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Demonstration of the developed procedure for the computation of the nonlinear steady-state response on practical examples

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1. Introduction

In the rotor dynamics, complex behaviour occurs due to nonlinear properties of the rolling elements, rubbing contact between disc and stator, crack breathing, fluid effects of bearings and dampers, the development of efficient numerical methods for prediction of nonlinear steady-state periodic response [1, 2] and its stability [3] is needed.

The nonlinear equations of motion of a rotor dynamic system with the discs, rotor shaft, different types of bearings, dampers, and supports elements, can be stated in the following form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{B}(\omega)\dot{\mathbf{x}}(t) + \mathbf{K}(\omega)\mathbf{x}(t) = \mathbf{f_{NL}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \omega) + \mathbf{f_P}(t, \omega) + \mathbf{f_S}. \tag{1}$$

$\mathbf{M}$, $\mathbf{B}(\omega)$, $\mathbf{K}(\omega)$ denote the $n \times n$ matrices of mass, damping, and stiffness, respectively, which depend on the excitation frequency ω due to the rotational effect, $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$, and $\mathbf{x}$ are the vectors of generalized accelerations, velocities, and displacements, respectively, $\mathbf{f_{NL}}$, $\mathbf{f_P}$, and $\mathbf{f_S}$ are the vectors of the nonlinear forces, periodic external excitation forces, and static load, respectively, t is time, and $\dot{(\cdot)}$ denotes time derivation.

The paper shows the application of the computational procedure for the determination of the steady-state response of the nonlinear motion equations. The created procedure is based on the harmonic balance method with the utilization of the arc-length parametrization and Floquet's theory. In addition, the selected steady-state responses were verified by direct integration of the motion equations and the solution shows a good agreement.

2. Approximation of the periodic response by the harmonic balance method

The periodic response of the motion equation (1) can be approximated by Fourier series [1]

$$\mathbf{x}(t) = \mathbf{q}_0 + \sum_{k=1}^{n_\mathrm{H}} \mathbf{q}_{\mathbf{C}k} \cos(k\omega t) + \mathbf{q}_{\mathbf{S}k} \sin(k\omega t), \tag{2}$$

where n_H stands for the number of the harmonic terms. For convenience, the Fourier coefficients can be arranged into $(2n_\mathrm{H} + 1)n \times 1$ vector

$$\mathbf{q} = \begin{bmatrix} \mathbf{q_0} & \mathbf{q_{C1}} & \mathbf{q_{S1}} & \dots & \mathbf{q}_{\mathbf{C}k} & \mathbf{q}_{\mathbf{S}k} & \dots & \mathbf{q}_{\mathbf{C}n_\mathrm{H}} & \mathbf{q}_{\mathbf{S}n_\mathrm{H}} \end{bmatrix}^\mathrm{T}, \tag{3}$$

the trigonometric Fourier basis can be arranged into $n \times (2n_\mathrm{H} + 1)n$ transformation matrix [4]

$$\mathbf{T} = [\mathbf{I} \; \cos(\omega t)\mathbf{I} \; \sin(\omega t)\mathbf{I} \dots \cos(k\omega t)\mathbf{I} \; \sin(k\omega t)\mathbf{I} \dots \cos(n_\mathrm{H}\omega t)\mathbf{I} \; \sin(n_\mathrm{H}\omega t)\mathbf{I}], \tag{4}$$

which is often called inverse discrete Fourier transform (DFT) matrix and where $\mathbf{I}$ is $n \times n$ identity matrix. Now, instantaneous displacement values can obtained in a compact form

$$\mathbf{x}(t) = \mathbf{T}(\omega t)\mathbf{q}. \tag{5}$$

For the derivatives with respect to time, $(2n_{\mathrm{H}}+1)n \times (2n_{\mathrm{H}}+1)n$ frequential derivative operator matrix

$$\nabla = \mathrm{diag}\left(\mathbf{0}_{n\times n}\nabla_1 \dots \nabla_k \dots \nabla_{n_{\mathrm{H}}}\right), \quad \text{where} \quad \nabla_k = k \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{6}$$

can be assembled. The matrix deals with chain rule products (ω is intentionally omitted) and modification of DFT transformation matrix. Therefore, the r-th derivatives can be obtained as

$$\mathbf{x}^{(r)} = \omega^r \mathbf{T}(\omega t)\nabla^r \, \mathbf{q}\,. \tag{7}$$

