Modified Tellegen Principle Used for Power and Energy Systems Modeling

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***Abstract* – The paper deals with a problem of modeling of dynamic systems. The proposed approach to the problem solution is based on modified Tellegen's theorem well known from electrical engineering. The novelty of this approach is that it is based on the instantaneous power calculation for real linear and nonlinear systems e.g. electrical circuits, mechanical systems, heat transfer etc. Consequently, mathematically as well as physically correct results are obtained. Some known and often used system representation structures are discussed from the developed point of view as an addition. The examples are also included. The mathematical derivation and results of simulations are presented in this paper.**

Keywords – dissipation; energy; power; scalar product; state space; Tellegen

I. Introduction

It is familiar that there are two basic approaches to system modeling. The first one consists in using mathematical formulas and physical tools (a causality principle, different forms of conservation laws, power balance relations, etc.) in order to describe appropriate system behavior. It has successfully been used in many fields of science and engineering so far. However, there are also situations where physical laws are not known or cannot be expressed in a proper mathematically exact form. In that case the second basic approach to system modeling can be used. It is based on identification methods working in terms of experimentally gained data. The main aim of this contribution is to formulate power and energetic approach which can be used for different types of system. The method starts from the assumption that any physically correct system representation holds power conservation principle. Such approach can be used for different type of systems.

II. Modified Tellegen Principle

Let N be a physically correct electrical circuit with the k lumped parameters. Parameters of N can be linear or nonlinear, hysteresis or non-hysteresis, time-constant or variable. The currents in the branches are $i_k(t)$ and the voltage on these branches are $v_k(\ )$. Theorem 1. (Classic Tellegen´s theorem [1]). For branch currents $i_k(t)$ and branch voltages $v_k(t)$ holds true:

$$\left\langle i(t)^T,\ v(t)\right\rangle = \sum_{k=1}^{b} i_k(t)v_k(t) = 0 \tag{1}$$

It is worth noticing a close relation between first and second Kirchhoff´s laws (ensuring physical correctness of N) and Tellegen´s theorem. It is also important to note that inner product according (1) include instantaneous power dissipated on resistors and instantaneous power on inductors and capacitors. Finally, let's note that Tellegen's theorem applies not only to electrical circuits but to any model of a physical correct system with lumped parameters, for example mechanical, thermal, etc [2 - 4]. Therefore if the system is described appropriately by state space equations, voltages and currents are substituted by state space variables and it's derivations.

$$\left\langle x(t)^T,\ \dot{x}(t)\right\rangle = x_1(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t) .... = 0 \tag{2}$$

It is worth noticing a close relation between physical correctness and Tellegen's theorem. It is possible describe power or energy in some system by means of Tellegen‘s theorem. It is shown in first simple example.

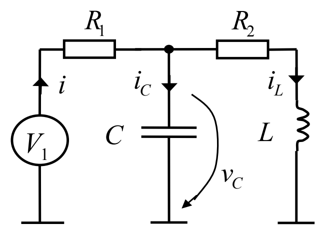

Figure 1. Simple circuit used for example 1

Example1: The state space equations of Fig. 1 are

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$$L\frac{di_L}{dt} = -R_2 i_L + v_C$$
$$C\frac{dv_C}{dt} = -i_L - \frac{1}{R_1}v_C + \frac{V_1}{R_1} \tag{3}$$

The system according by eq. (3) can be described in state space as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t);\; y(t) = C(t)x(t) \tag{4}$$

therefore

$$\frac{dx_1}{dt} = -\frac{R_2}{L}x_1 + \frac{1}{L}x_2$$
$$\frac{dx_2}{dt} = -\frac{1}{C}x_1 - \frac{1}{CR_1}x_2 + \frac{V_1}{CR_1} \tag{5}$$

in matrix forms

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R_2}{L} & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{CR_1} \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{CR_1} \end{bmatrix}}_{B} \cdot V_1 \tag{6}$$

