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Dispersion errors for wave propagation in thin plate due to the finite element method

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Widespread use of ultrasonic guided waves in non-destructive testing increases demand for the efficient and reliable numerical modeling of this phenomena [4]. Here the effect of spatial and temporal discretisation in finite element modeling on the accuracy of the numerical predictions is illustrated.

In former works [1, 2] the analytical solution for dispersion errors estimation in finite element modeling of wave propagation was derived. These works include the solution of the regular mesh and the plane strain elements. Here these relations are illustrated by the example of modeling of Lamb's waves propagating in a plate.

Lamb's wave is an elastic wave which propagates through a solid thin plate with free boundaries. In a homogeneous isotropic plate of a finite thickness two set of waves propagates with finite speed, these sets are called symmetric and antisymmetric waves, according to its shape, Fig. 1. Their velocities depends on the relationship between the thickness of the plate and the wavelength of the wave. These waves are highly dispersive, but their phase and group velocities can be stated analytically [3]. Their small wavelengths in higher frequency range and only a small loss of amplitude magnitude make them very popular for nondestructive testing. Thus, the properties of the finite element method modeling are desirable, see [4, 5].

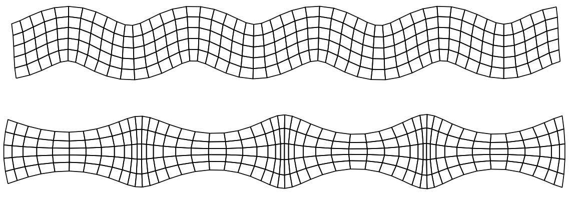

Fig. 1. Antisymmetric (*top*) and symmetric (*bottom*) mode of the Lamb's wave in thin plane

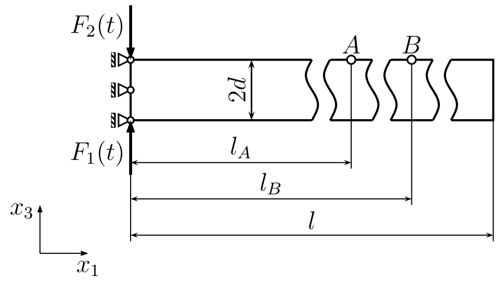

Fig. 2. Model of the plate

Both the spatial and temporal discretisation in finite element modeling lead to numerical dispersion errors. This means that the phase velocity of the numerical solution of the wave propagation is frequency dependent even in the absence of any dispersion in the actual media. When a dispersive waves are modeled, the dispersion error caused the difference between the analytical and numerical solution.

Characteristic equations determining the relation between the wave number $k = 2\pi/\lambda$, λ is the wavelength, and the angular velocity $\omega = 2\pi f$, f denoting frequency, of Lamb's wave are

$$\frac{\tan(\beta\, d/2)}{\tan(\alpha\, d/2)} = -\frac{4\alpha\beta k^2}{(k^2-\beta^2)^2} \tag{1}$$

and

$$\frac{\tan(\beta\, d/2)}{\tan(\alpha\, d/2)} = -\frac{(k^2-\beta^2)^2}{4\alpha\beta k^2}, \tag{2}$$

where

$$\alpha^2 = \frac{\omega^2}{c_{\mathrm{L}}^2} - k^2 \qquad \text{and} \qquad \beta^2 = \frac{\omega^2}{c_{\mathrm{T}}^2} - k^2. \tag{3}$$

c_{L}, c_{T} are the velocities of the longitudinal and shear waves, respectively.

Numerical methods are used to find phase velocity $c_{\mathrm{p}} = \omega/k$ and the group velocity $c_{\mathrm{g}} = \mathrm{d}\omega/\mathrm{d}k$. Solutions of Eq. (1) are waves with a symmetric shape and solution of Eq. (2) are waves with an antisymmetric shape. See Fig. 1.

The dispersion curves were calculated from (1) and (2), see Fig. 3.

