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Inverse problem for nonlinear Gao beam and foundation

J. Radová, J. Machalová

Faculty of Science, Palacký University Olomouc, 17. listopadu 12, 779 00 Olomouc, Czech Republic

This contribution deals with the identification problem for a nonlinear beam and two types of foundation – an elastic deformable and perfectly rigid foundation. We study the nonlinear model beam known as a Gao beam that was published by Prof. D. Y. Gao in [1]. A small correction of the Gao beam model was presented in the recent paper [3]. On the basis of this correction the Gao beam model equation reads as follows

$$E\,I\,w^{IV} - \,E\,\alpha\,(w')^2\,w'' + P\,\mu\,w'' \;=\; f \qquad \text{in}\;\;(0,L), \tag{1}$$

where

$$\alpha \;=\; 3\,t\,b\,(1-\nu^2), \qquad \mu \;=\; (1-\nu^2), \qquad f \;=\; (1-\nu^2)\,q$$

and E is Young's elastic modulus, $I = \frac{2}{3}t^3 b$ denotes a constant area inertia moment of the cross-section, where $2t$ is a thickness, and b width of the beam. Further, the transverse displacement is denoted by w, ν is the Poisson ratio and q denotes the applied traversal load per unit length L of the beam. Finally, P stands for the constant axial force acting at the end point $x = L$. We shall distinguish two cases: $P > 0$ causing the compression of the beam and $P < 0$ causing its tension. The beam model needs to be completed by one of the following stable and unstable boundary conditions:

(B1) simply supported beam: $w(0) = w(L) = 0;\; w''(0) = w''(L) = 0,$

(B2) clamped beam: $w(0) = w'(0) = w(L) = w'(L) = 0,$

(B3) propped cantilever beam: $w(0) = w'(0) = w(L) = 0;\; w''(L) = 0,$

(B4) cantilever beam: $w(0) = w'(0) = 0;$

$$w''(L) = \,E\,I\,w'''(L) - \tfrac{1}{3}E\,\alpha\,(w'(L))^3 + P\,\mu\,w'(L) = 0.$$

Firstly, we consider the contact problem for the Gao beam situated above a perfectly rigid obstacle. The following analysis is restricted to the clamped Gao beam with the boundary conditions (B2), see Fig. 1. The gap between the beam and the foundation is defined by a function $g(x)$ and the contact force is denoted by $T(w)$. If $w(x) = g(x)$ at $x \in (0, L)$, then the beam is in a contact with the foundation at x and $T(w(x)) \geq 0$. If $w(x) > g(x)$ at $x \in (0, L)$, then there is no contact at x and thus $T(w(x)) = 0$. Therefore, Eq. (1) and the contact conditions on $(0, L)$ read as

$$E\,I\,w^{IV} - \,E\,\alpha\,(w')^2\,w'' + P\,\mu\,w'' \;=\; f \,+\, T(w) \qquad \text{in} \qquad (0,L), \tag{2}$$

$$\left.\begin{array}{ll} w & \geq\; g \\ T(w) & \geq\; 0 \\ (w-g)\,T(w) & =\; 0 \end{array}\right\} \quad \text{in}\;\;(0,L). \tag{3}$$

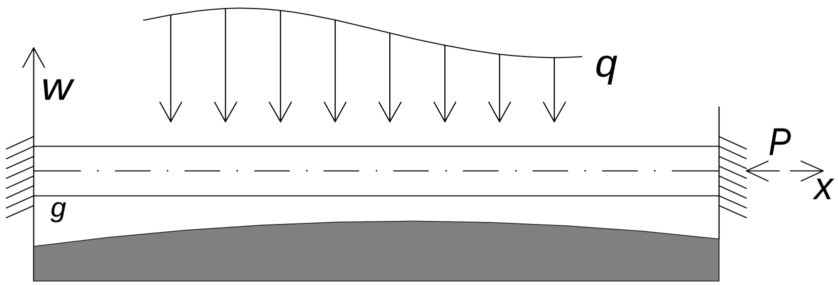

Fig. 1. Clamped beam situated above the foundation

The relations (3) are well known as the Signorini conditions. The variational formulation reads as follows

