mathbb{T}}$ does not contain any right-dense point, i.e., $[t_0, T]_{\mathbb{T}}$ contains only isolated points, then the part (a) of the theorem follows immediately from Lemma 5.1.

   (ii) Let there exist a right-dense point in $[t_0, \bar{t})_{\mathbb{T}}$. Denote

   $$\mathcal{P}_{rd} = \{t \in [t_0, \bar{t})_{\mathbb{T}} : \ t \text{ is right-dense}\} \neq \emptyset,$$

   and $t_{rd} = \sup \mathcal{P}_{rd}$. From the fact that $\bar{t}$ is left-scattered and from the definition of supremum we obtain $t_{rd} < \bar{t}$. Moreover, since $\mathbb{T}$ is closed, there is $t_{rd} \in \mathbb{T}$. Further, we show that t_{rd} is right-dense as well. Indeed, let us assume that t_{rd} is right-scattered, i.e., $t_{rd} \notin \mathcal{P}_{rd}$. Then

t_{rd} is an unattained supremum of $\mathcal{P}_{rd}$ and there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset \mathcal{P}_{rd}$ such that $t_n \nearrow t_{rd}$. This would imply that t_{rd} is left-dense, a contradiction. Thus, t_{rd} is right-dense.
From the definition of t_{rd}, the sequence of predecessors of $\bar{t}$, namely

$$\left\{\rho^j_{\mathbb{T}}(\bar{t})\right\}_{j=1}^{\infty} \subseteq (t_{rd}, \bar{t}]_{\mathbb{T}},$$

is well-defined and satisfies $\rho^j_{\mathbb{T}}(\bar{t}) \searrow t_{rd}$. Let us assume that $x \in \mathbb{Z}$ is arbitrary but fixed, i.e., $x = \bar{x} + i_0$ or $x = \bar{x} - i_0$ for some $i_0 \in \mathbb{N}_0$. We consider the case $x = \bar{x} + i_0$; the other case is similar. Lemma 5.1 implies that for all $j \geq i_0$, there is $u\left(x, \rho^j_{\mathbb{T}}(\bar{t})\right) = u\left(\bar{x} + i_0, \rho^j_{\mathbb{T}}(\bar{t})\right) = R$. Then the continuity of the function $u(x, \cdot)$ yields that

$$R = \lim_{j \to \infty} u\left(x, \rho^j_{\mathbb{T}}(\bar{t})\right) = u(x, t_{rd}),$$

and since $x \in \mathbb{Z}$ is arbitrary, there is $u(x, t_{rd}) = R$ for all $x \in \mathbb{Z}$.
Now we prove that $u(x,t) = R$ for $x \in \mathbb{Z}$ and $t \in [t_0, t_{rd}]_{\mathbb{T}}$. We use the backward induction principle in the variable t (see [4, Theorem 1.7 and Remark 1.8]):

- Above we have shown that for $t = t_{rd}$ there is $u(x, t_{rd}) = R$ for all $x \in \mathbb{Z}$.
- Let $t \in (t_0, t_{rd}]_{\mathbb{T}}$ be left-scattered and $u(x,t) = R$ for all $x \in \mathbb{Z}$. Then Lemma 5.1 immediately implies that $u(x, \rho_{\mathbb{T}}(t)) = R$ for all $x \in \mathbb{Z}$.
- Let $t \in [t_0, t_{rd})_{\mathbb{T}}$ be right dense and $u(x,s) = R$ for all $x \in \mathbb{Z}$ and $s \in (t, t_{rd}]_{\mathbb{T}}$. Then again from the continuity of the functions $u(x, \cdot)$ we obtain

$$R = \lim_{s \to t+} u(x,s) = u(x,t) \quad \text{for all} \quad x \in \mathbb{Z}.$$

- We do not have to consider the case when $t \in (t_0, t_{rd}]_{\mathbb{T}}$ is left-dense, since we assume that $[t_0, t_{rd}]_{\mathbb{T}}$ does not contain any such point.

The backward induction principle implies that $u(x,t) = R$ for all $x \in \mathbb{Z}$ and $t \in [t_0, t_{rd}]_{\mathbb{T}}$.
Finally, it remains to prove that $u(x,t) = R$ for $x \in \mathbb{Z}$ and $t \in [t_{rd}, T]_{\mathbb{T}}$. Since $u(x, t_0) = u^0_x = R$ for all $x \in \mathbb{Z}$, there is $r = m = M = R$ and analogously as above, we can use Theorem 4.4 (weak maximum principle) to show that

$$R \leq u(x,t) \leq R, \quad \text{i.e.,} \quad u(x,t) = R, \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}}. \qquad \square$$

Corollary 5.5. *Assume that (H_1), (H_2), (H_3), $(\overline{H_4})$, $(\overline{H_5})$, $(\overline{H_6})$ hold with $r = m \leq M = R$ and $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of (2.1). If there is a point $t_d \in [t_0, T)_{\mathbb{T}}$ that is not isolated and if the initial condition u^0 is not constant, then*

$$r < u(x,t) < R \quad \textit{for all} \quad x \in \mathbb{Z}, \quad t \in (t_d, T]_{\mathbb{T}}.$$

Proof. Assume by contradiction that there exist $\bar{x} \in \mathbb{Z}$, $\bar{t} \in (t_d, T]_{\mathbb{T}}$ such that $u(\bar{x}, \bar{t}) \in \{r, R\}$. Since $t_d \in [t_0, \bar{t})_{\mathbb{T}}$ and t_d is not isolated, the part (b) of Theorem 5.3 yields that $r = m = M = R$, a contradiction with the assumption that u^0 is not constant. $\square$

The following remarks explain why the original conditions (H_4)–(H_6) are not sufficient to establish the strong maximum principle, and had to be replaced by their stronger counterparts $(\overline{H_4})$–$(\overline{H_6})$.

Remark 5.6. (H_4) is too weak for the strong maximum principle; we need the constants $a, c \in \mathbb{R}$ to be strictly positive. Indeed, let us consider the linear transport equation

$$\frac{\partial u}{\partial t}(x,t) = u(x+1,t) - u(x,t), \quad x \in \mathbb{Z}, \quad t \in [0,T],$$
$$u(x,0) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

i.e., the initial-value problem (2.1) with $a = 1$, $b = -1$, $c = 0$ and $f \equiv 0$. Then the unique bounded solution is given by (see [30, Corollary 4.3])

$$u(x,t) = \begin{cases} 1, & x \geq 0, \quad t \in [0,T], \\ 1 - \displaystyle\sum_{j=0}^{-x-1} \frac{t^j}{j!} \mathrm{e}^{-t}, & x < 0, \quad t \in [0,T]. \end{cases}$$

Thus, the strong maximum principle does not hold.

Remark 5.7. To see that (H_5) does not suffice, consider the time scale $\mathbb{T} = \mathbb{N}_0$ and the linear equation $(f \equiv 0)$

$$u^\Delta(x,t) = \frac{1}{2}u(x+1,t) - u(x,t) + \frac{1}{2}u(x-1,t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0,$$

which corresponds to (2.1) with $a = c = \frac{1}{2}$, $b = -1$, and $f \equiv 0$. The equation holds if and only if

$$u(x,t+1) = \frac{1}{2}u(x+1,t) + \frac{1}{2}u(x-1,t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{N}_0.$$

For the initial condition

$$u(x,0) = \begin{cases} 1, & x \text{ is even}, \\ 0, & x \text{ is odd}, \end{cases}$$

we obtain

$$u(x,1) = \begin{cases} 0, & x \text{ is even}, \\ 1, & x \text{ is odd}, \end{cases}$$

which violates the strong maximum principle.

Remark 5.8. Finally, let a, b, c be an arbitrary triple satisfying $(\overline{H_4})$, and $\mathbb{T} = \mu\mathbb{N}_0 = \{0, \mu, 2\mu, \ldots\}$, where $\mu > 0$ satisfies $(\overline{H_5})$. Consider the problem (2.1) with

$$u_x^0 = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad \text{and} \quad f(u,x,t) = \left(b + \frac{1}{\mu}\right)(1-u).$$

We have $m = 0$ and $M = 1$. For $r = 0$ and $R = 1$, the function f satisfies (H_6), but not $(\overline{H_6})$. Using (2.1), we calculate

$$u(0,\mu) = \mu a u(1,0) + (1+\mu b)u(0,0) + \mu c u(-1,0) + \mu f(u(0,0),0) = \mu(a+c) + (1+\mu b) \overset{(\overline{H_4})}{=} 1.$$

Therefore, $u(0,\mu) = 1 = R$, but $u(0,0) = 0$, which contradicts the strong maximum principle.

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APPENDIX E

Implicit discrete Nagumo equation and variational methods

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Variational methods and implicit discrete Nagumo equation

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A B S T R A C T

We analyze existence and uniqueness of ℓ^2-solutions of the implicit discrete Nagumo reaction–diffusion equation. We study the infinite-dimensional problem variationally and describe corresponding potentials which have either the convex or mountain pass geometry. Consequently, we show that the implicit Nagumo equation has a solution for all reaction parameters $\lambda \in \mathbb{R}$, at least for small time discretization steps h. Moreover, the solution is unique in the bistable case, $\lambda > 0$.

1. Introduction

Reaction–diffusion equations (RDEs) $v_t = v_{xx} + f(v)$ model a wide range of physical, chemical, biological and epidemiological phenomena in which two forces interact. On the one hand, the diffusion causes spatial spread of species/substance. On the other hand, the reaction function f describes the local dynamics (e.g., reproduction, or chemical reaction), [8].

Numerical methods for RDEs usually consist of two processes, [21]. First, a space discretization reduces partial differential equation(s) into a system of ordinary differential equations. Then a certain time discretization technique is applied. In the case of RDEs, implicit methods are often used because of the stiffness, [12].

Many studies considered preservation of various characteristics of RDEs through discretization processes. In contrast to the problem on the bounded domain, e.g., [11], the problem on the unbounded domain is infinite-dimensional and the corresponding dynamics is more complex, [2,3,10].

In this paper we consider a fully implicit discretization of Nagumo equation on an unbounded domain:

$$\begin{cases} \Delta_t v(x,t) = k\Delta^2_{xx} v(x-1,t+h) + \lambda v(x,t+h)\left(1 - v^2(x,t+h)\right), & \lambda \in \mathbb{R}, \\ v(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

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where $x \in \mathbb{Z}$, $t \in h\mathbb{N}_0 := \{0, h, 2h, \ldots\}$, $k > 0$, $h > 0$ and the partial differences are defined by $\Delta_t v(x,t) = \frac{v(x,t+h)-v(x,t)}{h}$, $\Delta^2_{xx} v(x-1,t+h) = v(x-1,t+h) - 2v(x,t+h) + v(x+1,t+h)$. For the sake of brevity, we assume that the space discretization step $h_x = 1$ but all our results are valid for arbitrary $h_x > 0$ by considering $\bar{k} = \frac{k}{h_x^2}$ instead of k in (1.1).

Explicit discrete Nagumo equation has been studied extensively by many authors, e.g., [4,23]. Note that in this case, existence and uniqueness follow trivially from the recursive scheme and consequently the interest is focused on (non)existence of traveling waves, pattern formation, etc.

We use variational methods to get existence (and uniqueness for $\lambda \geq 0$) of bounded solutions of the implicit problem (1.1) in $\ell^2 = \ell^2(\mathbb{Z})$ (note that [2,3] study (1.1) in weighted sequence spaces). Our technique provides results for solutions in ℓ^2 in certain cases which have not been studied so far (e.g., when the dissipativity condition used in [3] is violated).

Variational methods have proved to be a very efficient tool in the analysis of ordinary boundary discrete problems (e.g., [7,14,15]), whose solutions correspond to stationary solutions of (1.1). In this case, the finite dimension of function spaces is usually exploited. From this point of view, our paper moves these studies one step further. It not only considers non-stationary solutions of (1.1) but also formulates variationally discrete problems in infinite-dimensional spaces.

In more general terms, several properties of differential equations and dynamical systems have recently been identified that show a strong, rich and interesting dependence on partial/full discretizations. Besides the renowned properties of dynamical systems (e.g., [18,16]), let us mention spectral properties (e.g., [9,17,20]) or maximum principles (e.g., [13,19,22]).

The paper is organized as follows. In Sections 2–3 we provide functional-analytic background and formulate (1.1) variationally. In Section 4 we show global uniqueness of solutions of (1.1) for $\lambda \geq 0$ and sufficiently small h. In Section 5, we show the existence of solutions for $\lambda < 0$. Given the fact that in this case the functional has the mountain pass geometry we conjecture, in Section 6, about multiple solutions for $\lambda < 0$. We conclude with a summary of our results and a short list of open problems in Section 7.

2. Abstract formulation in ℓ^2

Since we are interested in the existence of solutions of (1.1) which form $\ell^2 := \ell^2(\mathbb{Z})$ sequences at every time level $t \in h\mathbb{N}_0$, we reformulate (1.1) as an abstract problem in ℓ^2 in this section.

First, we define operators L, $N : \ell^2 \to \ell^2$ by (for $u \in \ell^2$):

$$(Lu)_i := ku_{i-1} - 2ku_i + ku_{i+1} = k\Delta^2 u_{i-1}, \quad i \in \mathbb{Z}, \tag{2.1}$$

$$(N(u))_i := u_i \left(1 - u_i^2\right), \quad i \in \mathbb{Z}. \tag{2.2}$$

We prove basic properties of operators L and N. First, we analyze the linear operator L (note that for $k = 1$ it is the central second difference operator). We observe that L is bounded with $\|L\|_* = 4k$ (see [19, Section 3]). Further, we prove that L is self-adjoint and negative.

Lemma 2.1. *The operator $L \in \mathcal{L}(\ell^2)$ defined by (2.1) is self-adjoint.*

Proof. Let $u, w \in \ell^2$ be arbitrary. Then the series

$$\sum_{i\in\mathbb{Z}} (u_{i-1} - 2u_i + u_{i+1})w_i$$

is absolutely convergent. Indeed, we can use the triangle and Cauchy–Schwarz inequalities to prove that:

$$\sum_{i\in\mathbb{Z}} |u_{i-1} - 2u_i + u_{i+1}| \cdot |w_i| \leq \sum_{i\in\mathbb{Z}} (|u_{i-1}| + 2|u_i| + |u_{i+1}|) \cdot |w_i| \leq 4\|u\|_2\|w\|_2.$$

Consequently, the following reordering is correct:

$$\begin{aligned}(Lu,w)_2 &= k\sum_{i\in\mathbb{Z}}(u_{i-1}-2u_i+u_{i+1})w_i\\ &= k\left(\ldots+(u_{i-2}-2u_{i-1}+u_i)w_{i-1}+(u_{i-1}-2u_i+u_{i+1})w_i+(u_i-2u_{i+1}+u_{i+2})w_{i+1}+\ldots\right)\\ &= k\left(\ldots+u_{i-1}(w_{i-2}-2w_{i-1}+w_i)+u_i(w_{i-1}-2w_i+w_{i+1})+u_{i+1}(w_i-2w_{i+1}+w_{i+2})+\ldots\right)\\ &= k\sum_{i\in\mathbb{Z}}u_i(w_{i-1}-2w_i+w_{i+1})=(u,Lw)_2\end{aligned}$$

and therefore, L is self-adjoint. □

Lemma 2.2. *The operator $L\in\mathcal{L}(\ell^2)$ defined by (2.1) is negative.*

Proof. We prove that $K:=-\frac{1}{k}L$ is a positive operator. Let $u\in\ell^2$ be arbitrary. Then,

$$\begin{aligned}(Ku,u)_2 &= -\sum_{i\in\mathbb{Z}}(u_{i-1}-2u_i+u_{i+1})u_i\\ &= 2\sum_{i\in\mathbb{Z}}u_i^2-\sum_{i\in\mathbb{Z}}u_i(u_{i-1}+u_{i+1})\\ &= 2\|u\|_2^2-(u,w)_2\\ &\ge 2\|u\|_2^2-\|u\|_2\|w\|_2\end{aligned}$$

when we apply the Cauchy–Schwarz inequality for $u,w\in\ell^2$, $w_i:=u_{i-1}+u_{i+1}$. The last term of the estimate above is nonnegative if

$$2\|u\|_2-\|w\|_2\ge 0\quad\text{or equivalently}\quad \|w\|_2^2\le 4\|u\|_2^2.$$

With the help of the inequality $(a+b)^m\le 2^{m-1}(a^m+b^m)$, $a,b\ge 0$, $m\ge 1$, we can estimate

$$\|w\|_2^2=\sum_{i\in\mathbb{Z}}|w_i|^2=\sum_{i\in\mathbb{Z}}|u_{i-1}+u_{i+1}|^2\le 2\sum_{i\in\mathbb{Z}}\left(|u_{i-1}|^2+|u_{i+1}|^2\right)\le 4\|u\|_2^2.$$

Hence, $(Ku,u)_2\ge 0$, i.e., K is positive operator and therefore, L is negative. □

Next, we focus on the nonlinear Nemytskii (substitution) operator N and prove that it is well-defined and continuous.