The system according (6) can be transformed by state-space similarity transformation

$$A_1 = T\cdot A\cdot T^{-1};\; B_1 = T\cdot B;\; C_1 = C\cdot T^{-1} \tag{7}$$

The transformation matrices are

$$T = \begin{bmatrix} \sqrt{L} & 0 \\ 0 & \sqrt{C} \end{bmatrix};\; T^{-1} = \begin{bmatrix} 1/\sqrt{L} & 0 \\ 0 & 1/\sqrt{C} \end{bmatrix} \tag{8}$$

Matrices A and B after transformation (marked as A_1 and B_1) creates generalized system

$$A_1 = \begin{bmatrix} -\alpha_{11} & \alpha_2 \\ -\alpha_2 & -\alpha_{22} \end{bmatrix};\; B_1 = \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} \tag{9}$$

where

$$\alpha_{11} = \frac{R_2}{L}; \alpha_2 = \frac{1}{\sqrt{CL}}; \alpha_{22} = \frac{1}{CR_1}; \beta_2 = \frac{1}{\sqrt{C}R_1} \tag{10}$$

where relations between $x_1 \Leftrightarrow i_L$ and $x_2 \Leftrightarrow v_C$ are

$$\bar{x}_1 = \sqrt{L}\cdot i_L;\; \bar{x}_2 = \sqrt{C}\cdot v_C \tag{11}$$

Power is given as inner product

$$P(t) = \left\langle \bar{x}(t)^T, \frac{d\bar{x}(t)}{dt} \right\rangle = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}\left(A_1 \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + B_1 V_1 \right)$$
$$= -\alpha_{11}\bar{x}_1^2 + \underbrace{\alpha_2\bar{x}_1\bar{x}_2 - \alpha_2\bar{x}_1\bar{x}_2}_{0} - \alpha_{22}\bar{x}_2^2 + \beta_2 V_1\cdot\bar{x}_2 \tag{12}$$

After substitution $P(t)$ is

$$P(t) = -\frac{R_2}{L}\bar{x}_1^2 + \underbrace{\frac{\bar{x}_1\bar{x}_2}{\sqrt{LC}} - \frac{\bar{x}_1\bar{x}_2}{\sqrt{LC}}}_{0} - \frac{\bar{x}_2^2}{CR_1} + \frac{V_1\cdot\bar{x}_2}{\sqrt{C}\cdot R_1} \tag{13}$$

Using (11) and (13) for powers evaluation

$$P(t) = -\frac{R_2}{L}\bar{x}_1^2 - \frac{\bar{x}_2^2}{CR_1} + \frac{V_1\cdot\bar{x}_2}{\sqrt{C}\cdot R_1}$$
$$= -\frac{R_2}{L}\left(\sqrt{L}\cdot i_L\right)^2 - \frac{\left(\sqrt{C}\cdot v_C\right)^2}{CR_1} + \frac{V_1\cdot\left(\sqrt{C}\cdot v_C\right)}{\sqrt{C}\cdot R_1} \tag{14}$$

After some manipulations

$$P(t) = \underbrace{V_1\frac{V_1 - v_C}{R_1}}_{P_i} \underbrace{- R_2 i_L^2 - \frac{\left(V_1 - v_C\right)^2}{R_1}}_{P_D} = 0 \tag{15}$$

The power $P(t)$ described by (15) consists from input power P_i and two dissipations powers P_D (dissipation on R_1 and R_2).

The real RLC system (see Fig. 1) after similarity transformation is artificial. Moreover the transformation is complicated for high order systems or nonlinear systems. Therefore the new approach (generalized and modified Tellegen's theorem) based was derived. The main difference is that matrix Q contains of energy storage elements is used

$$P_M(t) = \left\langle \left(Q\cdot x(t)\right)^T, \frac{dx(t)}{dt} \right\rangle \tag{16}$$

where $P_M(t)$ is dissipated power. The matrix Q is square diagonal matrix size of n. For example 1, the matrix Q is

