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Some aspects of isogeometric analysis discretization of the incompressible fluid flow problem

J. Egermaier, H. Horníková

Department of Mathematics and European Centres of Excellence New Technologies for the Information Society, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 301 00, Plzeň, Czech Republic

1. Introduction

We focus on efficient numerical solution of the incompressible Navier-Stokes equations using our in-house solver based on the isogeometric analysis (IgA) approach. IgA uses the isoparametric approach, i.e., the same basis functions are used for description of the computational domain geometry and also for representation of the solution. The primary goal of using isogeometric analysis is to be always geometrically exact, independently of the discretization, and to avoid generation of computational meshes which is often a very time-consuming step for finite element (FEM) and finite volume (FVM) methods. Since the computational domains are usually designed as B-spline or NURBS objects in the industrial practice, IgA relies on B-spline/NURBS basis for representation of the solution. The B-spline/NURBS discretization basis has several specific properties different from standard finite element basis, most importantly a higher interelement continuity leading to denser matrices. Our aim is also to developed efficient solver of these systems by a preconditioned Krylov subspace method. Therefore, the efficiency of the ideal and approximate versions of suitable state-of-the-art block preconditioners for the Navier-Stokes equations is also discussed.

2. Navier-Stokes equations

The mathematical model is based on the incompressible Navier-Stokes equations. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, d being the number of spatial dimensions, with the boundary $\partial\Omega$ consisting of two complementary parts, Dirichlet $\partial\Omega_D$ and Neumann $\partial\Omega_N$. The steady-state incompressible Navier-Stokes problem is given as a system of $d + 1$ differential equations together with boundary conditions

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{0} && \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}_D && \text{on } \partial\Omega_D, \\ \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p &= \mathbf{0} && \text{on } \partial\Omega_N, \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the flow velocity, p is the kinematic pressure, ν is the kinematic viscosity and $\mathbf{g}_D$ is a given function. If the velocity is specified everywhere on the boundary, the pressure solution is only unique up to a hydrostatic constant.

3. Numerical model

The nonlinear problem (1) is linearized by Picard method and discretized using isogeometric analysis approach, see [1] for details. IgA is a relatively new discretization approach [2] based on Galerkin method, where the basis of the discrete solution space is taken from the B-spline/NURBS representation of the computational domain Ω. We limit ourselves to the B-spline discretization basis in this work. The discretization, similarly to finite element method, leads to a sparse non-symmetric linear system of saddle-point type

$$\begin{bmatrix} \mathbf{F} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \tag{2}$$

where $\mathbf{F}$ is block diagonal with the diagonal blocks consisting of the discretization of the viscous term and the linearized convective term, $\mathbf{B}^T$ and $\mathbf{B}$ are discrete gradient and negative divergence operators, respectively. An efficient iterative solver is necessary for solving large systems (2) because direct solvers are very time and memory consuming. Krylov subspace methods are the most commonly used in similar applications and can be very efficient if combined with a good preconditioning technique. Since our matrices are non-symmetric, we choose a Krylov subspace method GMRES.

4. Preconditioning techniques

In contrast of standard finite element method, the B-spline basis is generally of higher continuity across the element boundaries. This leads to denser matrices, which makes the linear system more expensive to solve. We are interested in the convergence behavior of the preconditioned GMRES with several block preconditioners, which were developed for finite element discretizations, especially its dependence on the B-spline basis degree and continuity.

Due to the form of linear system (2) we use some block preconditioners based on the decomposition

$$\begin{bmatrix} \mathbf{F} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B}\mathbf{F}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{F}^{-1}\mathbf{B}^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{3}$$

where $\mathbf{S} = -\mathbf{B}\mathbf{F}^{-1}\mathbf{B}^T$ is the Schur complement, which is approximated in different ways. The tested precoditiners are LSC (Least-Squares Commutator), PCD (Pressure Convection-Diffusion), AL (Augmented Lagrangian) and SIMPLE (Semi-Implicit Method for Pressure Linked Equations) type preconditioners. An overview of these preconditioners can be found e.g. in [3].