Lemma 2.3. *The operator $N:\ell^2\to\ell^2$ defined by (2.2) is continuous on* $\mathrm{Dom}(N)=\ell^2$.

Proof. Firstly, we prove that $\mathrm{Dom}(N)=\ell^2$, i.e., we have to verify that $\|N(u)\|_2<\infty$ for all $u\in\ell^2$. Let $u\in\ell^2$ be arbitrary, then

$$\begin{aligned}\|N(u)\|_2^2=\sum_{i\in\mathbb{Z}}|(N(u))_i|^2 &= \sum_{i\in\mathbb{Z}}|u_i-u_i^3|^2\\ &\le \sum_{i\in\mathbb{Z}}2\left(|u_i|^2+|u_i|^6\right)\\ &= 2\left(\|u\|_2^2+\|u\|_6^6\right)\\ &\le 2\left(\|u\|_2^2+\|u\|_2^6\right)<\infty,\end{aligned}$$

when we use $(a+b)^m \leq 2^{m-1}(a^m + b^m), a, b \geq 0, m \geq 1$, in the first inequality and the embedding $\ell^2 \hookrightarrow \ell^p$, $p \geq 2$, and the fact that for all $u \in \ell^2$ there is $\|u\|_p \leq \|u\|_2$ in the second one.

Secondly, we prove that N is continuous. Let $u^n \to u$ in ℓ^2 (we use upper indices for the sequences in ℓ^2). Therefore, there exists $q > 0$ such that $\|u^n\|_2 \leq q$ for all $n \in \mathbb{N}$. Since u is the limit of the sequence $\{u^n\}$, there is also $\|u\|_2 \leq q$. Further, we use that $\sup_{i\in\mathbb{Z}} |w_i| = \|w\|_\infty \leq \|w\|_2$ for all $w \in \ell^2$ and hence, there is:

$$\sup_{i\in\mathbb{Z}} |u_i^n| \leq \|u^n\|_2 \leq q, \quad \text{i.e.,} \quad |u_i^n| \leq q \quad \text{for all} \quad n \in \mathbb{N}, i \in \mathbb{Z}.$$

Analogically, for the limit u we obtain $|u_i| \leq q$ for all $i \in \mathbb{Z}$. Consequently, we can estimate:

$$\begin{aligned}
\|N(u^n) - N(u)\|_2^2 &= \sum_{i\in\mathbb{Z}} |u_i^n - (u_i^n)^3 - u_i + u_i^3|^2 \\
&\leq \sum_{i\in\mathbb{Z}} 2\left(|u_i^n - u_i|^2 + |(u_i^n)^3 - u_i^3|^2\right) \\
&= 2\|u^n - u\|_2^2 + 2\sum_{i\in\mathbb{Z}} |u_i^n - u_i|^2 \cdot |(u_i^n)^2 + u_i^n u_i + u_i^2|^2 \\
&\leq 2\|u^n - u\|_2^2 + 2\sum_{i\in\mathbb{Z}} |u_i^n - u_i|^2 \cdot 9q^4 \\
&= 2(1 + 9q^4)\|u^n - u\|_2^2
\end{aligned} \tag{2.3}$$

and since the right-hand side converges to zero, $N(u^n) \to N(u)$ in ℓ^2, i.e., the Nemytskii operator N is continuous. □

Going back to the implicit reaction–diffusion equation (1.1), we assume that $\varphi := \{\varphi(x)\}_{x\in\mathbb{Z}} \in \ell^2$ and look for solutions of (1.1) for which $v(\cdot, t) := \{v(x, t)\}_{x\in\mathbb{Z}} \in \ell^2$ for all $t \in h\mathbb{N}_0$. Then the problem (1.1) is equivalent to the difference equation in the Hilbert space ℓ^2:

$$\begin{cases} \Delta_t v(\cdot, t) = L(v(\cdot, t+h)) + \lambda N(v(\cdot, t+h)), & \lambda \in \mathbb{R}, \\ v(\cdot, 0) = \varphi, \end{cases} \tag{2.4}$$

where $\Delta_t v(\cdot, t) = \frac{1}{h}\left(v(\cdot, t+h) - v(\cdot, t)\right)$ and L, N are defined in (2.1), (2.2) respectively.

First, we consider a local problem, i.e., for a fixed $t \in h\mathbb{N}_0$ and a given $v(\cdot, t) \in \ell^2$ (i.e., a priori known) we look for a solution $v(\cdot, t+h) \in \ell^2$ of (2.4) (later, in Sections 4–5 we extend this approach via mathematical induction and analyze a global problem, in which we study the existence of solutions $v(\cdot, t)$ of (2.4) for all time instances $t \in h\mathbb{N}_0$). We can rewrite the equation in (2.4) into

$$v(\cdot, t+h) = v(\cdot, t) + hL(v(\cdot, t+h)) + h\lambda N(v(\cdot, t+h)), \qquad \lambda \in \mathbb{R}.$$

If we denote $b := v(\cdot, t)$, $u := v(\cdot, t+h)$, then the problem (2.4) for a fixed $t \in h\mathbb{N}_0$ is equivalent to a fixed-point problem in ℓ^2:

$$u = b + hL(u) + h\lambda N(u), \qquad \lambda \in \mathbb{R}. \tag{2.5}$$

3. Variational formulation

We proceed from the abstract fixed-point problem formulation (2.5) of the local problem and introduce the variational setting in this section. Problem (2.5) is equivalent to the following operator equation:

$$F(u) := u - b - hL(u) - h\lambda N(u) = o. \tag{3.1}$$

The operator $F : \ell^2 \to \ell^2$ has a potential $\mathcal{F} : \mathrm{Dom}(\mathcal{F}) = \ell^2 \to \mathbb{R}$ given by (we use Lemma 2.1, i.e., that the operator L is self-adjoint):

$$\begin{aligned} \mathcal{F}(u) &:= \frac{1}{2}\sum_{i\in\mathbb{Z}} u_i^2 - \sum_{i\in\mathbb{Z}} b_i u_i - \frac{h}{2}\sum_{i\in\mathbb{Z}}(Lu)_i u_i - \frac{h\lambda}{2}\sum_{i\in\mathbb{Z}} u_i^2 + \frac{h\lambda}{4}\sum_{i\in\mathbb{Z}} u_i^4 \\ &= \frac{1-h\lambda}{2}\|u\|_2^2 - (b,u)_2 - \frac{h}{2}(Lu,u)_2 + \frac{h\lambda}{4}\|u\|_4^4. \end{aligned} \tag{3.2}$$

The next lemma reveals the connection between critical points of the potential $\mathcal{F}$ and solutions of the equation (3.1).

Lemma 3.1. *$\tilde{u} \in \ell^2$ is a critical point of $\mathcal{F}$ given by (3.2) if and only if $\tilde{u}$ is the solution of (3.1).*

Proof. The functional $\mathcal{F}$ given by (3.2) is differentiable and for each $w \in \ell^2$ we have

$$\begin{aligned} (\nabla\mathcal{F}(u), w)_2 &= \sum_{i\in\mathbb{Z}} u_i w_i - \sum_{i\in\mathbb{Z}} b_i w_i - h\sum_{i\in\mathbb{Z}}(Lu)_i w_i - h\lambda\sum_{i\in\mathbb{Z}} u_i w_i + h\lambda\sum_{i\in\mathbb{Z}} u_i^3 w_i \\ &= (u - b - hL(u) - h\lambda N(u), w)_2\,. \end{aligned} \tag{3.3}$$

Therefore, $\tilde{u}$ is a critical point of $\mathcal{F}$ if and only if $\nabla\mathcal{F}(\tilde{u}) = o$, i.e., if $\tilde{u} - b - hL(\tilde{u}) - h\lambda N(\tilde{u}) = o$. □

Consequently, we look for critical points of the potential $\mathcal{F}$. Our analysis is based on the following two variational principles.

Theorem 3.2. *(See [6, Theorem 7.2.11, Proposition 7.1.8].) Let M be a closed, nonempty, bounded and convex subset of a Hilbert space H. Let $\mathcal{F}$ be a convex and continuous functional on M. Then $\mathcal{F}$ is bounded below and there exists $\tilde{u} \in M$ such that $\mathcal{F}(\tilde{u}) = \inf_{u\in M} \mathcal{F}(u)$. Moreover, if $\mathcal{F}$ is strictly convex and Gâteaux differentiable on M and $\tilde{u} \in \mathrm{Int}(M)$ then $\tilde{u}$ is the unique critical point of $\mathcal{F}$ on M.*

Theorem 3.3. *(See [6, Theorem 7.2.12, Proposition 7.1.8].) Let $\mathcal{F}$ be a convex, continuous and weakly coercive functional on a Hilbert space H. Then $\mathcal{F}$ is bounded below on H and there exists $\tilde{u} \in H$ such that $\mathcal{F}(\tilde{u}) = \inf_{u\in H} \mathcal{F}(u)$. Moreover, if $\mathcal{F}$ is strictly convex and Gâteaux differentiable on H then $\tilde{u}$ is the unique critical point of $\mathcal{F}$ on H.*

In order to apply Theorems 3.2 and 3.3 we study the properties of the potential $\mathcal{F}$ given by (3.2). First, let us study the part of the functional $\mathcal{F}$ given by the linear operator L:

$$\mathcal{A} : \ell^2 \to \mathbb{R}, \quad \mathcal{A}(u) := -(Lu, u)_2. \tag{3.4}$$

Lemma 3.4. *The functional $\mathcal{A} : \ell^2 \to \mathbb{R}$ given by (3.4) is convex on ℓ^2.*

Proof. First, let us notice that the bounded linear operator L is self-adjoint (Lemma 2.1) and negative (Lemma 2.2), i.e., the bounded linear operator $A := -L$ is self-adjoint and positive. We can rewrite the definition of the quadratic functional $\mathcal{A}$ as follows:

$$\mathcal{A}(u) = -(Lu, u)_2 = ((-L)u, u)_2 = (Au, u)_2.$$

The functional $\mathcal{A}$ is continuously differentiable and its gradient is given by:

$$\nabla \mathcal{A}(u) = 2A(u).$$

From the positivity of the linear operator A we obtain:

$$(\nabla \mathcal{A}(u) - \nabla \mathcal{A}(w), u - w)_2 = 2(A(u - w), u - w)_2 \geq 0$$

for all $u, w \in \ell^2$ and therefore, the gradient $\nabla \mathcal{A}$ is the monotone mapping. Consequently, the operator $\mathcal{A}$ is convex on ℓ^2 (see, e.g., [6, Exercise 7.2.30]). $\square$

Since we want to apply Theorems 3.2 and 3.3, we need the continuity of the potential $\mathcal{F}$. We show that $\mathcal{F}$ is even continuously differentiable.

Lemma 3.5. *The potential $\mathcal{F} : \ell^2 \to \mathbb{R}$ given by (3.2) is continuously differentiable on ℓ^2.*

Proof. The functional $\mathcal{F}$ is differentiable from Lemma 3.1. Therefore, we prove that the gradient $\nabla \mathcal{F} : \ell^2 \to \ell^2$ is a continuous mapping. From (3.3) there is:

$$\nabla \mathcal{F}(u) = u - b - hLu - h\lambda N(u).$$

The fact that $L \in \mathcal{L}(\ell^2)$ and Lemma 2.3 yield that $\nabla \mathcal{F}$ is continuous and hence, $\mathcal{F}$ is continuously differentiable on ℓ^2. $\square$

4. Existence and uniqueness for $\lambda \geq 0$

In this section we consider the Nagumo equation (1.1) with the bistable nonlinearity, i.e., the case $\lambda \geq 0$. We prove the global existence and uniqueness for small values of time discretization step $h > 0$ with the help of Theorem 3.3.

Theorem 4.1. *Let $\lambda \geq 0$ and assume $h(\lambda + 4k) < 1$ and $\varphi \in \ell^2$. Then the problem (1.1) has a unique solution $v(x,t)$ that exists for all $x \in \mathbb{Z}$, $t \in h\mathbb{N}_0$ and satisfies*

$$\left(\sum_{x \in \mathbb{Z}} |v(x,t)|^2 \right)^{\frac{1}{2}} < \infty \quad \textit{for all} \quad t \in h\mathbb{N}_0.$$

Proof. We prove the statement by mathematical induction. For $t = 0$ we put $v(x,0) = \varphi(x)$ for all $x \in \mathbb{Z}$, i.e., $v(\cdot,0) = \varphi \in \ell^2$. Let us assume that we have a unique solution $v(\cdot,t)$ at time $t \in h\mathbb{N}_0$ satisfying $\|v(\cdot,t)\|_2 < \infty$. Let us prove the existence of a unique solution of (1.1) at time $t + h$. As we showed in the previous section, this local problem is equivalent to finding critical points of the associated potential $\mathcal{F}$ given by (3.2) (we use the same notation, e.g., $u = v(\cdot, t+h)$, $b = v(\cdot,t)$, etc.).

Recall that the potential $\mathcal{F}$ given by

$$\mathcal{F}(u) = \frac{1 - h\lambda}{2} \|u\|_2^2 - (b,u)_2 - \frac{h}{2}(Lu,u)_2 + \frac{h\lambda}{4}\|u\|_4^4, \quad \lambda \geq 0,$$

is continuously differentiable (see Lemma 3.5) and it is strictly convex on ℓ^2. Indeed,

- $\frac{1-h\lambda}{2}\|u\|_2^2 = \frac{1-h\lambda}{2} \sum\limits_{i \in \mathbb{Z}} u_i^2$ is strictly convex since $h\lambda - 1 < -4hk < 0$ and the real function $t \mapsto t^2$ is strictly convex,
- $-(b,u)_2$ is convex since it is the linear form,

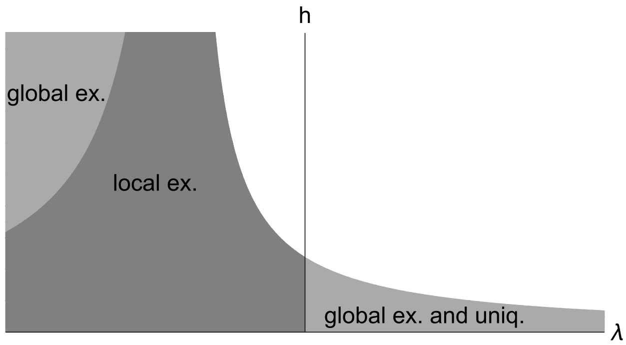

Fig. 1. Graphical illustration of existence and uniqueness results for implicit Nagumo RDE (1.1). See Table 1 for more details.

