COMPUTATIONAL MECHANICS 2018
34th conference with international participation
Srní
October 31 - November 2, 2018

Estimation of Lyapunov exponents of discrete data series

C. Fischer[a], J. Náprstek[a]

[a] Institute of Theoretical and Applied Mechanics, CAS v.v.i., Prosecká 76, CZ-190 00 Praha, Czech Republic

The dynamical systems describing general non-linear structures represent an interesting and demanding topic in various branches of engineering. This regards the both cases of mathematical and experimental models or an analysis of results from measurements in situ. In case of a complex behaviour of a structure or a non-linear mathematical model, the measured response or computed data series can exhibit wide range of response types, from stationary and periodic to diverging or chaotic behaviour. The stability in the sense of sensitivity to small perturbations, however, is the key property of each type of the system response.

The concept of Lyapunov exponents (LE) is the most usable and most robust stability measure, despite of numerous new methods and modifications. However, practical estimation of LE for both continuous systems and discrete data is still a demanding task. Even if the topic was addressed by numerous papers in the past it seems that the practical usage of recommended methods usually raise additional questions. It is natural because the theoretical results are mostly substantiated by an limiting relation, assumptions of which are hardly fulfilled in the practice.

The case of continuous systems is often dealt in the literature. As an interestin review can serve, e.g., paper [4]. A very promising approach for continuous systems is presented by Dieci et al. in [1]. On the other hand, the literature is meagre when it regards the case of discrete data set. The present contribution extends the previous work of the authors [2] and aims at presenting, comparison and analysis of two approaches derived specifically for the case when the dynamical system is represented only as a discrete data series (procedures due to Wolf et al. [6] and Rosenstein et al. [5]) and a possible extension of the mentioned algorithm due Dieci et al. [1] to certain discrete cases.

Let us consider the continuous dynamical system

$$\dot{x}(t) = f(x,t)\,, \qquad x(0) = x_0\,. \tag{1}$$

Stability of its solution $\tilde{x}_{x0}$ can be deduced from increasing separation of two nearby orbits, initial distance of which is δ_0 in $t = 0$:

$$\delta(t) = \tilde{x}_{x0}(t) - \tilde{x}_{x0+\delta_0}(t).$$

The commonly used estimates of LE fall into two main categories. The first uses a heuristic approach based on the relation

$$||\delta(t)|| = \mathrm{e}^{\lambda_1 t}||\delta(0)||. \tag{2}$$

Although usage this approach is not limited to the cases where only the discrete data are available, this methods are used mostly in such case. The second group, on the other hand, is based on so called variational equation

$$\mathbf{P}'(t) = A(x,t)\mathbf{P}(t)\,, \qquad \mathbf{P}(0) = \mathbf{I}\,, \tag{3}$$

where $\mathbf{P}(t)$ is the derivative of $\tilde{x}_{x0}$ with respect to initial condition x_0. Since this approach is based on the explicit knowledge of Jacobian $A(\tilde{x}, t)$, it is naturally aimed at analysis of continuous systems.

The popular algorithm belonging to the first group is the implementation which accompanies paper due to Wolf et al. [6]. The algorithm follows the nature of the problem: It is based on identification of close points on the orbit. Such points are considered as close or perturbed initial conditions and separation of corresponding orbital sections is measured. The largest LE λ_1 is computed from the growth of distance of both orbits. When the separation becomes large, a new trajectory is chosen near the reference trajectory considering close distance and direction.

The more recent procedure described by Rosenstein et al. [5] is similarly based on identifying different yet similar sections in the data series, which are used subsequently to simulate separation of close orbits. Result of the procedure is returned as dependence of the averaged distance of two orbits on the increasing time lag to initial "close" point. The distance should increase linearly in the logarithmic scale up to size of the attractor. The slope of the linear ramp then represents an estimate of the largest LE.

The weak point of the Rosenstein's approach is identification of the determinative part of the resulting dependence which is used for estimation of the average slope, see description in [2]. The authors successfully used a simple detection of the "corner sample" based on the horizontal direction of the upper plateau. A number of alternative approaches could be proposed, however, they mostly require some ad hoc intervention.

The main problem in algorithms based on the variational equation and belonging to the second group is that the auxiliary matrix $\mathbf{P}(t)$ has to be kept orthogonal. This requirement implies necessity of reorthogonalization in every iteration step. The work presented by Dieci et al. [1] is based on keeping the system $\mathbf{P}(t)$ in triangular form using the time-dependent orthogonal discrete or continuous QR transformation. Usage of this procedure claims certain prerequisites to the discrete data representing the dynamical system. Namely, the data set has to be capable of continuous interpolation.

Numerical experiments with the mentioned algorithms show that in the case of discrete data, namely those obtained experimentally, the functionality of all available approaches is limited and closely reflects quality of the data.

Acknowledgement

The kind support of the Czech Science Foundation project No. 17-26353J and of the RVO 68378297 institutional support are gratefully acknowledged.

References

[1] Dieci, L., Jolly, M.S., Van Vleck, E. S., Numerical techniques for approximating Lyapunov exponents and their implementation, Journal of Computational and Nonlinear Dynamics 6 (2011) 011003, doi: 10.1115/1.4002088.

[2] Fischer, C., Náprstek, J., Lyapunov exponents — practical computation, Proceedings of the conference Engineering Mechanics 2017, Svratka, 2017, pp. 310-313.

[3] Fischer, C., Náprstek, J., Numerical assessment of stability of the ball vibration absorber, Proceedings of the conference Computatinal Mechanics 2017, Špičák, 2017, pp. 23-24.

[4] Ramasubramanian, K., Sriram, M.S., A comparative study of computation of Lyapunov spectra with different algorithms, Physica D: Nonlinear Phenomena 139 (1) (2000) 72-86.

[5] Rosenstein, M.T., Collins, J.J., De Luca, C.J., A practical method for calculating largest Lyapunov exponents from small data sets, Physica D: Nonlinear Phenomena 65 (1-2) (1993) 117-134.

[6] Wolf, A., Swift, J.B., Swinney, H.L., Vastano, J.A., Determining Lyapunov exponents from a time series, Physica D: Nonlinear Phenomena 16 (3) (1985) 285-317.