$$Q = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \tag{17}$$

Therefore $P_M(t)$ is

$$P_M(t) = \left\langle \left(Q\cdot x(t)\right)^T, \frac{dx(t)}{dt} \right\rangle =$$
$$\left\langle \left(\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)^T, \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \right\rangle =$$
$$\left\langle \begin{bmatrix} Lx_1 & Cx_2 \end{bmatrix}, \begin{bmatrix} -\frac{R_2}{L}x_1 + \frac{1}{L}x_2 \\ -\frac{1}{C}x_1 - \frac{1}{CR_1}x_2 + \frac{V_1}{CR_1} \end{bmatrix} \right\rangle \tag{18}$$

result of the scalar product is

$$P_M(t) = \left\langle Qx(t), \frac{dx(t)}{dt} \right\rangle =$$
$$-R_2 x_1^2 + \underbrace{x_1 x_2 - x_1 x_2}_{0} - \frac{1}{R_1}x_2^2 + \frac{V_1}{R_1}x_2 \tag{19}$$

where x_1=i_L and x_2=v_C and therefore result is the same as eq. (15) without unpleasant transformation. For system modelling it is possible use 2 types of models: M1-RLC for systems with complex conjugate poles and also for systems with only real poles, M2 for systems with real poles (RC or RL models), circuit diagram of the models are shown in Fig. 2, 3 and 4.

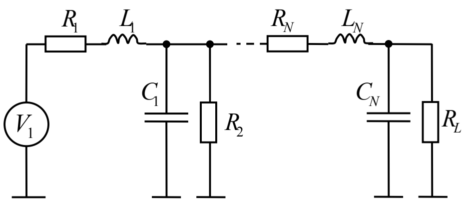

Figure 2. The ladder structure of RLC model with complex conjugate and also real poles. Model M1-RLC

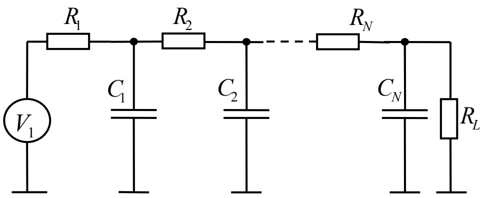

Figure 3. The ladder structure of RC model with real poles. Model M2-RC

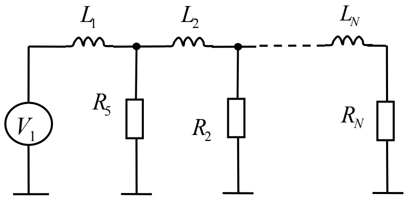

Figure 4. The ladder structure of RL model with real poles. Model M2-RL

The state space equation for model M1-RLC is eq. (20), for model M2-RC it is eq. (29) – only matrix A is presented. Eq. for model M2-RL is similar (not shown here). Equations are for 4th orders models only.

$$\begin{bmatrix} \frac{di_{L1}}{dt} \\ \frac{du_{C1}}{dt} \\ \frac{di_{L2}}{dt} \\ \frac{du_{C2}}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & -\frac{1}{L_1} & 0 & 0 \\ \frac{1}{C_1} & -\frac{1}{C_1R_2} & -\frac{1}{C_1} & 0 \\ 0 & \frac{1}{L_2} & -\frac{R_3}{L_2} & -\frac{1}{L_2} \\ 0 & 0 & \frac{1}{C_2} & -\frac{1}{C_2R_4} \end{bmatrix} \begin{bmatrix} i_{L1} \\ u_{C1} \\ i_{L2} \\ u_{C2} \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} V_1 \qquad (20)$$

with property (for linear systems)

$$\sum_{j=1}^{n} a_{ij} = 0; \quad i = 2,3...n-1; \quad i \neq j \qquad (21)$$

where matrix Q for M1-RLC - i.e. eq. (20) is

$$Q = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & C_2 \end{bmatrix} \qquad (22)$$