- $-\frac{h}{2}(Lu,u)_2 = \frac{h}{2}\mathcal{A}(u)$ is convex since $\frac{h}{2} > 0$ and $\mathcal{A}$ is convex from Lemma 3.4,
- $\frac{h\lambda}{4}\|u\|_4^4 = \frac{h\lambda}{4}\sum_{i\in\mathbb{Z}} u_i^4$ is convex since $h\lambda \geq 0$ and the real function $t \mapsto t^4$ is strictly convex.

Moreover, $\mathcal{F}$ is weakly coercive on ℓ^2 since for $u^n \to u$ in ℓ^2 the following holds:

$$\mathcal{F}(u) \geq \frac{1-h\lambda-h\|L\|_*}{2}\|u\|_2^2 - \|b\|_2\|u\|_2 = \frac{1-h(\lambda+4k)}{2}\|u\|_2^2 - \|b\|_2\|u\|_2 \to \infty,$$

where we use the Cauchy–Schwarz inequality, $\|L\|_* = 4k$ and $h(\lambda+4k) < 1$.

Consequently, from Theorem 3.3 there exists a unique minimizer $\tilde{u} \in \ell^2$ of the potential $\mathcal{F}$ and it is the unique critical point of $\mathcal{F}$ on ℓ^2. Hence, there exists the unique solution $v(x,t+h)$ of (1.1) at time $t+h$ such that

$$v(\cdot,t+h) = \tilde{u} \quad \text{and} \quad \left(\sum_{x\in\mathbb{Z}} |v(x,t+h)|^2\right) = \|v(\cdot,t+h)\|_2 = \|\tilde{u}\|_2 < \infty. \qquad \square$$

Remark 4.2. We note that for given $\lambda \geq 0$ and $k > 0$ there always exist sufficiently small values of time discretization step $h > 0$ which satisfy $h(\lambda+4k) < 1$, see Fig. 1. Obviously, the stronger reaction (or the stronger diffusion) the smaller $h > 0$ is required.

5. Existence for $\lambda < 0$

For negative values of λ in (1.1) we lose the globally convex and weakly coercive geometry of the potential $\mathcal{F}$. We prove that in this case the functional $\mathcal{F}$ is convex at least locally (i.e., on a closed ball). Consequently, we use Theorem 3.2 to obtain the existence of a local minimizer of $\mathcal{F}$. Moreover, in Section 6 we show that $\mathcal{F}$ has the mountain pass geometry.

For the sake of brevity, let us define the auxiliary real valued function $\xi : \mathbb{R} \to \mathbb{R}$:

$$\xi(s) := \frac{1-h\lambda-4hk}{2}s + \frac{h\lambda}{4}s^3 \tag{5.1}$$

and the positive constant (assuming that $\lambda < 0$ and $h(\lambda+4k) < 1$):

$$R := \min\left\{\left(\frac{h\lambda-1}{3h\lambda}\right)^{\frac{1}{2}}, \left(\frac{2(4hk+h\lambda-1)}{3h\lambda}\right)^{\frac{1}{2}}\right\}. \tag{5.2}$$

Lemma 5.1. *Let $\lambda < 0$, $h(\lambda+4k) < 1$ and*

$$\|b\|_2 < \xi(R). \tag{5.3}$$

Then the functional $\mathcal{F}$ given by (3.2) has a local minimizer $\tilde{u} \in \ell^2$ which is the unique critical point of $\mathcal{F}$ with the property

$$\|\tilde{u}\|_2 < R. \tag{5.4}$$

Proof. Let us consider the ball $M := \overline{B(o,r)} \subset \ell^2$ with $r = \left(\frac{h\lambda-1}{3h\lambda}\right)^{\frac{1}{2}}$ and verify the assumptions of Theorem 3.2. We show that $\mathcal{F}$ is strictly convex on M. From its definition (3.2) we rewrite $\mathcal{F}$ as

$$\mathcal{F}(u) = \sum_{i\in\mathbb{Z}} \left(\frac{1-h\lambda}{2} u_i^2 + \frac{h\lambda}{4} u_i^4 \right) - (b,u)_2 - \frac{h}{2}(Lu,u)_2. \tag{5.5}$$

Observing the sum we define the function

$$\psi(s) := \frac{1-h\lambda}{2} s^2 + \frac{h\lambda}{4} s^4. \tag{5.6}$$

Recalling that $\lambda < 0$ and $h(\lambda + 4k) < 1$ and differentiating twice we deduce that ψ is strictly convex on the interval $I = \left[-\left(\frac{h\lambda-1}{3h\lambda}\right)^{\frac{1}{2}}, \left(\frac{h\lambda-1}{3h\lambda}\right)^{\frac{1}{2}} \right] = [-r, r]$. Since $\|u\|_\infty \le \|u\|_2$ for all $u \in \ell^2$, there is

$$\sup_{i\in\mathbb{Z}} |u_i| = \|u\|_\infty \le \|u\|_2 \le \left(\frac{h\lambda - 1}{3h\lambda} \right)^{\frac{1}{2}} = r \quad \text{for all} \quad u \in M.$$

Applying the strict convexity of the function ψ defined by (5.6) on I, the convexity of the linear form $-(b,u)_2$ and the convexity of the functional $\mathcal{A}(u) = -(Lu,u)_2$ (Lemma 3.4) we observe that $\mathcal{F}$ defined by (5.5) is strictly convex (using the same arguments as in the proof of Theorem 4.1).

Moreover, $\mathcal{F} \in C^1(\ell^2, \mathbb{R})$ from Lemma 3.5. The strict convexity and continuity of $\mathcal{F}$ on M imply (see Theorem 3.2) that for all $\rho \in (0, r]$ there exists a unique $\tilde{u}(\rho) \in \overline{B(o,\rho)} \subset \ell^2$ such that $\mathcal{F}(\tilde{u}(\rho)) = \inf_{u \in \overline{B(o,\rho)}} \mathcal{F}(u)$.

Secondly, we show that $\tilde{u}(\rho)$ (for some $\rho \in (0, r]$) is a critical point of $\mathcal{F}$ by excluding the possibility that $\tilde{u}(\rho)$ lies on the boundary $\partial B(o,\rho)$, possibly for maximal $\|b\|_2$. Since $\mathcal{F}(o) = 0$, it suffices to show that $\mathcal{F}(u) > 0$ for all $u \in \partial B(o,\rho)$, i.e., for all $u \in \ell^2$ such that $\|u\|_2 = \rho$.

Applying the Cauchy–Schwarz inequality and the fact that $\|u\|_4 \le \|u\|_2$ for all $u \in \ell^2$ we can estimate $\mathcal{F}(u)$ on $\partial B(o,\rho)$:

$$\begin{aligned}
\mathcal{F}(u) &= \frac{1-h\lambda}{2} \|u\|_2^2 - (b,u)_2 - \frac{h}{2}(Lu,u)_2 + \frac{h\lambda}{4}\|u\|_4^4 \\
&\ge \frac{1-h\lambda-h\|L\|_*}{2} \|u\|_2^2 - \|b\|_2 \|u\|_2 + \frac{h\lambda}{4}\|u\|_2^4 \\
&= \frac{1-h\lambda-4hk}{2} \rho^2 - \|b\|_2 \rho + \frac{h\lambda}{4} \rho^4 \\
&= \rho \left(\frac{1-h\lambda-4hk}{2} \rho - \|b\|_2 + \frac{h\lambda}{4}\rho^3 \right).
\end{aligned} \tag{5.7}$$

Consequently, $\mathcal{F}(u) \ge \rho \left(\frac{1-h\lambda-4hk}{2} \rho - \|b\|_2 + \frac{h\lambda}{4}\rho^3 \right) > 0$ for all $u \in \partial B(o,\rho)$ if

$$\|b\|_2 < \frac{1-h\lambda-4hk}{2}\rho + \frac{h\lambda}{4}\rho^3 \overset{(5.1)}{=} \xi(\rho).$$

Analyzing the function ξ defined by (5.1) we observe that it has a positive maximum $\xi(\rho_{\max})$ at

$$\rho_{\max} = \left(\frac{2(4hk + h\lambda - 1)}{3h\lambda}\right)^{\frac{1}{2}}. \tag{5.8}$$

Moreover, it is strictly increasing for $\rho \in [0, \rho_{\max}]$ and strictly decreasing for $\rho \in [\rho_{\max}, \infty)$. Since we do not know if $r \le \rho_{\max}$ or $r > \rho_{\max}$, we have to assume (5.3):

$$\|b\|_2 < \xi(\min\{r, \rho_{\max}\}) \overset{(5.2)}{=} \xi(R).$$

Consequently, $\tilde{u}(R) \in \mathrm{Int}(B(o, R))$ and therefore, it is a local minimizer of $\mathcal{F}$ and the unique critical point of $\mathcal{F}$ on $\overline{B(o, R)}$ (see Theorem 3.2 again). $\square$

Lemma 5.1 provides a sufficient condition for the existence of a local solution of (1.1) with $\lambda < 0$.

Theorem 5.2. *Let $\lambda < 0$ and assume $h(\lambda + 4k) < 1$ and $v(x,t)$ is a solution of (1.1) at a fixed time $t \in h\mathbb{N}_0$ such that*

$$\left(\sum_{x\in\mathbb{Z}} |v(x,t)|^2\right)^{\frac{1}{2}} < \xi(R).$$

Then there exists a solution $v(x, t+h)$ of the problem (1.1) at time $t+h$ such that

$$\left(\sum_{x\in\mathbb{Z}} |v(x,t+h)|^2\right)^{\frac{1}{2}} < R.$$

Proof. The existence of such solution is equivalent to the existence of a critical point of the potential $\mathcal{F}$ defined by (3.2) which follows from Lemma 5.1. $\square$

We extend this result, under an additional assumption on h and λ, to the global existence.

Theorem 5.3. *Let $\lambda < 0$ and assume $h(\lambda + 4k) \le -2$ and $\varphi \in \ell^2$ satisfies*

$$\|\varphi\|_2 < \xi(R).$$

Then the problem (1.1) has a solution $v(x,t)$ that exists for all $x \in \mathbb{Z}$, $t \in h\mathbb{N}_0$ and is unique with the property

$$\left(\sum_{x\in\mathbb{Z}} |v(x,t)|^2\right)^{\frac{1}{2}} < R \quad \textit{for all} \quad t \in h\mathbb{N}. \tag{5.9}$$

Proof. Since $\varphi \in \ell^2$, we can apply Theorem 5.2 to get a solution $v(x,h)$ at time $t = h$ that satisfies the inequality in (5.9) (observe that since $h(\lambda+4k) \le -2$ the assumption on λ and h in Theorem 5.2 is satisfied).

Next, we proceed by mathematical induction. Let us assume that we have a solution $v(x,t)$ at a fixed time t satisfying (5.9). To prove the existence of $v(x, t+h)$ at time $t+h$ we need $\|v(\cdot,t)\|_2 < \xi(R)$. The induction hypothesis implies that $\|v(\cdot,t)\|_2 < R$. Since $\rho_{\max} \le \xi(\rho_{\max})$ for $h > 0$, $\lambda < 0$ satisfying $h(\lambda + 4k) \le -2$ ($\rho_{\max}$ is given by (5.8)) and the function ξ given by (5.1) is concave on $[0, \infty)$ then:

$$s \le \xi(s) \quad \text{for} \quad s \in [0, R].$$

Consequently, $\|v(\cdot,t)\|_2 < R \le \xi(R)$. Thus, Theorem 5.2 implies that there exists a solution $v(x, t+h)$ which is unique with the property (5.9). $\square$

Remark 5.4. For the illustration of admissible values of $h > 0$ and $\lambda < 0$ in Theorem 5.2 and Theorem 5.3 see Fig. 1 again.

6. Mountain pass geometry and conjectures about multiplicity for $\lambda < 0$

In addition to the local convexity shown in the previous section, we show that the potential $\mathcal{F}$ has the mountain pass geometry for $\lambda < 0$. A natural question arises – can we apply the Mountain Pass Theorem for proving the existence of another critical point of $\mathcal{F}$, which would imply the existence of another solution of (1.1)?

Theorem 6.1 *(Mountain Pass Theorem). (See Ambrosetti, Rabinowitz [1], [6, Theorem 7.4.5].) Let H be a Hilbert space and $\mathcal{F} \in C^1(H, \mathbb{R})$, $e \in H$ and $\rho > 0$ be such that $\|e\|_H > \rho$ and*

$$\inf_{\|u\|_H = \rho} \mathcal{F}(u) > \mathcal{F}(o) \geq \mathcal{F}(e). \tag{6.1}$$

Let

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{F}(\gamma(t)) \quad \text{and} \quad \Gamma := \{\gamma \in C([0,1], H) : \; \gamma(0) = o, \; \gamma(1) = e\}.$$

Let $\mathcal{F}$ satisfy the Palais–Smale condition on the level c:

$(\mathrm{PS})_c$ *Any sequence $\{u^n\} \subset H$ such that*

$$\mathcal{F}(u^n) \to c \in \mathbb{R}, \quad \nabla \mathcal{F}(u^n) \to o \in H,$$

has a convergent subsequence in the norm of H.

Then c is a critical value of $\mathcal{F}$.

We conjecture that in the case of $\lambda < 0$ the functional $\mathcal{F}$ has at least two critical points.

Conjecture 6.2. *Let $\lambda < 0$ and assume $h(\lambda + 4k) < 1$ and $b \in \ell^2$ satisfy (5.3). Then the functional $\mathcal{F}$ given by (3.2) has at least two critical points.*

The existence of the first critical point of $\mathcal{F}$ follows from Lemma 5.1. We want to show that $\mathcal{F}$ has another critical point which is of the mountain pass-type. Theorem 6.1 has three assumptions – the continuous differentiability of $\mathcal{F}$, the mountain pass geometry (6.1) and the Palais–Smale condition $(\mathrm{PS})_c$. Firstly, the continuous differentiability of $\mathcal{F}$ has already been proven in Lemma 3.5. Secondly, the following statement shows that under the assumptions of Lemma 5.1 the potential $\mathcal{F}$ has the mountain pass geometry.