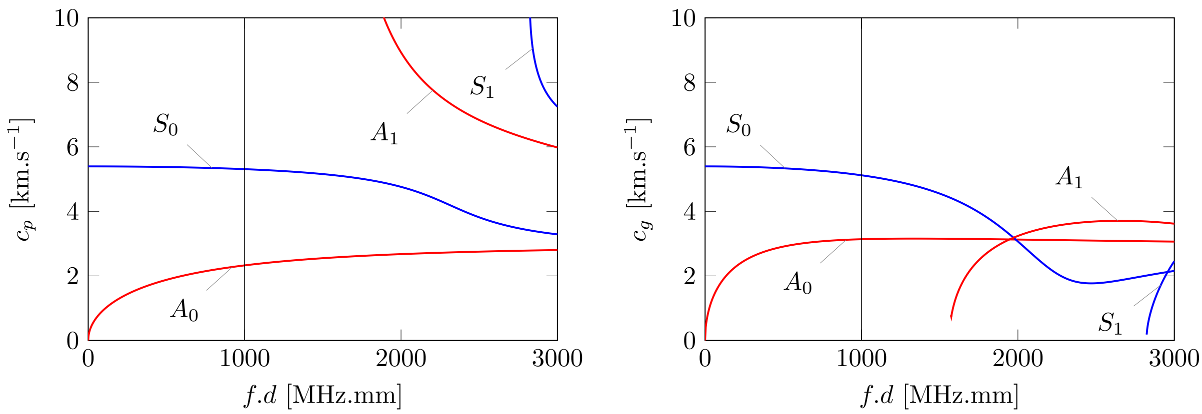

Fig. 3. Dispersion curves of Lamb's wave. The phase velocity vs. frequency (*left*) and the group velocity vs. frequency (*right*). Red lines show the first and second wave with an antisymmetric mode, the blue lines show the first and second wave with the symmetric mode. Black line shows the velocities of waves used in the example with frequency f of 0.5 MHz and the plate thickness d of 2 mm.

Since the Lamb's wave propagation suppose the plain strain problem, the plain strain 2D elements can be used. The thickness of the plate $2d$ is 2 mm and the modelled plate's length l is 500 mm, see Fig. 2. The plate is made from aluminium with Young's modulus of elasticity $E = 70$ GPa, $\nu = 0.33$, $\rho = 2.7$ g.cm^{-3}. Then shear modulus G is 26.3 GPa.

The Lamb's wave in the plate can be excited by applying forces

$$F_1(t) = \hat{F} \sin(\omega t) \sin^2\left(\frac{\omega t}{2n}\right), \tag{4}$$

where $\omega = 2\pi f$ at two points at the opposite sides of the plate, as is shown in Fig. 2. The loading frequency f is 500 000 Hz and $n = 16$ is the number of cycles and it determines the width of the signal around the central frequency. For the symmetric mode force amplitudes are $F_2 = -F_1$ and for the antisymmetric mode $F_2 = F_1$. The signal is shown on the left side in Fig. 4.

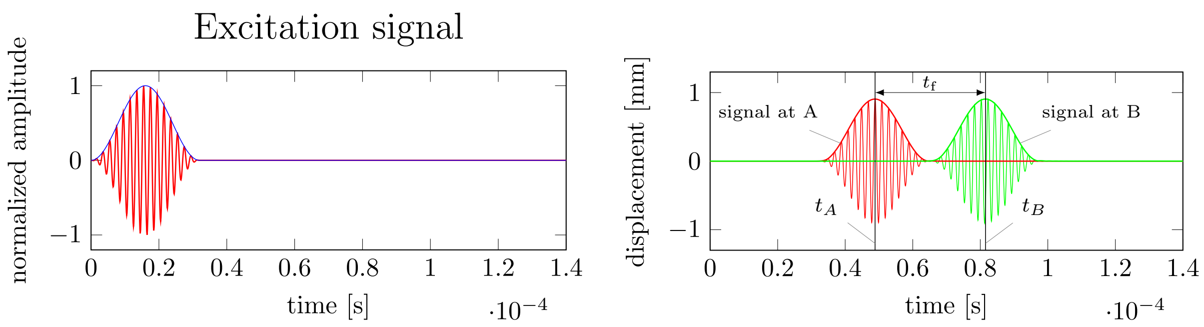

Fig. 4. (*Left*) Time history of applied force F. (*Right*) Calculated displacement at points A and B, plotted with the red and green line, respectively. The envelope of the signal. The centroid of both envelopes defines the time when the wave reach point A and B, denoted t_A and t_B, respectively. The flight time of the wave is then $t_f = t_B - t_A$.

The signal is read as a displacement in two points, point A in distance of $l_A = 0.1$ m from the loading force and point B in distance of $l_B = 0.2$ m of the loading force, see Fig. 2. The plane is long enough that no reflections of the free end of the plane occur.

The loading is supposed to be in the middle of the plane, so the displacement in the direction of the wave propagation of all nodes of the cross section of the plane on one side (loading side) is prescribed to be zero.

For wave propagation modeling, three different finite element meshes were used. All of them contain square 4-node plain strain elements with the element size H equal to 0.67 mm, 0.4 mm and 0.2 mm, respectively, thus we have 3, 5 and 10 elements for the thickness of the plate. Part of the mesh with 5 elements to thickness are shown in Fig. 1.

The time integration method used in our example was the Newmark method with parameters $\gamma = 1/2$ and $\beta = 1/4$ and consistent mass matrix. Here only the results for antisymmetric mode, when for given frequency f the theoretical value of phase and group velocities are $c_{p\,\mathrm{A}} = 2.3266\,\mathrm{km.s^{-1}}$ and $c_{g\,\mathrm{A}} = 3.1371\,\mathrm{km.s^{-1}}$. The time step used in finite element simulation was for prescribed Courant number $C = \Delta t\, c_{g\,\mathrm{A}}/H$ calculated from $\Delta t = C_\mathrm{L} H / c_{g\,\mathrm{A}}$, it differs for different element size H.