Power P_M for M1-RLC is calculated as

$$P_M = \left\langle (Qx)^T, \frac{dx}{dt} \right\rangle = L_1x_1\left(-\frac{R_1x_1}{L_1} - \frac{x_2}{L_1} + \frac{V_1}{L_1}\right) + C_1x_2\left(\frac{x_1}{C_1} - \frac{x_2}{C_1R_2} - \frac{x_3}{C_1}\right) + L_2x_3\left(\frac{x_2}{L_2} - \frac{R_3x_3}{L_2} + \frac{x_4}{L_2}\right) + C_2x_4\left(\frac{x_3}{C_2} - \frac{x_4}{C_2R_4}\right) \qquad (23)$$

and result after some manipulation is

$$P_M = -R_1x_1^2 + V_1x_1 - \frac{x_2^2}{R_2} - R_3x_3^2 - \frac{x_4^2}{R_4} = \underbrace{V_1x_1}_{Pi} \underbrace{-R_1x_1^2 - \frac{x_2^2}{R_2} - R_3x_3^2 - \frac{x_4^2}{R_4}}_{Pd} \qquad (24)$$

Example2: Differential equation which represents some dynamical system is

$$x^{(4)} + x^{(3)} + 3x^{(2)} + 0.8\dot{x} + x = 0 \qquad (25)$$

From eq. (20) and generalized eq. (26) (M1-RLC) it is possible find many solutions of eq. (25) in state space representations. The energy storage elements are 4 and dissipation elements can be from 1 to 4.

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{d_1}{q_1} & -\frac{1}{q_1} & 0 & 0 \\ \frac{1}{q_2} & -\frac{d_2}{q_2} & -\frac{1}{q_2} & 0 \\ 0 & \frac{1}{q_3} & -\frac{d_3}{q_3} & -\frac{1}{q_3} \\ 0 & 0 & \frac{1}{q_4} & -\frac{d_4}{q_4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{q_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} V_1 \qquad (26)$$

d_i – dissipation, q_i – energy storage element. One of many possible solutions is (matrix has the same eigenvalues): d_1=1; d_2=0; d_3=0; d_4=0; q_1=1; q_2=0.4545; q_3=6.37; q_4=0.346. The system with one dissipation is given in eq. (27)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 2.2 & 0 & -2.2 & 0 \\ 0 & 0.157 & 0 & -0.157 \\ 0 & 0 & 2.89 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_1 \qquad (27)$$

or C_2=parameter and L_1=1/(2.89·C_2); C_1=C_2/0.76; L_2=1/(0.4545*C_2); R_1=L_1; see eq. (19). Other solution, i.e. system with 2 dissipations is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}-0.85 & -2.35 & 0 & 0\\ 1 & 0 & -1 & 0\\ 0 & 0.08 & 0 & -0.08\\ 0 & 0 & 5.13 & -0.154\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}2.35\\ 0\\ 0\\ 0\end{bmatrix}V_1 \quad (28)$$

The systems according eq. (27) and (28) (M1-RLC) are real systems with *RLC* components, e.g. for eq. (28) and according eq. (20): R_1=0.362; L_1=0.43; C_1=1; L_2=12.5; C_2=0.195; R_4=33.3.

For M2-RC (only matrix A is presented) matrix is

$$A=\begin{bmatrix}\frac{-1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{1}{C_1R_2} & 0 & 0\\ \frac{1}{C_2R_2} & \frac{-1}{C_2}\left(\frac{1}{R_2}+\frac{1}{R_3}\right) & \frac{1}{C_2R_3} & 0\\ 0 & \frac{1}{C_3R_3} & \frac{-1}{C_3}\left(\frac{1}{R_3}+\frac{1}{R_4}\right) & \frac{1}{C_3R_4}\\ 0 & 0 & \frac{1}{C_4R_4} & \frac{-1}{C_4}\left(\frac{1}{R_4}+\frac{1}{R_5}\right)\end{bmatrix} \quad (29)$$

with property (for linear systems)

$$\sum_{j=1}^{n} a_{ij}=0;\quad i=2,3...n-1 \quad (30)$$

Example 3: Differential equation for M2-RC is given in eq. (31)

$$x^{(4)}+10x^{(3)}+35x^{(2)}+50\dot{x}+24x=0 \quad (31)$$

Equation (31) can be converted to system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}-2.82 & 0.62 & 0 & 0\\ 1.84 & -2.16 & 0.32 & 0\\ 0 & 2.73 & -2.85 & 0.12\\ 0 & 0 & 2.17 & -2.17\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0.62\\ 0\\ 0\\ 0\end{bmatrix}\cdot V_1 \quad (32)$$