Lemma 6.3. *Let $\lambda < 0$ and assume $h(\lambda + 4k) < 1$ and $b \in \ell^2$ satisfy (5.3). Then there exist $e \in \ell^2$ and $\rho > 0$ such that $\|e\|_2 > \rho$ and the functional $\mathcal{F}$ given by (3.2) satisfies (6.1).*

Proof. From the proof of Lemma 5.1 (see (5.7) and below) we get:

$$\mathcal{F}(u) \geq R\left(\frac{1 - h\lambda - 4hk}{2} R - \|b\|_2 + \frac{h\lambda}{4} R^3\right) =: a > 0 \quad \text{for} \quad u \in \partial B(o, R). \tag{6.2}$$

Let $e \in \ell^2$ be such that:

$$e_i = \begin{cases} s > 0, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

Consequently, for $s \to \infty$ there is (recall that $\lambda < 0$):

$$\mathcal{F}(e) = \frac{1}{2}s^2 - b_0 s + hks^2 - \frac{h\lambda}{2}s^2 + \frac{h\lambda}{4}s^4 = \frac{1 + hk - h\lambda}{2}s^2 - b_0 s + \frac{h\lambda}{4}s^4 \to -\infty.$$

Hence, for sufficiently large $s > R > 0$ there is $\mathcal{F}(e) < 0$, i.e., if we put $\rho := R$ then $\|e\|_2 = s > \rho$ and from (6.2):

$$\inf_{\partial B(o,\rho)} \mathcal{F}(u) \geq a > 0 = \mathcal{F}(o) > \mathcal{F}(e). \qquad \square$$

The main obstacle in the application of Theorem 6.1 in this case is to prove its third assumption, the Palais–Smale condition $(\mathrm{PS})_c$. This condition requires that all sequences $\{u^n\} \subset \ell^2$ satisfying:

$$\mathcal{F}(u^n) \to c \in \mathbb{R} \quad \text{and} \quad \nabla\mathcal{F}(u^n) \to o \in \ell^2, \tag{6.3}$$

contain a strongly convergent subsequence. Usually, the proof proceeds in two steps. Firstly, one proves that $\{u^n\}$ contains a bounded subsequence. Secondly, since a Hilbert space is considered, one passes to a weakly convergent subsequence and shows that it converges strongly as well.

The first step, showing that all sequences $\{u^n\}$ satisfying (6.3) contain a bounded subsequence, is not too complicated.

Lemma 6.4. *Let $\lambda < 0$, $h(\lambda + 4k) < 1$, $b \in \ell^2$ and the functional $\mathcal{F}$ be given by (3.2). Then every sequence $\{u^n\} \subset \ell^2$ satisfying (6.3) contains a bounded subsequence.*

Proof. Let the sequence $\{u^n\} \subset \ell^2$ satisfy (6.3). Hence, for a given $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ there is:

$$\mathcal{F}(u^n) < c + \varepsilon \quad \text{and} \quad \|\nabla\mathcal{F}(u^n)\|_2 \leq 1. \tag{6.4}$$

Therefore, from the Cauchy–Schwarz inequality we obtain that for all $n > \bar{n}$ there is:

$$\frac{1}{4}|(\nabla\mathcal{F}(u^n), u^n)_2| \leq \|\nabla\mathcal{F}(u^n)\|_2 \|u^n\|_2 \leq \|u^n\|_2. \tag{6.5}$$

We use (6.4) and (6.5) to estimate (again $n > \bar{n}$):

$$\begin{aligned} c + \varepsilon + \|u^n\|_2 &\geq \mathcal{F}(u^n) - \frac{1}{4}(\nabla\mathcal{F}(u^n), u^n)_2 \\ &= \frac{1}{2}\|u^n\|_2^2 - (b, u^n)_2 - \frac{h}{2}(Lu^n, u^n)_2 - \frac{h\lambda}{2}\|u^n\|_2^2 + \frac{h\lambda}{4}\|u^n\|_4^4 \\ &\quad - \frac{1}{4}\|u^n\|_2^2 + \frac{1}{4}(b, u^n)_2 + \frac{h}{4}(Lu^n, u^n)_2 + \frac{h\lambda}{4}\|u^n\|_2^2 - \frac{h\lambda}{4}\|u^n\|_4^4 \\ &= \frac{1 - h\lambda}{4}\|u^n\|_2^2 - \frac{3}{4}(b, u^n)_2 - \frac{h}{4}(Lu^n, u^n)_2 \\ &\geq \frac{1 - h\lambda - 4hk}{4}\|u^n\|_2^2 - \frac{3}{4}\|b\|_2\|u^n\|_2, \end{aligned}$$

which is equivalent to

$$c + \varepsilon + \left(1 + \frac{3}{4}\|b\|_2\right)\|u^n\|_2 \geq \frac{1 - h\lambda - 4hk}{4}\|u^n\|_2^2$$

where $\frac{1-h\lambda-4hk}{4} > 0$ from the assumption $h(\lambda + 4k) < 1$. If we assume, without loss of generality, that $c + \varepsilon > 0$, then the quadratic function $\frac{1-h\lambda-4hk}{4}\|u^n\|_2^2$ is bounded from above by the linear function $c + \varepsilon + \left(1 + \frac{3}{4}\|b\|_2\right)\|u^n\|_2$ for $n > \bar{n}$ and therefore, $\|u^n\|_2$ is bounded. $\square$

Remark 6.5. Lemma 6.4 yields that every sequence $\{u^n\} \subset \ell^2$ satisfying (6.3) contains a bounded subsequence. Since we are on the separable Hilbert space ℓ^2 we can pass to a weakly convergent subsequence (denoted for simplicity by $\{u^n\}$ as well) satisfying

$$u^n \rightharpoonup u \in \ell^2. \tag{6.6}$$

Typical mountain-pass arguments at this stage exploit the convergence of either $(\nabla\mathcal{F}(u^n) - \nabla\mathcal{F}(u), u^n - u)_2$ or $(\nabla\mathcal{F}(u^n) - \nabla\mathcal{F}(u^m), u^n - u^m)_2$. The analysis of these expressions could yield the convergence of $\|u^n - u\|_2 \to 0$ or alternatively

$$\|u^n\|_2 \to \|u\|_2. \tag{6.7}$$

In turn, the weak convergence (6.6) and the convergence of norms (6.7) would imply the desired convergence $u^n \to u$.

In the case of the functional $\mathcal{F}$ defined by (3.2) we observe that

$$\begin{aligned}
(\nabla\mathcal{F}(u^n) - \nabla\mathcal{F}(u), u^n - u)_2 &= (1 - h\lambda)(u^n, u^n - u)_2 - (b, u^n - u)_2 \\
&\quad - h(Lu^n, u^n - u)_2 + h\lambda \sum_{i\in\mathbb{Z}} (u_i^n)^3 (u_i^n - u_i) \\
&\quad - (1 - h\lambda)(u, u^n - u)_2 + (b, u^n - u)_2 \\
&\quad + h(Lu, u^n - u)_2 - h\lambda \sum_{i\in\mathbb{Z}} u_i^3 (u_i^n - u_i) \\
&= (1 - h\lambda)\|u^n - u\|_2^2 - h(L(u^n - u), u^n - u)_2 \\
&\quad + h\lambda \sum_{i\in\mathbb{Z}} \left((u_i^n)^3 - u_i^3\right)(u_i^n - u_i).
\end{aligned}$$

Using the fact that the linear bounded operator L is negative (see Lemma 2.2) we can estimate:

$$(1 - h\lambda)\|u^n - u\|_2^2 \leq (\nabla\mathcal{F}(u^n) - \nabla\mathcal{F}(u), u^n - u)_2 - h\lambda \sum_{i\in\mathbb{Z}} \left((u_i^n)^3 - u_i^3\right)(u_i^n - u_i). \tag{6.8}$$

Unfortunately, we are unable to show that the last term tends to zero (note that the term is nonnegative due to the fact that $\lambda < 0$ and the nonnegativity of each term $\left((u_i^n)^3 - u_i^3\right)(u_i^n - u_i)$ in the sum). Similarly, we are unable to use the estimate

$$(1 - h\lambda + h\lambda K)\|u^n - u\|_2^2 \leq (\nabla\mathcal{F}(u^n) - \nabla\mathcal{F}(u), u^n - u)_2, \tag{6.9}$$

where K is a bound on the norm of the sequence $\left\{(u_i^n)^2 + u_i^n u_i + u_i^2\right\}$. The estimate (6.9) then follows from (6.8) once we use the equality

$$\left((u_i^n)^3 - u_i^3\right)(u_i^n - u_i) = (u_i^n - u_i)^2 \left((u_i^n)^2 + u_i^n u_i + u_i^2\right).$$

Table 1
Summary of our results for implicit Nagumo RDE (1.1), also see Fig. 1.

λ	$\lambda < 0$		$\lambda \geq 0$	
	$(-\infty, -\frac{2}{h} - 4k]$	$(-\frac{2}{h} - 4k, \frac{1}{h} - 4k)$	$[0, \frac{1}{h} - 4k)$	$[\frac{1}{h} - 4k, \infty)$
Geometry of $\mathcal{F}$	mountain pass	mountain pass	globally convex	?
Existence	global (Theorem 5.3)	local (Theorem 5.2)	global (Theorem 4.1)	?
Uniqueness	? (Conjecture 6.7)	? (Conjecture 6.7)	yes (Theorem 4.1)	?

Despite the first impressions it is not clear whether we can ensure that the term $(1 - h\lambda + h\lambda K)$ is positive because $h > 0$ and $\lambda < 0$ are given and the bound K depends both on λ and h (via the geometry of the functional $\mathcal{F}$). Consequently, the open question is whether a finer analysis of the interplay among K, λ and h can yield the positivity of the term $(1 - h\lambda + h\lambda K)$, at least for some values of h, λ (see Fig. 1). The estimate (6.9) would then directly yield $\|u^n - u\|_2 \to 0$.

Therefore, the following conjecture remains open and essential to the proof of the existence of another critical point of $\mathcal{F}$.

Conjecture 6.6. *Let $\lambda < 0$, $h(\lambda + 4k) < 1$, $b \in \ell^2$ and the functional $\mathcal{F}$ be given by (3.2). Then every sequence satisfying (6.3) contains a strongly convergent subsequence.*

From another point of view, note that the difficulty is caused by the fact that the space variable x in the problem (1.1) is from the unbounded domain, i.e., $x \in \mathbb{Z}$. In the abstract formulation it means that the underlying Hilbert space (ℓ^2 in our case) is infinite-dimensional. If we solved the initial–boundary value problem, i.e., the problem (1.1) with $x \in [a, b] \cap \mathbb{Z}$ and with some boundary conditions at points $x = a$ and $x = b$, the abstract problem would be finite-dimensional and the proof of the Palais–Smale condition would be restricted to the proof of boundedness of $\{u^n\}$ (since in the finite-dimensional space every bounded sequence contains a convergent subsequence) which is done in Lemma 6.4.

Finally, note that if Conjecture 6.2 holds then the problem (1.1) has at least two solutions. We sum up this in the following conjecture.

Conjecture 6.7. *Let $\lambda < 0$, $h(\lambda + 4k) < 1$ and $v(x, t)$ be a solution of (1.1) at a fixed time $t \in h\mathbb{N}_0$ such that:*

$$\left(\sum_{x \in \mathbb{Z}} |v(x, t)|^2 \right)^{\frac{1}{2}} < \xi(R).$$

Then the problem (1.1) has at least two solutions $v_1(x, t + h)$, $v_2(x, t + h)$ at time $t + h$ such that:

$$\left(\sum_{x \in \mathbb{Z}} |v_j(x, t + h)|^2 \right)^{\frac{1}{2}} < \infty, \quad j = 1, 2.$$

7. Conclusion and open problems

We studied implicit discretization of the Nagumo RDE via variational methods. Our results (which are summed up in Table 1 and illustrated in Fig. 1) leaned on the geometry of the corresponding potential $\mathcal{F}$ which is convex for the bistable case $\lambda > 0$ and has the mountain pass geometry for the case $\lambda < 0$.

There are several open questions related to our conclusions:

1. In the case $\lambda < 0$, can we prove the existence of another solution at least for some values of $h > 0$? Or, can we obtain nonuniqueness if we study the problem in weighted sequence spaces which have been used, e.g., in [3]?
2. We have no results for the case in which $\lambda \geq \frac{1}{h} - 4k$. What is the geometry of the potential in this case?
3. Is there a global solution for $\lambda < 0$ and $h \to 0+$?
4. Can our results be extended to general RDE with other nonlinearities?
5. It is known (e.g., [5]) that in certain cases the existence of sub- and supersolutions ensures a special variational structure. Can this connection be established for implicit RDEs and used to obtain improved results without assumptions on the initial condition $\varphi(x)$?

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APPENDIX F

Landesman-Lazer conditions for difference equations involving sublinear perturbations

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Landesman–Lazer conditions for difference equations involving sublinear perturbations

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ABSTRACT

We study the existence and uniqueness for discrete Neumann and periodic problems. We consider both ordinary and partial difference equations involving sublinear perturbations. All the proofs are based on reformulating these discrete problems as a general singular algebraic system. Firstly, we use variational techniques (specifically, the Saddle Point Theorem) and prove the existence result based on a type of Landesman–Lazer condition. Then we show that for a certain class of bounded nonlinearities this condition is even necessary and therefore, we specify also the cases in which there does not exist any solution. Finally, the uniqueness is discussed.

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1. Introduction

We investigate the existence and uniqueness for discrete Neumann and periodic problems. We analyze both ordinary and partial difference equations. The motivation for the study of discrete problems is miscellaneous. For example, the discrete approach is quite natural in economic, social or biological modelling – we can mention population dynamics or analyzing processes in cell (e.g. neural) nets, etc., [3]. Naturally, the difference equations are important for the numerical reasons, since they arise from the differential equations for example by the finite difference method, [8].

Our results are based on reformulating discrete boundary value problems as an algebraic system

$$Au = G(u), \quad u \in \mathbb{R}^N, \tag{P}$$

where the matrix A represents corresponding linear difference operator with particular boundary conditions and G is the superposition vector function representing a nonlinear perturbation. Generally, Dirichlet problems correspond to systems with regular positive definite matrices [12,21], Neumann and periodic problems involve singular positive semi-definite matrices [18,20].

There are many papers that deal with the existence and uniqueness for discrete Neumann and periodic problems. A lot of these works use topological approach. Let us mention, e.g. [1,5–7], where the authors present a nice application of the Brouwer topological degree or Brouwer fixed-point theorem together with the method of lower and

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upper solutions. They obtain conditions for the existence and multiplicity of solutions for problems involving one dimensional discrete Laplacian or ϕ-Laplacian. Moreover, the papers [5–7] contain versions of Landesman–Lazer type conditions (firstly studied by Landesman and Lazer [14] for elliptic partial differential equations in resonance).

Our approach is motivated by [18], where the author shows a nice comparison of topological and variational methods for discrete periodic problems and emphasizes advantages of analyzing these problems variationally. The expanding literature shows that the critical point theory is generally an efficient tool in the analysis of discrete problems. One can see the application of variational techniques for ordinary difference equations in [12,18], for problems involving discrete p-Laplacian in [4,9,19], and for partial difference equations in [2,15].

Many works mentioned above investigate particular boundary value problems, in other words, use a concrete form of the matrix A in (P). Nonetheless, we prefer the general representation (P) and apply the properties of A that are common for all possible choices - ordinary/partial difference equation, Neumann/periodic boundary conditions. Let us note that our results can also be applied for the so-called difference equations on graphs. On the other hand, the formulation via (P) does not allow the investigation of problems involving nonlinear difference operators like p- or general ϕ-Laplacian.

Consequently, we study (P) with a positive semi-definite A and with a nonlinear function G which has sublinear growth (Section 2). We apply the Saddle Point Theorem which is due to P.H. Rabinowitz [16] to prove the existence result based on a type of Landesman–Lazer condition (Sections 3–4). Furthermore, if we restrict ourselves to a certain class of bounded nonlinearities, we show that this condition is even necessary. Therefore, we also specify the cases in which there does not exist any solution (Section 5). Finally, we discuss the uniqueness (Section 6).

2. Problem formulation

We formulate in examples below discrete Neumann and periodic problems as the algebraic system (P) on $\mathbb{R}^N$, $N \geq 2$. We show that for these problems the appropriate matrices A satisfy the following general conditions (see [15]):

- (A_1) A is a symmetric and positive semi-definite matrix.
- (A_2) $\lambda_1 = 0$ is an eigenvalue of A with the multiplicity one.
- (A_3) $\varphi_1 = \dagger 1, 1, \dots, 1 \ddagger^{\mathrm{T}} \in \mathbb{R}^N$ is the eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$.