For frequency $f = 0.5$ MHz, the wave number $k = 2\pi f / c_{p\,\mathrm{A}}$ is 1350 and the product $k.H/\pi$ is equal to 0.29, 0.17 and 0.086, respectively. This corresponds to 7, 11.6 and 23 elements modeling the whole wavelength.

A group velocity of FEM model was determined from displacement signal in point A and B using Hilbert transform to obtain the envelope of the signal and the speed of the centroid of the signal is the group velocity in the model, see the right side of Fig. 4.

The table with calculated group velocities for different Courant number and different k is given (in $\mathrm{km.s^{-1}}$) as:

Courant number C	0.1	0.5	1	2
$kH/\pi = 0.29$	3.6301	3.5406	3.2692	2.3285
$kH/\pi = 0.17$	3.1311	3.1017	3.0103	2.6650
$kH/\pi = 0.09$	3.1354	3.1281	3.1054	3.0145

The relative error can be calculated as a ratio of the calculated group velocity and theoretical group velocity $c_{g\,\mathrm{A}} = 3.1371\,\mathrm{km.s^{-1}}$, see Fig. 3. Since the antisymmetric mode in plate corresponds with its shape more to the shear mode of the wave propagating in space, in Fig. 5 the relative error is shown together with theoretical dispersive errors of the shear wave in FEM model of space, see [2].

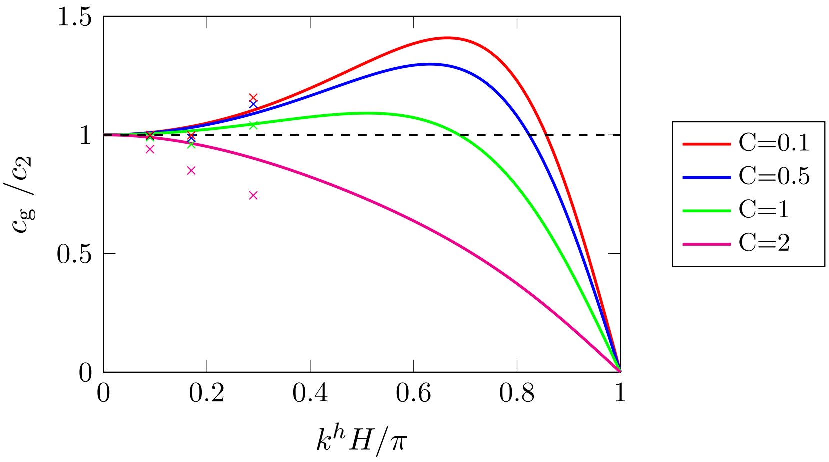

Fig. 5. The comparison of dispersion errors of finite element models in space (solid lines) and in plate ($\times$ mark). Different values of Courant number denoted by different colors.

In Fig. 5, the non-dispersive relative group velocity with dashed line is plotted together with the theoretical relative dispersive group velocities of a shear wave in space with solid lines (different colors for different Courant number). The size of the dispersive error is given by the deviation of the curve to the horizontal dashed line.

From the ratio of the calculated and theoretical value of group velocity of the antisymmetric mode of a Lamb's wave the relative dispersive errors were calculated and plotted in the same graph by cross mark. The relative error is here the distance between the horizontal line and the cross for particular values of kH/π and C.

Here it is verified that the theoretical expression for dispersion errors in space [2] serves well for the estimation of dispersion errors in other example, when the plain strain element can be used.

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References

[1] Kolman, R., Plešek, J., Červ, J., Okrouhlík, M., Pařík, P., Temporal-spatial dispersion and stability analysis of finite element method in explicit elastodynamics, International Journal for Numerical Methods in Engineering 106 (2016) 113-128.

[2] Kruisová, A., Kolman, R., Mračko, M., Okrouhlík, M., Temporal-spatial dispersion analysis of finite element method in implicit time integration, Proceedings of the 25th conference Engineering Mechanics 2019, Svratka, Institute of Thermomechanics of the Czech Academy of Sciences, 2019, pp. 215-218.

[3] Rose, J. L., Ultrasonic guided waves in solid media, Cambridge University Press, 2014.

[4] Wan, X., Xu, G., Zhang, Q., Tse, P. W., Tan, H., A quantitative method for evaluating numerical simulation accuracy of time-transient Lamb wave propagation with its application to selecting appropriate element size and time step, Ultrasonics 64 (2016) 25-42.

[5] Willberg, C., Duczek S., Vivar Perez J. M., Schmicker D., Gabbert, U., Comparison of different higher order finite element schemes for the simulation of Lamb waves, Computer Methods in Applied Mechanics and Engineering 241-244 (2012) 246-261.