It is possible calculate values of resistors and capacitors from (29) and (32) for chosen C_1=1 (as parameter). Other values are: R_1=0.45; C_2=0.33; R_2=1.62; C_3=0.039; R_3=9.29; C_4=0.0022; R_4=211 and matrix Q for M2-RC is

$$Q=\begin{bmatrix}C_1 & 0 & 0 & 0\\ 0 & C_2 & 0 & 0\\ 0 & 0 & C_3 & 0\\ 0 & 0 & 0 & C_4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0.33 & 0 & 0\\ 0 & 0 & 0.039 & 0\\ 0 & 0 & 0 & 0.0022\end{bmatrix} \quad (33)$$

therefore according M2-RC (28), (32) and (15) P_M is

$$P_M=\left\langle (Qx)^T,\frac{dx}{dt}\right\rangle=C_1\cdot x_1\cdot\frac{dx_1}{dt}+ C_2\cdot x_2\cdot\frac{dx_2}{dt}+C_3\cdot x_3\cdot\frac{dx_3}{dt}+C_4\cdot x_4\cdot\frac{dx_4}{dt} \quad (34)$$

after adjustments (34) according to the equation (23)

$$P_M=V_1\frac{V_1-v_{C1}}{R_1}-\frac{(V_1-v_{C1})^2}{R_1}-\frac{(v_{C1}-v_{C2})^2}{R_2} -\frac{(v_{C2}-v_{C3})^2}{R_3}-\frac{(v_{C3}-v_{C4})^2}{R_4} \quad (35)$$

Energy of the system is calculated according

$$E(t)=\int_0^\infty P_M(t)dt \quad (36)$$

where q_{ii} are diagonal elements of Q.

Example 4: Nonlinear autonomous dissipative Duffing system differential equation is

$$\ddot{x}+b\dot{x}-x+x^3=0 \quad (37)$$

where dissipative parameter b=0.1. In state space

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=-by+x-x^3\end{aligned} \quad (38)$$

Previous equations can be rewritten as nonlinear electrical system

$$\begin{aligned}\frac{dv_C}{dt}&=\frac{1}{C}i_L\\ \frac{di_L}{dt}&=\frac{1}{L}\left(-R\cdot i_L+v_C-v_C^3\right)\end{aligned} \quad (39)$$

Power P_M given by scalar product is

$$P_M=\left\langle\left(\begin{bmatrix}C & 0\\ 0 & L\end{bmatrix}\cdot\begin{bmatrix}v_C\\ i_L\end{bmatrix}\right)^T,\begin{bmatrix}\dot{v}_C\\ \dot{i}_L\end{bmatrix}\right\rangle =v_C\cdot i_L+i_L\cdot v_C-i_L\cdot v_C^3-R\cdot i_L^2 \quad (40)$$

The simulation results are shown in Fig. 5 to Fig. 6 which presents time evolution of signals, Fig. 7 and 8 concerning power P_M and only dissipated power $-R\cdot i_L^2$, Fig, 9 and 10 which displays energy of the system and dissipated energy respective. It should be noticed that energy is calculated by integration of power.