Example 2.1: Let us consider the discrete Neumann problem

$$\begin{cases} -\Delta^2 u(t-1) = \tilde{g}(t, u(t)), \quad t = 1, 2, \dots, N, \\ \Delta u(0) = c_1, \\ \Delta u(N) = c_2, \end{cases} \tag{2.1}$$

where $u : \{0, 1, \dots, N, N+1\} \to \mathbb{R}$, $\Delta^2 u(t-1) = u(t-1) - 2u(t) + u(t+1)$ is the second central difference of u, $\Delta u(t) = u(t+1) - u(t)$ is the first forward difference of u, $\tilde{g} : \{1, 2, \dots, N\} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $c_1, c_2 \in \mathbb{R}$.

We use the values $u(t)$ for $t = 1, 2, \ldots, N$ to define a vector

$$u = \left[u(1), u(2), \ldots, u(N)\right]^{\mathrm{T}} \in \mathbb{R}^N \tag{2.2}$$

(T denotes the transposition of a vector). The boundary conditions are equivalent to $u(0) = u(1) - c_1$ and $u(N+1) = u(N) + c_2$. Therefore, we find out that (2.1) is equivalent to the algebraic problem (P) with the vector u defined by (2.2) and

$$A = \begin{bmatrix} 1 & -1 & 0 & & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{bmatrix} \quad \text{and} \quad G(u) = \begin{bmatrix} g(1, u(1)) \\ g(2, u(2)) \\ g(3, u(3)) \\ \vdots \\ g(N-1, u(N-1)) \\ g(N, u(N)) \end{bmatrix}, \tag{2.3}$$

where

$$g(t, u) = \begin{cases} \tilde{g}(1, u) - c_1, & t = 1, \\ \tilde{g}(t, u), & t = 2, 3, \ldots, N-1, \\ \tilde{g}(N, u) + c_2, & t = N. \end{cases}$$

The matrix A in (2.3) satisfies (A_1)–(A_3).

Indeed, the symmetry is clear. Let us prove the positive semi-definiteness of A. Firstly, the matrix A is singular because, e.g. the sum of its rows is the zero vector. Secondly, we verify that for all $u \in \mathbb{R}^N$ there is $(Au, u) \geq 0$. Let $u \in \mathbb{R}^N$ and $N \geq 3$ (for $N = 2$ it is obvious), then

$$\begin{aligned}
(Au, u) &= (u(1) - u(2))u(1) + \sum_{t=2}^{N-1} \left[(- u(t-1) + 2u(t) - u(t+1))u(t)\right] \\
&\quad + (- u(N-1) + u(N))u(N) \\
&= u^2(1) - u(1)u(2) + \sum_{t=2}^{N-1} \left[\left(u^2(t) - u(t)u(t+1)\right) + \left(u^2(t) - u(t-1)u(t)\right)\right] \\
&\quad + u^2(N) - u(N-1)u(N) \\
&= \sum_{t=1}^{N-1} \left(u^2(t) - u(t)u(t+1)\right) + \sum_{t=2}^{N} \left(u^2(t) - u(t-1)u(t)\right) \\
&= \sum_{t=1}^{N-1} \left(u^2(t) - u(t)u(t+1) + u^2(t+1) - u(t)u(t+1)\right) \\
&= \sum_{t=1}^{N-1} \left(u(t) - u(t+1)\right)^2 \\
&\geq 0.
\end{aligned}$$

Therefore, A satisfies (A_1). The positive semi-definitness of A implies also that $\lambda_1 = 0$ is the minimal eigenvalue. Let us prove that $\lambda_1 = 0$ has the multiplicity one. The eigenvectors corresponding to $\lambda_1 = 0$ are nontrivial solutions of the homogeneous linear algebraic system $Au = o$. Since an eigenvector multiplied by a real constant is also an eigenvector, we can assume without loss of generality that $u(1) = \rho$ where $\rho \neq 0$ is a real parameter. Therefore, from the first equation of the system $Au = o$, we obtain that $u(2) = \rho$. Then we can proceed inductively to show that all eigenvectors corresponding to $\lambda_1 = 0$ have the form $[\rho, \rho, \dots, \rho]^{\mathrm{T}} \in \mathbb{R}^N$. Hence, the dimension of the eigenspace corresponding to $\lambda_1 = 0$ is one which implies that $\lambda_1 = 0$ has the multiplicity one and (A_2) is proved. Moreover, if we put $\rho = 1$ we obtain that $\varphi_1 = [1, 1, \dots, 1]^{\mathrm{T}} \in \mathbb{R}^N$ is an eigenvector corresponding to $\lambda_1 = 0$ and thus, (A_3) holds.

Let us note that the fact that A satisfies (A_1)–(A_3) could be derived also from more general Example 2.5.

Example 2.2: Let us consider the discrete periodic problem (see [15,18])

$$\begin{cases} -\Delta^2 u(t-1) = g(t, u(t)), \quad t = 1, 2, \dots, N, \\ u(0) = u(N), \\ \Delta u(0) = \Delta u(N). \end{cases} \tag{2.4}$$

Analogously as in Example 2.1, (2.4) can be rewritten as the algebraic problem (P) where A is defined by

$$A = \begin{bmatrix} 2 & -1 & 0 & & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

and G is given via the function g in the same way as in (2.3). The matrix A satisfies (A_1)–(A_3) (see Example 2.5).

The following two examples show that also partial difference equations can be considered.

Example 2.3: Let us consider the Neumann problem for the difference Poisson equation (see [15])

$$\begin{cases} -\Delta_s^2 u(s-1, t) - \Delta_t^2 u(s, t-1) = \tilde{g}(s, t, u(s,t)), \quad t, s = 1, 2, \dots, N, \\ \Delta_s u(0, t) = c_1(t) \quad \text{and} \quad \Delta_s u(N, t) = c_2(t) \quad \text{for all} \quad t = 1, 2, \dots, N, \\ \Delta_t u(s, 0) = d_2(s) \quad \text{and} \quad \Delta_t u(s, N) = d_2(s) \quad \text{for all} \quad s = 1, 2, \dots, N, \end{cases} \tag{2.5}$$

where $u : \{0, 1, \dots, N, N+1\}^2 \to \mathbb{R}$, $\Delta_s^2 u(s-1, t)$, $\Delta_t^2 u(s, t-1)$ are the second partial central differences of u, $\Delta_s u(s,t)$, $\Delta_t u(s,t)$ are the first partial forward differences of u with respect to s and t, $g : \{1, 2, \dots, N\}^2 \times \mathbb{R} \to \mathbb{R}$, and $c_1, c_2, d_1, d_2 : \{1, 2, \dots, N\} \to \mathbb{R}$.

We follow the approach, e.g. from [8] or [15]. Using the values $u(s,t)$ for $s, t = 1, 2, \dots, N$ we define a vector

$$u = \left[u(1,1), \dots u(1,N), u(2,1), \dots, u(2,N), \dots, u(N,1), \dots, u(N,N)\right]^{\mathrm{T}} \in \mathbb{R}^{N^2}. \tag{2.6}$$

Consequently, we obtain that (2.5) is equivalent to the algebraic problem (P) on $\mathbb{R}^{N^2}$ with the vector u defined by (2.6) and with a block matrix $A \in \mathbb{R}^{N^2 \times N^2}$ given by

$$A = \begin{bmatrix} B_1 & -I & 0 & & 0 & 0 \\ -I & B_2 & -I & \dots & 0 & 0 \\ 0 & -I & B_2 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & B_2 & -I \\ 0 & 0 & 0 & \dots & -I & B_1 \end{bmatrix},$$

where $I \in \mathbb{R}^{N \times N}$ is the identity matrix and $B_1, B_2 \in \mathbb{R}^{N \times N}$ are given by

$$B_1 = \begin{bmatrix} 2 & -1 & 0 & & 0 & 0 \\ -1 & 3 & -1 & \dots & 0 & 0 \\ 0 & -1 & 3 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 3 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 & -1 & 0 & & 0 & 0 \\ -1 & 4 & -1 & \dots & 0 & 0 \\ 0 & -1 & 4 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 4 & -1 \\ 0 & 0 & 0 & \dots & -1 & 3 \end{bmatrix}.$$

The nonlinear function G can be established involving boundary conditions analogously as in Example 2.1. The matrix A satisfies (A_1)–(A_3) (see Example 2.5).

Example 2.4: Let us consider the periodic problem for the difference Poisson equation

$$\begin{cases} -\Delta_s^2 u(s-1,t) - \Delta_t^2 u(s,t-1) = g(s,t,u(s,t)), \quad s,t = 1,2,\dots,N, \\ u(0,t) = u(N,t) \quad \text{and} \quad \Delta_s u(0,t) = \Delta_s u(N,t) \quad \text{for all} \quad t = 1,2,\dots,N, \\ u(s,0) = u(s,N) \quad \text{and} \quad \Delta_t u(s,0) = \Delta_t u(s,N) \quad \text{for all} \quad s = 1,2,\dots,N. \end{cases} \tag{2.7}$$

Analogously as in Example 2.3, we find out that (2.7) can be reformulated as the algebraic problem (P) on $\mathbb{R}^{N^2}$ with a block matrix $A \in \mathbb{R}^{N^2 \times N^2}$ given by

$$A = \begin{bmatrix} B & -I & 0 & & 0 & -I \\ -I & B & -I & \dots & 0 & 0 \\ 0 & -I & B & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & B & -I \\ -I & 0 & 0 & \dots & -I & B \end{bmatrix},$$

where $B \in \mathbb{R}^{N \times N}$ is defined by

$$B = \begin{bmatrix} 4 & -1 & 0 & & 0 & -1 \\ -1 & 4 & -1 & \dots & 0 & 0 \\ 0 & -1 & 4 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 4 & -1 \\ -1 & 0 & 0 & \dots & -1 & 4 \end{bmatrix}.$$

The matrix A satisfies (A_1)–(A_3) (see Example 2.5).

The last example shows the possible application of our results to general difference equations on graphs.

Example 2.5: Let $\mathcal{G} = (V, E)$ be an undirected graph with a set of vertices $V = \{1, 2, \ldots, N\}$ and a set of edges $E \subset \{\{s, t\} : s, t \in V, s \neq t\}$. The set $\mathcal{N}(t) = \{i \in V : \{i, t\} \in E\}$ is the neighbourhood of the vertex $t \in V$ and the number $d_{\mathcal{G}}(t) = |\mathcal{N}(t)|$ is the degree of vertex $t \in V$ (see, e.g. [13] for details about the graph theory).

Let $u : V \to \mathbb{R}$ be a function defined on the set of vertices V and define the difference operator on the graph $\mathcal{G}$

$$\Delta_{\mathcal{G}} u(t) = d_{\mathcal{G}}(t)u(t) - \sum_{i \in \mathcal{N}(t)} u(i).$$

Consequently, we consider the nonlinear difference equation on the graph $\mathcal{G}$

$$\Delta_{\mathcal{G}} u(t) = g(t, u(t)), \quad t \in V, \tag{2.8}$$

with $g : V \times \mathbb{R} \to \mathbb{R}$. The problem (2.8) is equivalent to the algebraic system (P) with A being the so-called Laplace matrix of $\mathcal{G}$. The entries of A are given by

$$A(s, t) = \begin{cases} d_{\mathcal{G}}(t), & s = t, \\ -1, & s \neq t \quad \text{and} \quad \{s, t\} \in E, \\ 0, & s \neq t \quad \text{and} \quad \{s, t\} \notin E. \end{cases} \tag{2.9}$$

If $\mathcal{G}$ is a connected graph then A satisfies (A_1)–(A_3) (see [13, Section 13]).

Let us conclude the example with an interesting relationship of difference equations on graphs with Neumann and periodic boundary value problems for difference equations. It follows from the algebraic formulations of boundary value problems (Examples 2.1 – 2.4) that:

- the Neumann problem for ordinary difference equation (2.1) is equivalent to (2.8) with $\mathcal{G}$ being a path (see Figure 1(a)),
- the periodic boundary value problem for ordinary difference equation (2.4) corresponds to (2.8) with $\mathcal{G}$ being a cycle (see Figure 1(b)),
- the Neumann problem for the difference Poisson equation (2.5) (for the sake of simplicity let $N = 3$) corresponds to (2.8) with $\mathcal{G}$ given in Figure 1(c),
- the periodic problem for the difference Poisson equation (2.7) (again let $N = 3$) is equivalent to (2.8) with $\mathcal{G}$ given in Figure 1(d).

Therefore, the difference equations on graphs generalize the boundary value problems for ordinary and partial difference equations. Since all graphs in Figure 1 are connected, the matrices A in Examples 2.1 – 2.4 satisfy (A_1)–(A_3).

Remark 2.6: Let us note that we do not have to restrict ourselves to discrete problems of second order. One can show that the reformulation into (P) also works for problems of $2n$th order ($n \in \mathbb{N}$), see [18].

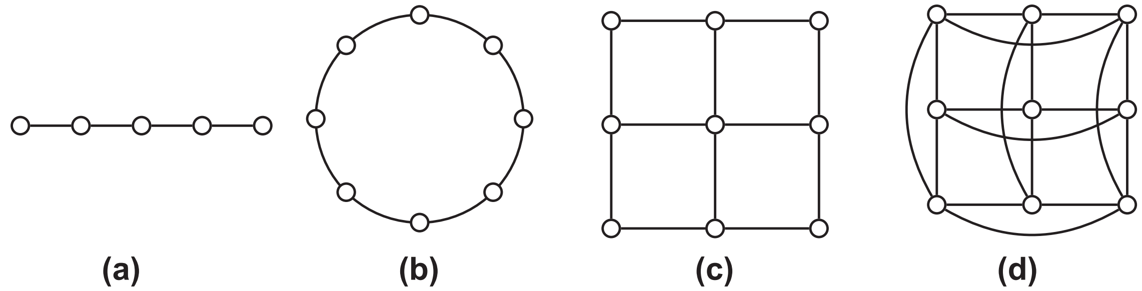

Figure 1. The graphs $\mathcal{G}$ from Example 2.5 that are related with Neumann and periodic discrete boundary value problems (2.1), (2.4), (2.5) and (2.7).

Motivated by Examples 2.1 – 2.5, we study the general algebraic problem for $u \in \mathbb{R}^N$, $N \geq 2$,

$$Au = G(u), \tag{P}$$

where $A \in \mathbb{R}^{N\times N}$ is an $N \times N$ matrix satisfying (A_1)–(A_3) and $G : \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear superposition vector function given by

$$G(u) = \left[g(1, u(1)), g(2, u(2)), \ldots, g(N, u(N))\right]^{\mathrm{T}},$$

where $g : \{1, 2, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$.

From (A_1) and (A_2), there is $\lambda_1 = 0$ the minimal eigenvalue of A and we can understand the problem (P) as

$$Au = \lambda_1 u + G(u),$$

i.e. as the algebraic problem in resonance. Therefore, one can expect an orthogonality condition of Landesman–Lazer type on G for the existence. Since $G : \mathbb{R}^N \to \mathbb{R}^N$ is the superposition vector function defined via the function $g : \{1, 2, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$, we formulate our conditions for G via the function g as well:

(H_1) The functions $g(t, \cdot)$ are continuous on $\mathbb{R}$ for each $t = 1, 2, \ldots, N$.