Substituting (5) and (7) into (1), one can obtain the nonlinear algebraic residual equation

$$\mathbf{h}(\omega, \mathbf{q}) = \mathbf{P}(\omega)\mathbf{q} - \mathbf{T}^{+}\mathbf{f}_{\mathbf{NL}}(\mathbf{Tq}, \omega\mathbf{T}\nabla\mathbf{q}, \omega^2\mathbf{T}\nabla^2\mathbf{q}, \omega) - \mathbf{u}_{\mathbf{P}}(\omega) - \mathbf{g}_{\mathbf{S}}, \tag{8}$$

where $\mathbf{P}(\omega) = \omega^2(\mathbf{I}\otimes\mathbf{M})\nabla^2 + \omega(\mathbf{I}\otimes\mathbf{B})\nabla + \mathbf{I}\otimes\mathbf{K}$ is the dynamical stiffness matrix, $\mathbf{I}$ is the identity matrix of order $2n_{\mathrm{H}}+1$, and the vectors $\mathbf{u}_{\mathbf{P}}$, $\mathbf{g}_{\mathbf{S}}$ contain amplitudes of the periodic unbalance forces and the static forces. The $\mathbf{T}^{+}\mathbf{f}_{\mathbf{NL}}(\mathbf{Tq}, \omega\mathbf{T}\nabla\mathbf{q}, \omega^2\mathbf{T}\nabla^2\mathbf{q}, \omega)$ term represents so called alternating frequency-time (AFT) technique [1]. The $(\,^{+}\,)$ stands for Moore-Penrose pseudoinverse and the $\otimes$ denotes the Kronecker product.

For obtaining the response of a nonlinear systems, it is often mandatory to use a continuation technique [1]. In general, the continuation consists of predictor and corrector steps. The predictor was based on secant, which passed through previous solutions and determined the direction of the next initial guess. The length of this predictor vector was normalised to appropriate arc length value s. For the corrector phase, Crisfield's arc-length parametrization [1] was used in the form of the additional residual equation

$$\mathbf{p}(\omega, \mathbf{q}) = (\mathbf{q} - \mathbf{q}_{\mathrm{prev}})^{\mathrm{T}}(\mathbf{q} - \mathbf{q}_{\mathrm{prev}}) + (\omega - \omega_{\mathrm{prev}})^2 - s^2, \tag{9}$$

where $\mathbf{q}_{\mathrm{prev}}$ and ω_{prev} denote values acquired at the previous continuation step.

The harmonic balance method procedure with the utilisation of the arc-length parametrization can be described in the following steps:

1. Choose the initial Fourier coefficients vector and angular velocity.
2. Apply the inverse DFT on the Fourier coefficients vector and evaluate the $\mathbf{f}_{\mathbf{NL}}$.
3. Transform the nonlinear forces from the time-domain to the frequency-domain by DFT and assemble the dynamical stiffness matrix $\mathbf{P}$.
4. Solve the system given by nonlinear algebraic equations (8) and (9).
5. Compute the predictor and go to step 2.

3. Determination of the vibration stability by Floquet's theory

The vibration stability of the periodic response (2) was evaluated by Floquet theory [1] applied in the time domain. Therefore, the stability of a periodic solution is determined by the eigenvalues of the transition matrix [3], assembled over time of one period.

In the proposed procedure the transition matrix is obtained by a repeated solution of initial value problems for differently chosen initial conditions. It is known that with regard to either accuracy or computational time the transition matrix can be approximated by the product of exponential matrices, by the relationships of Newmark integration technique [3], and others.

4. Test cases

The first test case was the Duffing oscillator with nonlinear restoring force [2]. The equation of motion can be expressed as

$$\ddot{x}(t) + 2\xi\dot{x}(t) + \omega_0^2 x(t) + \alpha[x(t)]^3 = p_0 \cos(\omega t)\,. \tag{10}$$

Numerical simulations were carried out with parameters: the damping coefficient $\xi = 0.05\,\mathrm{s}^{-1}$, the natural frequency $\omega_0 = 1\,\mathrm{rad\,s}^{-1}$, the amplitude $p_0 = 0.1\,\mathrm{mm\,s}^{-2}$, and the nonlinear coefficient $\alpha = 0.02\,\mathrm{mm}^{-2}\,\mathrm{s}^{-2}$ orange or $\alpha = 10\,\mathrm{mm}^{-2}\,\mathrm{s}^{-2}$ blue color curve, respectively, see Fig. 1.