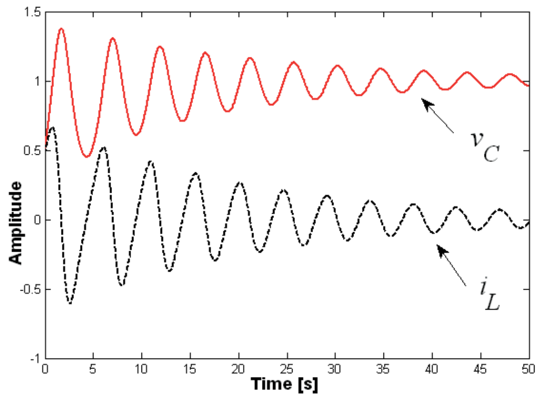

Figure 5. Dissipative Duffing system. Time evolution of v_C and i_L signals

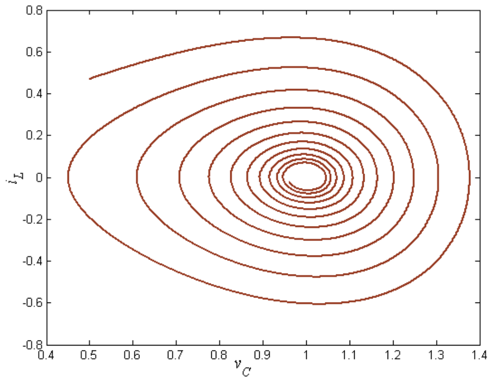

Figure 6. Dissipative Duffing system. Phase space evolution of v_C and i_L signals

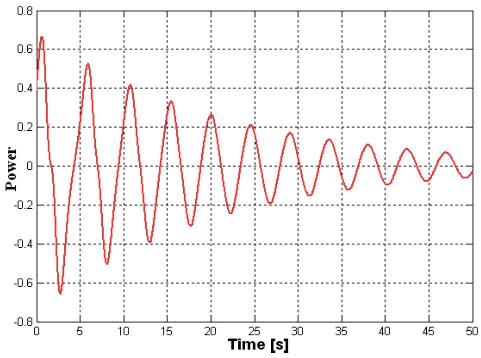

Figure 7. Dissipative Duffing system. Time evolution of power P_M according (40)

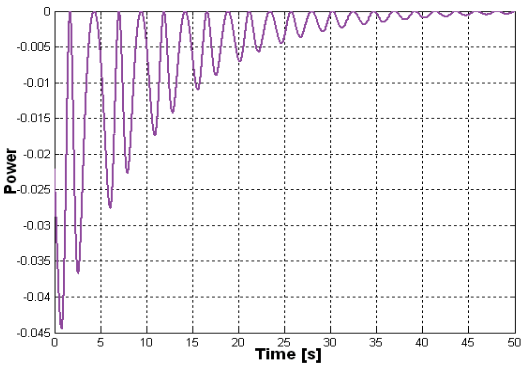

Figure 8. Dissipative Duffing system. Time evolution of dissipative power given by $-R \cdot i_L^2$

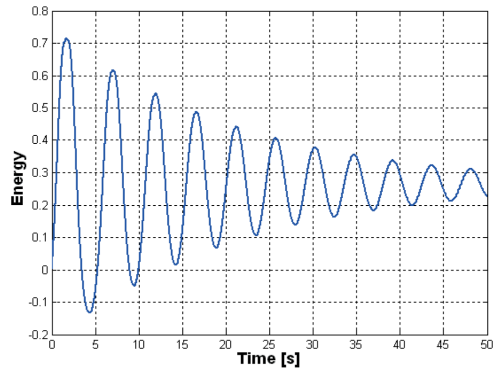

Figure 9. Dissipative Duffing system. Time evolution of energy given by integration of P_M

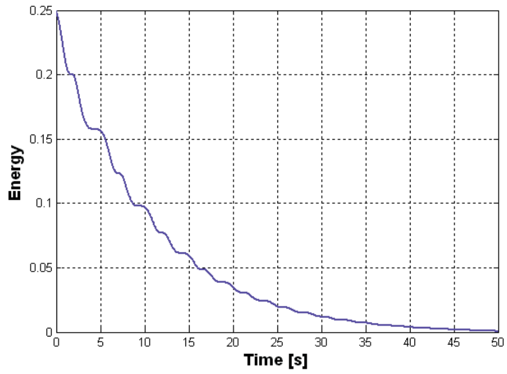

Figure 10. Dissipative Duffing system. Time evolution of dissipative energy given by integration $-R \cdot i_L^2$

III. Conclusion

In this paper the power and energy approach based on generalized Tellegen principle was used for linear and nonlinear dynamical systems. The theory was confirmed by simulation on several examples. Presented practice can be used for different types of systems which can be described by ordinary differential equation or set of differential equations.

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