(H_2) There exist $\alpha, \beta \in [0, 1)$ such that for each $t = 1, 2, \ldots, N$ there exist limits

$$g_{-\infty}(t) = \lim_{u\to-\infty} \frac{g(t, u)}{|u|^\alpha} \quad \text{and} \quad g_{+\infty}(t) = \lim_{u\to+\infty} \frac{g(t, u)}{|u|^\beta}.$$

(LL) The function g satisfies

$$\sum_{t=1}^{N} g_{-\infty}(t) < 0 < \sum_{t=1}^{N} g_{+\infty}(t).$$

Remark 2.7: The condition (LL) represents a variant of the Landesman–Lazer condition. It is a type of an orthogonality relation, since the inequalities in (LL) can be rewritten as

$$(g_{-\infty}, \varphi_1) < 0 < (g_{+\infty}, \varphi_1),$$

where the vectors $g_{\pm\infty} \in \mathbb{R}^N$ are defined by $g_{\pm\infty} = \left[g_{\pm\infty}(1), g_{\pm\infty}(2), \ldots, g_{\pm\infty}(N)\right]^{\mathrm{T}}$. The symbol

$$(u, v) = \sum_{t=1}^{N} u(t)v(t)$$

denotes the scalar product on $\mathbb{R}^N$.

Remark 2.8: Consider the problem (P) with A satisfying (A_1)–(A_3) when the function g is independent of t, i.e. $g(t, u) = g(u)$. Suppose that there exists $\rho \in \mathbb{R}$ such that $g(\rho) = 0$. Then obviously, (P) has the solution $u = \left[\rho, \rho, \ldots, \rho\right]^{\mathrm{T}}$, since $u = \rho\varphi_1$ and

$$Au = A(\rho\varphi_1) = \lambda_1\rho\varphi_1 = o = \left[g(\rho), g(\rho), \ldots, g(\rho)\right]^{\mathrm{T}} = G(u).$$

Consequently, we focus on the nonlinearities which depend also on the variable t.

3. Variational formulation

We discuss the variational formulation of (P) in this section and summarize the main theorems of the critical point theory that we use later.

Since we assume that A satisfies (A_1) and g satisfies (H_1), the potential $\mathcal{J} : \mathbb{R}^N \to \mathbb{R}$ associated with the problem (P) is well-defined and given by

$$\mathcal{J}(u) = \frac{1}{2}(Au, u) - \sum_{t=1}^{N} \int_0^{u(t)} g(t, s)\mathrm{d}s. \tag{3.1}$$

Lemma 3.1 **([21, Lemma 1]):** *A vector $u \in \mathbb{R}^N$ is a solution of* (P) *if and only if it is a critical point of the potential $\mathcal{J}$ given by* (3.1).

Theorem 3.2 **(Saddle Point Theorem, [10, Theorem 7.6.12]):** *Let $X = Y \oplus Z$ be a Banach space with Z closed and $0 < \dim Y < +\infty$. For $\bar{\rho} > 0$ define*

$$\mathcal{M} = \{u \in Y : \|u\| \leq \bar{\rho}\}, \quad \mathcal{M}_0 = \{u \in Y : \|u\| = \bar{\rho}\}.$$

Let $\mathcal{J} \in C^1(X, \mathbb{R})$ be such that

$$\inf_{u \in Z} \mathcal{J}(u) > \max_{u \in \mathcal{M}_0} \mathcal{J}(u).$$

Let

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}} \mathcal{J}(\gamma(u)) \quad \textit{where} \quad \Gamma = \left\{\gamma \in C(\mathcal{M}, X) : \gamma|_{\mathcal{M}_0} = \mathrm{id}\right\}$$

and $\mathcal{J}$ satisfy the Palais–Smale condition: 'Any sequence $\{u_n\} \subset X$ such that $\mathcal{J}(u_n) \to c$ and $\nabla\mathcal{J}(u_n) \to o$ has a convergent subsequence'. Then c is a critical value of $\mathcal{J}$.

The Saddle Point Theorem has three assumptions - the continuous differentiability of $\mathcal{J}$, the saddle type geometry of $\mathcal{J}$ and the Palais–Smale condition. To show that $\mathcal{J}$ has the saddle type geometry we need the following statement about the existence of a minimum for a functional.

Theorem 3.3 **([10, Thm. 7.2.8]):** *Let $\mathcal{J} : H \to \mathbb{R}$ be a weakly sequentially lower semi-continuous and weakly coercive functional on a Hilbert space H. Then $\mathcal{J}$ is bounded below on H and there exists $u_0 \in H$ such that $\mathcal{J}(u_0) = \inf_{u\in H} \mathcal{J}(u)$.*

Remark 3.4: Since we work on finite dimensional spaces where the weak and strong topologies coincide, the weak sequential lower semi-continuity is equivalent to the strong lower semi-continuity.

4. Existence of solution

In this section, we prove the existence for (P) applying the statements from Section 3. We work on the space $X = \mathbb{R}^N$. In order to apply Theorem 3.2 we define subspaces $Y, Z \subset \mathbb{R}^N$ as follows:

$$Y = \mathrm{Lin}\,\{\varphi_1\}, \quad Z = Y^{\perp}, \tag{4.1}$$

where $\varphi_1 = [1, 1, \dots, 1]^{\mathrm{T}}$ is the eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$ (see (A_3)). It is obvious that Y, Z satisfy the assumptions of Theorem 3.2. Let us start with the following three auxiliary lemmas.

Lemma 4.1: *Let A satisfy (A_1) and (A_2). Then the function $\|u\|_A = \left[(Au, u)\right]^{\frac{1}{2}}$ defines a norm on the subspace Z defined in (4.1).*

***Proof*:** The matrix A is symmetric from (A_1). If moreover A is positive definite on the subspace Z then the bilinear mapping $(\cdot,\cdot)_A : Z\times Z \to \mathbb{R}$ given by $(u, v)_A = (Au, v)$ defines a scalar product on Z and therefore, $\|\cdot\|_A$ is the norm induced by $(\cdot,\cdot)_A$. Indeed, the positive definiteness of A on Z follows from the fact that we are on the finite-dimensional space and from [10, Lemma 1.1.31] which guarantees that under (A_1) and (A_2) the restriction $A|_Z$ has only positive eigenvalues. □

In the following, $\|\cdot\|_p$ denotes the p-norm on $\mathbb{R}^N$ with $p \geq 1$, i.e.

$$\|u\|_p = \left(\sum_{t=1}^{N} |u(t)|^p\right)^{\frac{1}{p}}, \quad u \in \mathbb{R}^N.$$

Lemma 4.2: *For all $1 \leq r \leq p$ there is*

$$\|u\|_p \leq \|u\|_r \leq N^{\frac{1}{r}-\frac{1}{p}}\|u\|_p \quad \textit{for all} \quad u \in \mathbb{R}^N. \tag{4.2}$$

Moreover, if A satisfies (A_1) and (A_2) then for all $p \geq 1$ there exist $m_{p,A}, M_{p,A} > 0$ such that

$$m_{p,A}\|u\|_p \leq \|u\|_A \leq M_{p,A}\|u\|_p \quad \textit{for all} \quad u \in Z. \tag{4.3}$$

***Proof*:** The inequality (4.2) is well-known result for p-norms on $\mathbb{R}^N$. The inequality (4.3) follows from the fact that on finite-dimensional spaces all norms are equivalent (see [10, Corollary 1.2.11]) and from Lemma 4.1. □

Remark 4.3: For the sake of brevity, we denote henceforward the norm $\|\cdot\|_2$ induced by the scalar product $(\cdot,\cdot)$ only by $\|\cdot\|$.

The following lemma states that under (H_1) and (H_2), the function g has sublinear growth.

Lemma 4.4: *Let g satisfy (H_1) and (H_2) and $\gamma = \max\{\alpha, \beta\}$. Then there exists constants $M_1, M_2 \geq 0$ such that*

$$\left|g(t,u)\right| \leq M_1 \left|u\right|^{\gamma} + M_2 \quad \textit{for all} \quad t = 1, 2, \dots, N \quad \textit{and} \quad u \in \mathbb{R}.$$

***Proof*:** The statement is an immediate consequence of (H_1) and (H_2). □

Let us begin to verify the assumptions of Theorem 3.2. Firstly, we have to show that the potential $\mathcal{J}$ is continuously differentiable.

Lemma 4.5: *Let A satisfy (A_1) and g satisfy (H_1). Then the potential $\mathcal{J}$ given by (3.1) is continuously differentiable on $\mathbb{R}^N$.*

***Proof*:** The gradient $\nabla\mathcal{J} : \mathbb{R}^N \to \mathbb{R}^N$ of the potential $\mathcal{J}$ is given by

$$\nabla\mathcal{J}(u) = Au - G(u), \quad u \in \mathbb{R}^N.$$

The linear mapping $u \mapsto Au$ is trivially continuous. The mapping $u \mapsto G(u)$ is continuous, since

$$G(u) = \left[g(1,u(1)), g(2,u(2)), \dots, g(N,u(N))\right]^{\mathrm{T}},$$

and the entries $g(t,u(t))$ are continuous from (H_1). Consequently, the gradient $\nabla\mathcal{J} : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping and therefore, the potential $\mathcal{J}$ is continuously differentiable on $\mathbb{R}^N$ (see, e.g. [10, Proposition 3.2.15]). □

The following two lemmas describe the geometry of $\mathcal{J}$. The first one deals with the geometry of $\mathcal{J}$ on the subspace Z.

Lemma 4.6: *Let A satisfy (A_1) and (A_2) and g satisfy (H_1) and (H_2). Then there exists $u_Z \in Z$ such that*

$$\mathcal{J}(u_Z) = \min_{u\in Z} \mathcal{J}(u).$$

***Proof*:** We prove the statement by the application of Theorem 3.3. Let us verify that $\mathcal{J}$ is weakly coercive on Z. Let $\{u_n\} \subset Z$ be such that $\|u_n\| \to +\infty$. From the definition of $\mathcal{J}$ (3.1) we obtain

$$\mathcal{J}(u_n) \geq \frac{1}{2}(Au_n, u_n) - \sum_{t=1}^{N} \left| \int_0^{u_n(t)} |g(t,s)|\mathrm{d}s \right|.$$

Applying Lemmas 4.4 and 4.1 we continue with the estimate

$$\begin{aligned}
\mathcal{J}(u_n) &\geq \frac{1}{2}(Au_n, u_n) - \sum_{t=1}^{N} \left| \int_0^{u_n(t)} \left(M_1|s|^{\gamma} + M_2\right) \mathrm{d}s \right| \\
&\geq \frac{1}{2}(Au_n, u_n) - \frac{M_1}{\gamma+1} \sum_{t=1}^{N} |u_n(t)|^{\gamma+1} - M_2 \sum_{t=1}^{N} |u_n(t)| \\
&= \frac{1}{2}\|u\|_A^2 - \frac{M_1}{\gamma+1}\|u_n\|_{\gamma+1}^{\gamma+1} - M_2\|u_n\|_1.
\end{aligned}$$

Consequently, from Lemma 4.2 there is

$$\mathcal{J}(u_n) \geq \frac{m_{2,A}^2}{2}\|u_n\|^2 - \frac{M_1 N^{\frac{1-\gamma}{2}}}{\gamma+1}\|u_n\|^{\gamma+1} - M_2 N^{\frac{1}{2}}\|u_n\|.$$

Since $\|u_n\| \to +\infty$ and $\gamma = \max\{\alpha, \beta\} < 1$, there is $\mathcal{J}(u_n) \to +\infty$ and $\mathcal{J}$ is weakly coercive on Z.

Since we work on $\mathbb{R}^N$ and thus, the subspace Z is also finite dimensional, the weak and strong topologies on Z coincide. Therefore, the continuity of $\mathcal{J}$ (see Lemma 4.5) implies that $\mathcal{J}$ is also weakly sequentially lower semi-continuous on Z (see also Remark 3.4). Consequently, Theorem 3.3 yields the statement. □

The following lemma describes the geometry of $\mathcal{J}$ on the subspace Y.

Lemma 4.7: *Let A satisfy* (A_1)–(A_3) *and g satisfy* (H_1) *and* (H_2) *and (LL). Then*

$$\lim_{|\rho|\to+\infty} \mathcal{J}(\rho\varphi_1) = -\infty. \tag{4.4}$$

***Proof*:** Since $\varphi_1 = [1, 1, \dots, 1]^{\mathrm{T}} \in Y$ is the eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$, there is $(A\varphi_1, \varphi_1) = \lambda_1\|\varphi_1\|^2 = 0$. Therefore,

$$\mathcal{J}(\rho\varphi_1) = \frac{\rho^2}{2}(A\varphi_1, \varphi_1) - \sum_{t=1}^{N}\int_0^{\rho\varphi_1(t)} g(t,s)\mathrm{d}s = -\sum_{t=1}^{N}\int_0^{\rho} g(t,s)\mathrm{d}s. \tag{4.5}$$

Let us prove that $\mathcal{J}(\rho\varphi_1) \to -\infty$ for $\rho \to +\infty$. From (LL) there is $\sum_{t=1}^{N} g_{+\infty}(t) > 0$ and hence, there exists $\varepsilon > 0$ satisfying

$$\sum_{t=1}^{N}\big(g_{+\infty}(t) - \varepsilon\big) > 0. \tag{4.6}$$

Moreover, from (H_2) and from the definition of limits $g_{+\infty}(t)$ there exists $\bar{s} \in \mathbb{R}$ such that

$$g(t,s) > \big(g_{+\infty}(t) - \varepsilon\big)\,|s|^{\beta} \quad \text{for all} \quad t = 1, 2, \dots, N \quad \text{and} \quad s \geq \bar{s}. \tag{4.7}$$

Using (4.5) and (4.7), we estimate for $\rho \geq \bar{s}$

$$\begin{aligned}
\mathcal{J}(\rho\varphi_1) &= -\sum_{t=1}^{N}\int_0^{\bar{s}} g(t,s)\mathrm{d}s - \sum_{t=1}^{N}\int_{\bar{s}}^{\rho} g(t,s)\mathrm{d}s \\
&\leq -\sum_{t=1}^{N}\int_0^{\bar{s}} g(t,s)\mathrm{d}s - \sum_{t=1}^{N}\int_{\bar{s}}^{\rho} \big(g_{+\infty}(t) - \varepsilon\big)\,|s|^{\beta}\,\mathrm{d}s \\
&= -\sum_{t=1}^{N}\int_0^{\bar{s}} g(t,s)\mathrm{d}s - \int_{\bar{s}}^{\rho} |s|^{\beta}\,\mathrm{d}s \sum_{t=1}^{N}\big(g_{+\infty}(t) - \varepsilon\big) \\
&= -\sum_{t=1}^{N}\int_0^{\bar{s}} g(t,s)\mathrm{d}s - \frac{1}{\beta+1}\big(|\rho|^{\beta}\rho - |\bar{s}|^{\beta}\bar{s}\big)\sum_{t=1}^{N}\big(g_{+\infty}(t) - \varepsilon\big).
\end{aligned}$$

Observing that the former term on the right-hand side is constant, we pass to the limit for $\rho \to +\infty$ and applying (4.6) we obtain

$$\lim_{\rho \to +\infty} \mathcal{J}(\rho\varphi_1) = -\infty.$$

The second case $\lim_{\rho \to -\infty} \mathcal{J}(\rho\varphi_1) = -\infty$ can be shown in the same way using $g_{-\infty}(t)$ and the former inequality in (*LL*). Consequently, (4.4) holds. □

The last point before the application of Theorem 3.2 is the verification of the Palais–Smale condition.