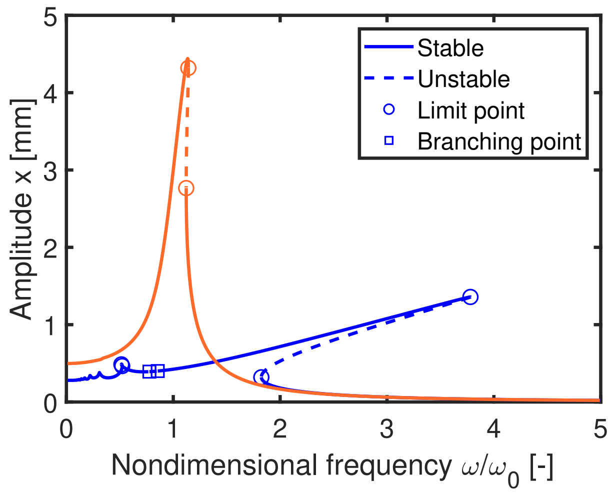

Fig. 1. Response curves of Duffing oscillator

To ensure good accuracy agreement with the time integration the $n_\mathrm{H} = 19$ harmonic terms were used to the approximation of the periodic response (2). Fig. 1 shows that the nonlinear effect is stronger for the higher value of the nonlinearity coefficient and that the limit and branching points on the response curve were detected.

Modified Jeffcott rotor according to the article [2] was employed as the second example, where different type of nonlinearity was tested. Due to the unbalance forces, the rotor can exceed clearance and interact with the stator modeled by stiffness. The equations of motion of the rotor system with contact between the disc and stator can be written as

$$m\ddot{x} + d\dot{x} + kx + k_\mathrm{c}\left(1 - \frac{h}{r}\right)[x - \mu y\,\mathrm{sign}(v_\mathrm{rel})] = p_\mathrm{b}\,\omega^2 \cos(\omega t), \tag{11}$$

$$m\ddot{y} + d\dot{y} + ky + k_\mathrm{c}\left(1 - \frac{h}{r}\right)[y + \mu x\,\mathrm{sign}(v_\mathrm{rel})] = p_\mathrm{b}\,\omega^2 \sin(\omega t), \tag{12}$$

where $r = \sqrt{x^2 + y^2}$ is the radial displacement and $v_\mathrm{rel} = \frac{x}{r}\dot{y} - \frac{y}{r}\dot{x} + R_\mathrm{disc}\omega$ is the relative velocity between the disc and stator surfaces.

The simulations were performed with parameters: the mass $m = 1\,\mathrm{kg}$, the damping coefficient $d = 5\,\mathrm{kg\,s}^{-1}$, the rotor stiffness $k = 100\,\mathrm{N\,m}^{-1}$, the stator stiffness $k_\mathrm{c} = 2500\,\mathrm{N\,m}^{-1}$, the clearance $h = 0.105\,\mathrm{mm}$, the unbalance $p_\mathrm{b} = 0.1\,\mathrm{kg\,m}$, the disc radius $R_\mathrm{disc} = 2.1\,\mathrm{mm}$, the natural frequency $\omega_0 = \sqrt{\frac{k_\mathrm{c}}{m}} = 50\,\mathrm{rad\,s}^{-1}$, and μ is the friction coefficient.

Simulation of the Jeffcott rotor was carried out with the $n_\mathrm{H} = 15$ harmonic terms and the resulting response curves are plotted in Fig. 2, where one can observe the influence of the friction coefficient value to the rotor response and locations of identified limit and Neimark-Sacker bifurcation points.

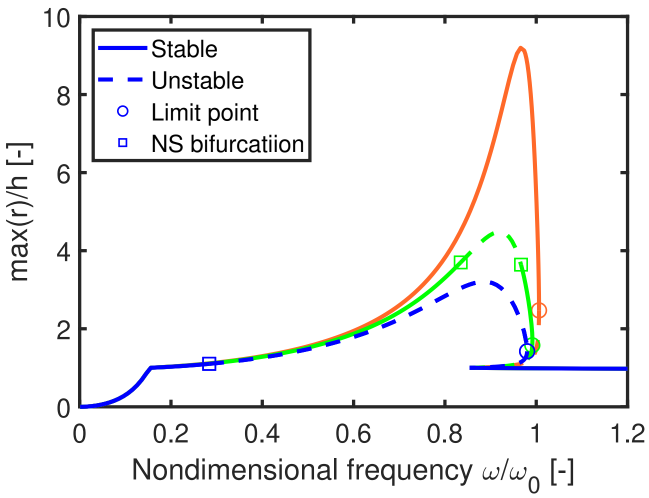

Fig. 2. Response curves of the modified Jeffcott rotor ($\mu = 0$, $\mu = 0.11$, and $\mu = 0.2$, orange, green, and blue color curve, respectively)

5. Conclusions

Procedure for the computing response curves of the nonlinear rotor dynamic models based on the harmonic balance method combined with the arc-length parametrization and Floquet's theory has been investigated. Numerical examples of Duffing oscillator and the modified Jeffcott rotor with enabled contact between the disc and stator are used for testing the developed procedure. The computed frequency responses, the vibration stability, and the locations of the limit and branch points are identical with the results presented in the article [2]. The results of the carried out study show the validity and capability of the created procedure for the computing whole frequency response curve and the determination of the vibration stability.

Acknowledgements

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