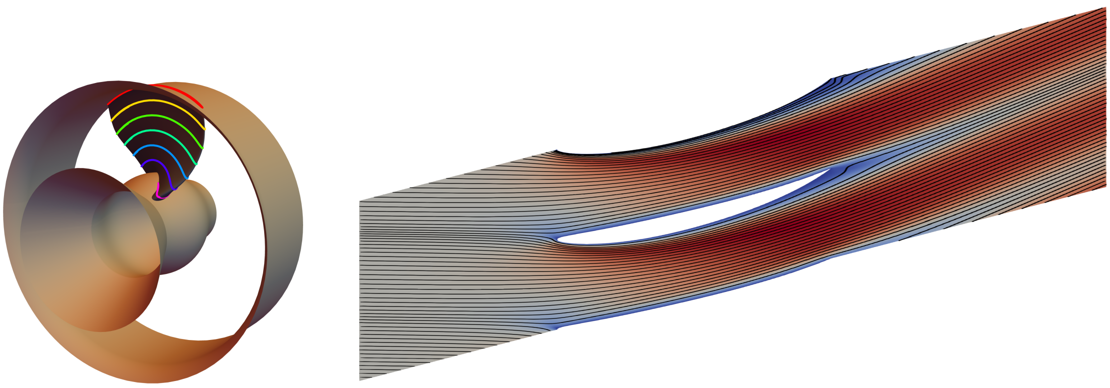

Fig. 1. Blade cascade flow

5. Numerical experiments

We chose several test problems to show and compare convergence properties of GMRES method with the particular preconditioners, especially dependence on the mesh refinements, Reynolds number, the degree of continuity of the solution across the element interfaces, etc.

One of the test examples is the 2D laminar blade profile cascade flow, see Fig. 1 (right). The computational domain is constructed by unfolding a cylindrical slice of the runner wheel domain, see Fig. 1 (left). The second test example is the 3D flow over a backward facing step domain. Both the steady and unsteady cases have been considered.

In Fig. 2 we can see example of convergence properties of selected preconditioners - residual norm of the solution dependency on the numbers of iterations over three levels of uniform mesh refinements. Third-order B-splines with C^2 interelement continuity are considered.

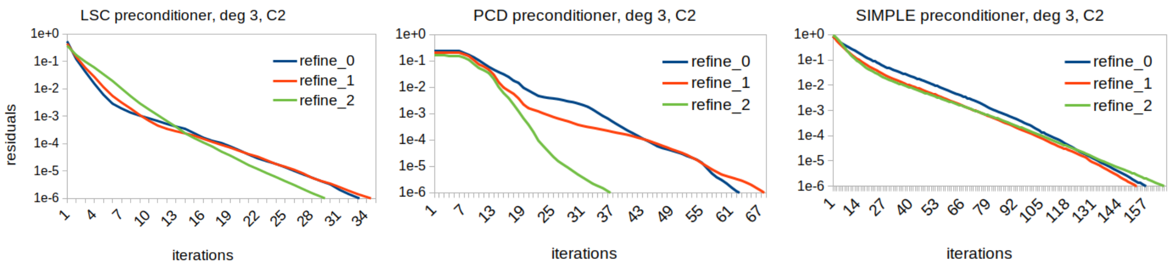

Fig. 2. Sensitivity of convergence to uniform mesh refinement of 2D blade profile domain

6. Conclusions

We tested the block preconditioners on several test problems discretized using B-spline basis of various degree and continuity. Based on these experiments, we can conclude, that higher interelement continutity of the discretization, which is typical for IgA, does not have a negative impact on the convergence of the preconditioned GMRES. Moreover, the opposite seems to be true in many cases. For example, some tested preconditioners were less sensitive to uniform mesh refinement for discretizations of high continuity. In the unsteady case, higher continuity even improves the convergence of GMRES for most preconditioners.

Acknowledgements

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