Lemma 4.8: *Let A satisfy (A_1)–(A_3) and g satisfy (H_1) and (H_2) and (LL). Then the potential $\mathcal{J}$ satisfies the Palais–Smale condition.*

***Proof*:** We prove a more general property:

$$\text{if} \quad \{\mathcal{J}(u_n)\} \subset \mathbb{R} \quad \text{is bounded} \quad \text{and} \quad \nabla\mathcal{J}(u_n) \to o, \tag{4.8}$$

then there exists a convergent subsequence $\{u_{n_k}\} \subset \{u_n\}$. In our case it is sufficient to prove that the sequence $\{u_n\}$ is bounded because we work on the finite-dimensional space $\mathbb{R}^N$ where every bounded sequence contains a convergent subsequence.

Motivated by the procedure from [11, Theorem 2], let us assume by contradiction that there exists a sequence $\{u_n\} \subset \mathbb{R}^N$ satisfying (4.8) and $\|u_n\| \to +\infty$. Without loss of generality, let us assume that $\|u_n\| > 0$ for all $n \in \mathbb{N}$. From the proof of Lemma 4.5 there is $\nabla\mathcal{J}(u_n) = Au_n - G(u_n)$. If we multiply this equality by $\frac{1}{\|u_n\|}$ and denote $v_n = \frac{u_n}{\|u_n\|}$ we obtain

$$\frac{\nabla\mathcal{J}(u_n)}{\|u_n\|} = Av_n - \frac{G(u_n)}{\|u_n\|}. \tag{4.9}$$

Since the sequence $\{v_n\} \subset \mathbb{R}^N$ is bounded and $\|v_n\| = 1$ for all $n \in \mathbb{N}$ there is $v_n \to v_0$ at least for a subsequence and $\|v_0\| = 1$.

For the left-hand side of (4.9) there is immediately $\frac{\nabla\mathcal{J}(u_n)}{\|u_n\|} \to o$, because $\nabla\mathcal{J}(u_n) \to o$. Let us investigate the term $\frac{G(u_n)}{\|u_n\|}$. We distinguish two possibilities:

- If $t \in \{1, 2, \dots, N\}$ is such that the sequence $\{u_n(t)\} \subset \mathbb{R}$ is bounded, then $\{g(t, u_n(t))\}$ is bounded as well from (H_1). Therefore, $\frac{g(t,u_n(t))}{\|u_n\|} \to 0$.
- Otherwise, if $t \in \{1, 2, \dots, N\}$ is such that the sequence $\{u_n(t)\} \subset \mathbb{R}$ is unbounded, then at least for a subsequence there is $|u_n(t)| \to +\infty$ and $u_n(t) \neq 0$ for all $n \in \mathbb{N}$. Applying Lemma 4.2, we estimate

$$\frac{|g(t, u_n(t))|}{\|u_n\|} \le \frac{N^{\frac{1}{2}} |g(t, u_n(t))|}{\|u_n\|_1} \le \frac{N^{\frac{1}{2}} |g(t, u_n(t))|}{|u_n(t)|} \le \frac{N^{\frac{1}{2}} |g(t, u_n(t))|}{|u_n(t)|^{\gamma} |u_n(t)|^{1-\gamma}},$$

where $\gamma = \max\{\alpha, \beta\}$. Since $\frac{|g(t,u_n(t))|}{|u_n(t)|^{\gamma}}$ is bounded from (H_1) and (H_2) and $\gamma < 1$, there is $\frac{|g(t,u_n(t))|}{\|u_n\|} \to 0$.

Consequently, if we put together these two cases, then $\frac{G(u_n)}{\|u_n\|} \to o$ in (4.9) at least for a subsequence. Therefore, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $v_{n_k} \to v_0$, $\frac{\nabla\mathcal{J}(u_{n_k})}{\|u_{n_k}\|} \to o$ and $\frac{G(u_{n_k})}{\|u_{n_k}\|} \to o$ in (4.9) which implies

$$Av_0 = o.$$

Hence, v_0 is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$, i.e. $v_0 = \rho\varphi_1$ for some $\rho \in \mathbb{R}$. Moreover, since $\|v_0\| = 1$ and $\varphi_1 = [1,1,\dots,1]^{\mathrm{T}}$, there is either $v_0 = \frac{1}{\sqrt{N}}\varphi_1 = \left[\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \dots, \frac{1}{\sqrt{N}}\right]^{\mathrm{T}}$ or $v_0 = -\frac{1}{\sqrt{N}}\varphi_1 = \left[-\frac{1}{\sqrt{N}}, -\frac{1}{\sqrt{N}}, \dots, -\frac{1}{\sqrt{N}}\right]^{\mathrm{T}}$.

Let us assume that $v_0 = \frac{1}{\sqrt{N}}\varphi_1$. From the definition of $\mathcal{J}$ (3.1) and from $\nabla\mathcal{J}(u) = Au - G(u)$, the following equality holds:

$$2\mathcal{J}(u_n) - (\nabla\mathcal{J}(u_n), u_n) = -2\sum_{t=1}^{N}\int_0^{u_n(t)} g(t,s)\mathrm{d}s + \sum_{t=1}^{N} g(t,u_n(t))u_n(t).$$

If we multiply this equality by $\frac{1}{\|u_n\|^{\beta+1}}$ and use $v_n = \frac{u_n}{\|u_n\|}$, we obtain

$$\frac{2\mathcal{J}(u_n)}{\|u_n\|^{\beta+1}} - \frac{(\nabla\mathcal{J}(u_n), v_n)}{\|u_n\|^{\beta}} = -\sum_{t=1}^{N}\frac{2}{\|u_n\|^{\beta+1}}\int_0^{u_n(t)} g(t,s)\mathrm{d}s + \sum_{t=1}^{N}\frac{g(t,u_n(t))}{\|u_n\|^{\beta}}v_n(t). \tag{4.10}$$

The left-hand side of (4.10) converges to zero because $\{\mathcal{J}(u_n)\}$ is bounded, $\nabla\mathcal{J}(u_n) \to o$ and $\{v_n\}$ is bounded. Let us analyze the right-hand side of (4.10) if we pass to the limit with respect to the above mentioned subsequence (we denote the index only by n for the simplicity):

- Firstly, we focus on the former term on the right-hand side of (4.10). L'Hôpital's rule and the convergence $u_n(t) \to +\infty$ for all $t = 1,2,\dots,N$ (because $v_n = \frac{u_n}{\|u_n\|} \to v_0 = \left[\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \dots, \frac{1}{\sqrt{N}}\right]^{\mathrm{T}}$ and $\|u_n\| \to +\infty$) yield

$$\begin{aligned}\lim_{n\to+\infty}&\sum_{t=1}^{N}\frac{2}{\|u_n\|^{\beta+1}}\int_0^{u_n(t)} g(t,s)\mathrm{d}s \\ &= \sum_{t=1}^{N}\left(\lim_{n\to+\infty}\frac{2v_n(t)\,|u_n(t)|^{\beta}}{\|u_n\|^{\beta}}\cdot\lim_{n\to+\infty}\frac{\int_0^{u_n(t)} g(t,s)\mathrm{d}s}{|u_n(t)|^{\beta}\,u_n(t)}\right)\\ &= \sum_{t=1}^{N}\left(\lim_{n\to+\infty}\left(2v_n(t)\,|v_n(t)|^{\beta}\right)\cdot\lim_{n\to+\infty}\frac{g(t,u_n(t))}{(\beta+1)\,|u_n(t)|^{\beta}}\right),\end{aligned}$$

provided the right-hand side makes sense. However, since $v_n(t) \to \frac{1}{\sqrt{N}}$, $u_n(t) \to +\infty$ for all $t = 1,2,\dots,N$ and from (H_1) and (H_2) we obtain

$$\lim_{n\to+\infty}\sum_{t=1}^{N}\frac{2}{\|u_n\|^{\beta+1}}\int_0^{u_n(t)} g(t,s)\mathrm{d}s = \frac{2}{N^{\frac{\beta+1}{2}}(\beta+1)}\sum_{t=1}^{N} g_{+\infty}(t). \tag{4.11}$$

- Secondly, we investigate the latter term on the right-hand side of (4.10). Applying similar arguments as above we derive that

$$\lim_{n\to+\infty}\sum_{t=1}^{N}\frac{g(t,u_n(t))}{\|u_n\|^{\beta}}v_n(t)=\sum_{t=1}^{N}\lim_{n\to+\infty}\left(\frac{g(t,u_n(t))}{|u_n(t)|^{\beta}}\cdot\frac{|u_n(t)|^{\beta}\,v_n(t)}{\|u_n\|^{\beta}}\right)$$
$$=\frac{1}{N^{\frac{\beta+1}{2}}}\sum_{t=1}^{N}g_{+\infty}(t). \tag{4.12}$$

Taking into account that the left-hand side of (4.10) converges to zero and limits (4.11), (4.12), we get

$$\frac{1}{N^{\frac{\beta+1}{2}}}\left(-\frac{2}{\beta+1}+1\right)\sum_{t=1}^{N}g_{+\infty}(t)=0,$$

a contradiction with the latter inequality in (LL) because $\beta<1$. Finally, if $v_0=-\frac{1}{\sqrt{N}}\varphi_1$ we obtain a contradiction with the former inequality in (LL) using the similar procedure with α instead of β. □

Consequently, we present the main result of this section – existence theorem for (P).

Theorem 4.9: *Let A satisfy (A_1)–(A_3) and g satisfy (H_1) and (H_2) and (LL). Then there exists a solution of* (P).

***Proof*:** According to Lemma 3.1, we show that $\mathcal{J}$ has a critical point. Auxiliary Lemmas 4.6 and 4.7 yield that $\mathcal{J}$ has a minimum $\mathcal{J}(u_Z)>-\infty$ on the subspace Z and $\lim_{|\rho|\to+\infty}\mathcal{J}(\rho\varphi_1)=-\infty$. Hence, there exists $\bar\rho>0$ sufficiently large such that

$$\mathcal{J}(u_Z)>\mathcal{J}(\pm\bar\rho\varphi_1). \tag{4.13}$$

Let us define

$$\mathcal{M}=\left\{u\in Y:\|u\|\le\bar\rho\sqrt{N}\right\}\quad\text{and}\quad\mathcal{M}_0=\left\{u\in Y:\|u\|=\bar\rho\sqrt{N}\right\}.$$

Then we obtain from (4.13)

$$\min_{u\in Z}\mathcal{J}(u)=\mathcal{J}(u_Z)>\max\{\mathcal{J}(-\bar\rho\varphi_1),\mathcal{J}(\bar\rho\varphi_1)\}=\max_{u\in\mathcal{M}_0}\mathcal{J}(u).$$

Moreover, $\mathcal{J}$ satisfies the Palais–Smale condition (Lemma 4.8). Theorem 3.2 then yields that

$$c=\inf_{\gamma\in\Gamma}\max_{u\in\mathcal{M}}\mathcal{J}(\gamma(u))\quad\text{where}\quad\Gamma=\left\{\gamma\in C(\mathcal{M},\mathbb{R}^N):\gamma|_{\mathcal{M}_0}=\mathrm{id}\right\}$$

is a critical value of $\mathcal{J}$ and there exists a critical point of $\mathcal{J}$. □

Example 4.10: Consider the boundary value problems (2.1), (2.4), (2.5), (2.7) or (2.8). Theorem 4.9 is applicable for example for the following nonlinear functions:

- $g(t,u)=\begin{cases}|u|^{p-2}u+f(t), & u<0,\quad p\in(1,2),\\ f(t), & u=0,\quad f:\{1,2,\dots,N\}\to\mathbb{R}\text{ arbitrary},\\ |u|^{q-2}u+f(t), & u>0,\quad q\in(1,2),\end{cases}$

- $g(t,u) = \begin{cases} |u|^{p-2}\, u + f(t), & u \le -1, & p \in (1,2), \\ -\sin\left(\frac{3\pi}{2}u\right) + f(t), & u \in (-1,1), & f : \{1,2,\ldots,N\} \to \mathbb{R} \text{ arbitrary}, \\ |u|^{q-2}\, u + f(t), & u \ge 1, & q \in (1,2), \end{cases}$
- $g(t,u) = \begin{cases} |u-t|^{p-2}\,(u-t) + \dfrac{\sin(u-t)}{u-t}, & u \ne t,\ \ p \in (1,2), \\ 1, & u = t. \end{cases}$

We can also consider bounded nonlinearities, e.g.:

- $g(t,u) = \mathrm{e}^{-u^2} + \tanh(u) + f(t)$ with $-1 < \frac{1}{N}\sum_{t=1}^{N} f(t) < 1$,
- $g(t,u) = (t-2)\arctan\big(u - \log(t)\big)$ with $N \ge 4$.

Example 4.10 motivates us to the following consequences of Theorem 4.9.

Corollary 4.11: *Let A satisfy (A_1)–(A_3) and g be defined by*

$$g(t,u) = h(u) + f(t),$$

where $f : \{1,2,\ldots,N\} \to \mathbb{R}$ is arbitrary and $h : \mathbb{R} \to \mathbb{R}$ satisfies (H_1) and (H_2) and (LL) with $\alpha, \beta \in (0,1)$. Then there exists a solution of (P).

***Proof*:** The statement is an immediate consequence of Theorem 4.9 and of the fact that for $\alpha, \beta \in (0,1)$ and each $t = 1,2,\ldots,N$ there is $\frac{f(t)}{|u|^\alpha} \to 0$ provided $u \to -\infty$ and $\frac{f(t)}{|u|^\beta} \to 0$ provided $u \to +\infty$. □

Corollary 4.12: *Let A satisfy (A_1)–(A_3) and g be defined by*

$$g(t,u) = h(u) + f(t),$$

where $f : \{1,2,\ldots,N\} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ satisfies (H_1) and (H_2) and (LL) with $\alpha = \beta = 0$ and $h_{\pm\infty} = \lim_{u\to\pm\infty} h(u)$. If f satisfies

$$-h_{+\infty} < \frac{1}{N}\sum_{t=1}^{N} f(t) < -h_{-\infty}, \tag{4.14}$$

then there exists a solution of (P).

***Proof*:** The statement follows immediately from $g_{\pm\infty}(t) = h_{\pm\infty} + f(t)$, $t = 1,2,\ldots,N$, and from Theorem 4.9. □

Remark 4.13: The inequalities in (4.14) are equivalent to $-h_{+\infty} < \frac{1}{N}(f, \varphi_1) < -h_{-\infty}$ where the vector $f \in \mathbb{R}^N$ is defined by $f = \big[f(1), f(2), \ldots, f(N)\big]^{\mathrm{T}}$.

5. Necessity of (*LL*) condition for bounded nonlinearities

Let us focus on (P) with bounded nonlinear functions g. We find out that for a certain class of bounded functions the Landesman–Lazer type condition (*LL*) is also necessary for the existence. We use the following additional condition:

(H_3) The function g satisfies

$$g_{-\infty}(t) < g(t,u) < g_{+\infty}(t) \quad \text{for all} \quad t = 1,2,\ldots,N \quad \text{and} \quad u \in \mathbb{R}.$$

Remark 5.1: If (H_1) and (H_3) hold together, then the functions $g(t,\cdot)$ are necessarily bounded for each $t = 1,2,\ldots,N$. It implies that (H_2) holds for all $\alpha,\beta \in \big(0,1\big)$. However, the strict inequalities in (H_3) yield that $g(t,\cdot)$ have to be bounded by the limits

$$g_{-\infty}(t) = \lim_{u\to-\infty} g(t,u) \quad \text{and} \quad g_{+\infty}(t) = \lim_{u\to+\infty} g(t,u),$$

i.e. by $g_{\pm\infty}(t)$ corresponding to $\alpha = \beta = 0$.

Theorem 5.2: *Let A satisfy (A_1)–(A_3) and g satisfy (H_1)–(H_3). Then (P) has a solution if and only if (LL) holds.*

Proof: The sufficiency of (LL) follows from Theorem 4.9. Conversely, assume that $u \in \mathbb{R}^N$ is a solution of (P). Therefore, u is a critical point of the potential $\mathcal{J}$ by Lemma 3.1, i.e. there is $(\nabla\mathcal{J}(u),v) = (Au - G(u),v) = 0$ for all $v \in \mathbb{R}^N$. If we put $v = \varphi_1$ then from the symmetry of A we get

$$(G(u),\varphi_1) = (Au,\varphi_1) = (u,A\varphi_1) = \lambda_1(u,\varphi_1) = 0. \tag{5.1}$$

Since $\varphi_1 = [1,1,\ldots,1]^{\mathrm{T}}$, the equality (5.1) is equivalent to $\sum_{t=1}^{N} g(t,u(t)) = 0$. Exploiting (H_3) we obtain

$$\sum_{t=1}^{N} g_{-\infty}(t) < \sum_{t=1}^{N} g(t,u(t)) = 0 < \sum_{t=1}^{N} g_{+\infty}(t).$$

□

Example 5.3: Consider the boundary value problems (2.1), (2.4), (2.5), (2.7) or (2.8) with the nonlinear function g defined by

$$g(t,u) = a \arctan(u) + f(t), \quad a > 0, \tag{5.2}$$

where $f : \{1,2,\ldots,N\} \to \mathbb{R}$. Obviously, the function g satisfies (H_1)–(H_3) with $\alpha = \beta = 0$ and $g_{\pm\infty}(t) = \pm\frac{a\pi}{2} + f(t)$. Therefore, (LL) is satisfied if and only if

$$-\frac{a\pi}{2} < \frac{1}{N}\sum_{t=1}^{N} f(t) < \frac{a\pi}{2}. \tag{5.3}$$

Consequently, Theorem 5.2 yields that the problems (2.1), (2.4), (2.5), (2.7) or (2.8) with g given by (5.2) have a solution if and only if f satisfies (5.3). In particular,

- for $a > \frac{2}{\pi N}\left|\sum_{t=1}^{N} f(t)\right|$ there exists a solution,
- for $a \leq \frac{2}{\pi N}\left|\sum_{t=1}^{N} f(t)\right|$ there does not exist any solution.

The following statement is an immediate consequence of Theorem 5.2 and is related to Corollary 4.12.

Corollary 5.4: *Let A satisfy (A_1)–(A_3) and g be defined as*

$$g(t, u) = h(u) + f(t), \tag{5.4}$$

where $f : \{1, 2, \ldots, N\} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ satisfies (H_1)–(H_3). Then (P) has a solution if and only if (4.14) holds.

Remark 5.5: Landesman–Lazer type conditions for difference equations have already been studied (among other things) in [5–7]. Our findings complement these results in the following way:

- Our approach via the algebraic formulation (P) is general in the sense that Neumann/periodic problems, ordinary/partial difference equations are considered at once, whereas in [5–7] specific boundary value problems are studied. On the other hand, in [6,7] more general problems involving discrete ϕ-Laplacian (even singular) are studied.
- Landesman–Lazer conditions in [5–7] are assumed to be sufficient. We show that for a certain class of bounded nonlinearities (LL) is even necessary and therefore, we obtain the nonexistence as well.
- All three papers [5–7] formulate the Landesman–Lazer conditions for nonlinear functions in separated form $g(t, u) = h(u) + f(t)$. We also study functions in general nonseparated form $g(t, u)$.

6. Uniqueness of solution

After the existence part of the paper we go further and analyze the uniqueness in this section. We use the following algebraic result for commuting matrices.

Theorem 6.1 ([17, Theorem 2.1]): *Let $A, B \in \mathbb{R}^{N\times N}$ be such that $AB = BA$ and $\lambda_s(A)$, $\lambda_s(B)$ be eigenvalues of A and B respectively and $\lambda_s(A + B)$ be eigenvalues of $A + B$ for $s = 1, 2, \ldots, N$. Then there exist permutations a and b of $\{1, 2, \ldots, N\}$ such that $\lambda_s(A+B) = \lambda_{a(s)}(A) + \lambda_{b(s)}(B)$ for all $s = 1, 2, \ldots, N$.*

In order to apply Theorem 6.1, we assume that the nonlinear function g satisfies the following conditions:

(H_4) The functions $g(t, \cdot)$ are continuously differentiable on $\mathbb{R}$ for each $t = 1, 2, \ldots, N$.

(H_5) Let $A \in \mathbb{R}^{N\times N}$ and $\lambda_s(A)$, $s = 1, 2, \ldots, N$, be eigenvalues of A. The function g satisfies

$$g_u(t, u) \neq \lambda_s(A) \quad \text{for all} \quad t = 1, 2, \ldots, N, \quad u \in \mathbb{R}, \quad \text{and} \quad s = 1, 2, \ldots, N.$$

Theorem 6.2: *Let A be arbitrary and g satisfy (H_4) and (H_5). Then (P) has at most one solution.*

***Proof*:** Suppose by contradiction that there are two distinct solutions $u, v \in \mathbb{R}^N$. Consequently, there is $A(u - v) = G(u) - G(v)$. From (H_4) and from the mean value theorem there has to exist $\xi \in \mathbb{R}^N$ such that

$$\begin{aligned}
G(u) - G(v) &= \begin{bmatrix} g(1,u(1)) - g(1,v(1)) \\ g(2,u(2)) - g(2,v(2)) \\ \vdots \\ g(N,u(N)) - g(N,v(N)) \end{bmatrix} = \begin{bmatrix} g_u(1,\xi(1))(u(1)-v(1)) \\ g_u(2,\xi(2))(u(2)-v(2)) \\ \vdots \\ g_u(N,\xi(N))(u(N)-v(N)) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} g_u(1,\xi(1)) & 0 & \dots & 0 \\ 0 & g_u(2,\xi(2)) & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_u(N,\xi(N)) \end{bmatrix}}_{=B} \begin{bmatrix} u(1)-v(1) \\ u(2)-v(2) \\ \vdots \\ u(N)-v(N) \end{bmatrix} \\
&= B(u-v).
\end{aligned}$$

If we denote $w = u - v$ we obtain that w has to be a solution of the homogeneous algebraic problem

$$(A - B)w = o. \tag{6.1}$$

Since B is diagonal, there is $AB = BA$. Theorem 6.1 and (H_5) yield that the eigenvalues $\lambda_s(A - B)$, $s = 1, 2, \dots, N$, are nonzero. Consequently, the matrix $A - B$ is regular and (6.1) has only the trivial solution $w = o$, a contradiction. □

The following lemma provides properties of the function g satisfying (H_4) and (H_5). Let us note that if (H_4) holds then (H_1) is obviously satisfied.

Lemma 6.3: *Let A satisfy (A_1) and (A_2) and $0 = \lambda_1(A) < \lambda_2(A) \le \dots \le \lambda_N(A)$ be eigenvalues of A. The following statements hold:*

(1) *If g satisfies (H_4) and (H_5) then the functions $g(t,\cdot)$ are strictly monotone for each $t = 1, 2, \dots, N$.*

(2) *If g satisfies (H_2) and (H_4), then (H_5) holds if and only if for each $t = 1, 2, \dots, N$ there is*

$$0 = \lambda_1(A) < g_u(t,u) < \lambda_2(A) \quad \text{or} \quad g_u(t,u) < \lambda_1(A) = 0 \quad \text{for all} \quad u \in \mathbb{R}.$$

(3) *If g satisfies (H_2)–(H_4), then (H_5) holds if and only if for each $t = 1, 2, \dots, N$ there is*

$$0 = \lambda_1(A) < g_u(t,u) < \lambda_2(A) \quad \text{for all} \quad u \in \mathbb{R}.$$

***Proof*:** Let us follow the structure of lemma:

(1) The first statement follows immediately from the continuous differentiability of the functions $g(t,\cdot)$ and from $g_u(t,u) \neq \lambda_1(A) = 0$ for any $t = 1, 2, \dots, N$ and $u \in \mathbb{R}$.

(2) Suppose (H_2), (H_4) and (H_5) and assume by contradiction that for some $t_2 \in \{1, 2, \dots, N\}$, $u_2 \in \mathbb{R}$ there is $g_u(t_2,u_2) > \lambda_2(A) > 0$. From (H_4) we know that $g_u(t_2,\cdot)$ is a continuous function. Since $g_u(t_2,u) \neq \lambda_2(A)$ for all $u \in \mathbb{R}$ (from (H_5)), there is $g_u(t_2,u) > \lambda_2(A) > 0$ for all $u \in \mathbb{R}$. Then we obtain for $u \ge 0$

$$\int_0^u g_s(t_2,s)\mathrm{d}s \ge \int_0^u \lambda_2(A)\mathrm{d}s,$$

and therefore, $g(t_2,u) \geq \lambda_2(A)u + g(t_2,0)$. This is a contradiction, since $g(t_2,\cdot)$ has sublinear growth from (H_2) and (H_4) (Lemma 4.4). The converse implication is obvious.

(3) The condition (H_3) implies that any function $g(t,\cdot)$, $t = 1,2,\ldots,N$, cannot be decreasing on $\mathbb{R}$ (see Remark 5.1). Then the last statement of lemma is an immediate consequence of the second one.

□

The following statements summarize the existence and uniqueness results from Theorems 4.9, 5.2 and Theorem 6.2.

Theorem 6.4: *Let A satisfy* (A_1)–(A_3) *and g satisfy* (H_2), (H_4) *and* (H_5) *and (LL). Then there exists a unique solution of* (P).

Theorem 6.5: *Let A satisfy* (A_1)–(A_3) *and g satisfy* (H_2)–(H_5). *Then* (P) *has a solution if and only if (LL) holds. Moreover, if the solution exists it has to be unique.*

Example 6.6: For simplicity, let us consider the Neumann problem (2.1) with $N = 3$ and $c_1 = c_2 = 0$. The corresponding matrix $A \in \mathbb{R}^{3\times 3}$ satisfies (A_1)–(A_3) and $\lambda_1(A) = 0$, $\lambda_2(A) = 1$ and $\lambda_3(A) = 3$ (see Example 2.1). Further, suppose that the function g is given by

$$g(t,u) = \left(\frac{t}{3} - a\right) \arctan(u) + bt, \quad a > 0, \quad b \in \mathbb{R}. \tag{6.2}$$

Let us investigate for which a, b we can apply Theorems 6.4 and 6.5.

- The condition (H_2) holds with $\alpha, \beta = 0$ and

$$g_{\pm\infty}(t) = \pm\frac{\pi}{2}\left(\frac{t}{3} - a\right) + bt. \tag{6.3}$$

- The condition (H_3) is satisfied if and only if the functions $g(t,\cdot)$ are strictly increasing for each $t = 1,2,\ldots,N$, i.e. if $a < \frac{1}{3}$.
- In order to satisfy (H_4) and (H_5), we compute the derivative

$$g_u(t,u) = \frac{\frac{t}{3} - a}{1 + u^2}.$$

  The condition (H_4) obviously holds. According to Lemma 6.3, (H_5) holds if and only if $0 \neq g_u(t,u) < 1$ for all $t = 1,2,\ldots,N$ and $u \in \mathbb{R}$. Since $0 < \frac{1}{1+u^2} \leq 1$, there has to be $0 \neq \frac{t}{3} - a < 1$ for $t = 1,2,3$. Hence, (H_5) is satisfied if and only if $a \in (0,+\infty) \setminus \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$.
- Since the limits $g_{\pm\infty}(t)$ are given by (6.3), there is

$$\sum_{t=1}^{3} g_{\pm\infty}(t) = \pm\frac{\pi}{2}\sum_{t=1}^{3}\left(\frac{t}{3} - a\right) + b\sum_{t=1}^{3} t = \pm\frac{\pi}{2}(2 - 3a) + 6b.$$

  Therefore, (LL) holds if and only if

$$-\pi\left(\frac{1}{6} - \frac{a}{4}\right) < b < \pi\left(\frac{1}{6} - \frac{a}{4}\right). \tag{6.4}$$

Let us notice, that (6.4) can be satisfied only for $a < \frac{2}{3}$.

Consequently, we can apply our results for $a \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \frac{2}{3}\right)$ and conclude:

(a) For $a \in \left(0, \frac{1}{3}\right)$ the conditions (H_2)–(H_5) are satisfied. Theorem 6.5 then yields that the problem (2.1) with $N = 3$ has a solution if and only if b satisfies (6.4). Moreover, the solution is unique provided it exists.

(b) For $a \in \left(\frac{1}{3}, \frac{2}{3}\right)$ the conditions (H_2) and (H_4) and (H_5) are satisfied but (H_3) not. Applying now Theorem 6.4 we obtain that the problem (2.1) with $N = 3$ has a unique solution at least for b satisfying (6.4).

In contrast to the previous Example 6.6, the next one shows the possible application of Theorem 6.4 for unbounded g.

Example 6.7: Consider the boundary value problems (2.1), (2.4), (2.5), (2.7) or (2.8) where g is defined by

$$g(t,u) = \begin{cases} |u|^{p-2}u + p - 2 + f(t), & u \geq 1, \\ (p-1)u + f(t), & |u| < 1, \\ |u|^{p-2}u - p + 2 + f(t), & u \leq -1, \end{cases} \tag{6.5}$$

where $p \in (1, 2)$ and $f : \{1, 2, \ldots, N\} \to \mathbb{R}$ is arbitrary. We obtain easily that g satisfies (H_2) and (LL) with $\alpha, \beta = p - 1$ and $g_{\pm\infty}(t) = \pm 1$. One can show that

$$g_u(t,u) = \begin{cases} (p-1)\,|u|^{p-2}, & |u| \geq 1, \\ p - 1, & |u| < 1. \end{cases}$$

The function $g_u(t, \cdot)$ is positive and continuous, i.e. (H_4) holds. Moreover, $g_u(t, \cdot)$ has a positive maximum $p - 1$. Let A be the appropriate matrix representing the difference operator in (2.1), (2.4), (2.5), (2.7) or (2.8). If $p - 1 < \lambda_2(A)$, then

$$\lambda_1(A) = 0 < g_u(t,u) \leq p - 1 < \lambda_2(A) \quad \text{for all} \quad t = 1, 2, \ldots, N \quad \text{and} \quad u \in \mathbb{R},$$

and hence, (H_5) holds. Therefore, Theorem 6.4 yields that if $p - 1 < \lambda_2(A)$, the problems (2.1), (2.4), (2.5), (2.7) or (2.8) with g given by (6.5) have a unique solution.

7. Concluding remarks and open problems

We conclude the paper with several open questions and possible future research directions arising from this study:

- If we assume that the nonlinear function is linear or superlinear, the geometry of the potential changes. Are we able to show existence results variationally even in these cases?
- Can we use some a priori bounds on a solution and apply our results for bounded nonlinearities (especially, necessary and sufficient existence condition) to prove the existence for other types of unbounded nonlinear functions?
- Motivated by [5–7] we can ask - is it possible and reasonable to define lower and upper solutions for the algebraic system (P) and apply this method to obtain new

existence results? Moreover, are there conditions guaranteeing the existence of such lower and upper solutions?

- One of the bases of this paper is the possibility of representing linear difference operators by a matrix A (see [12,18,20,21]). Straightforwardly, we cannot apply our results for problems involving nonlinear difference operators. Consequently, is it possible to extend our approach and reformulate problems with, e.g. discrete p-Laplacian (see [4,9,15,19]) or ϕ-Laplacian (see [6,7]) as an algebraic system and investigate the existence and uniqueness via this general representation?

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