$$\begin{cases} \text{Find } w \in K \text{ such that} \\ a(w, v-w) + \pi(w, v-w) \geq \mathcal{L}(v-w), \qquad \forall v \in K, \end{cases} \tag{$\mathcal{P}_R$}$$

where

$$a(w,v) = \int_0^L EIw''\, v''\mathrm{d}x - \int_0^L P(1-\nu^2)w'\, v'\mathrm{d}x, \tag{4}$$

$$\pi(w,v) = \int_0^L E\, t\, b\, (1-\nu^2)(w')^3 v'\mathrm{d}x, \qquad \mathcal{L}(v) = \int_0^L (1-\nu^2)\, q\, v\, \mathrm{d}x$$

and the set K of admissible deflection is defined by

$$K = \{v \in V : v \geq g \,\text{in}\, (0,L)\}, \tag{5}$$

where $V = H_0^2((0,L))$ is the space of admissible displacements.

Secondly, we consider the contact problem with an elastic deformable foundation, thus, i.e., the beam can penetrate into it in some parts of $(0,L)$. In this case, a contact force $T(w)$ is generated. We shall consider so called normal compliance model

$$T(w) = k_F\, b\, (1-\nu^2)(g-w)^+, \tag{6}$$

where $k_F > 0$ denotes the foundation modulus and $(u(x))^+ = \max\{0, u(x)\}$. The governing equation is given again by (2) with $T(w)$ defined by (6). The corresponding variational formulation is given by

$$\begin{cases} \text{Find } w \in V \text{ such that} \\ a(w,v) + \pi(w,v) - \kappa(w,v) = \mathcal{L}(v), \qquad \forall v \in V, \end{cases} \tag{$\mathcal{P}_D$}$$

where $a(w,v)$, $\pi(w,v)$ and $\mathcal{L}(v)$ are defined by (4),

$$\kappa(w,v) = \int_0^L k_F\, b(1-\nu^2)(g-w)^+\, v\, \mathrm{d}x. \tag{7}$$

The solution of similarly problem as ($\mathcal{P}_D$), we can find in [2], where the contact problems for the Gao beam with elastic deformable foundation without the gap $g(x)$ were analyzed.

The main idea of the identification problem is to determine the material coefficients (E, ν) of the static Gao beam equation in the contact problem with an elastic deformable or rigid foundation by using an *optimal control approach*.

The identification problem is formulated as the minimization of a least squares cost functional depending on the solution to problems $(\mathcal{P}_R)$ or $(\mathcal{P}_D)$. The set of admissible parameters U_{ad} is defined by

$$\begin{aligned} U_{ad} &:= \{p \in L^\infty((0,L)) \times L^\infty((0,L)) : \\ &\qquad 0 < p_{\min} \le p \le p_{\max} < \infty \text{ in } (0,L), p|_{K_i} \in P_0(K_i) \times P_0(K_i),\ i = 1, \dots, r\}, \end{aligned} \quad (8)$$

where $p = (E, \nu)$ and $p_{\min}, p_{\max}$ are given vectors. We suppose that the interval $(0, L)$ is decomposed into mutually disjoint open intervals K_i, $i = 1, \dots, r$, i.e. $K_i \cap K_j = \emptyset, \forall i \neq j$, and $\langle 0, L\rangle = \bigcup_{i=1}^{r} \overline{K}_i$. Further $P_0(K_i)$ is the set of constant functions on the subintervals K_i. The cost functional $\mathcal{J} : U_{ad} \longrightarrow \mathbb{R}$ is defined by

$$\mathcal{J}(w(p)) = \frac{1}{2}\|w(p) - z\|^2, \quad (9)$$

where $\|\cdot\|$ is L^2-norm, z is a target deflection and $w(p)$ is a solution of the contact problem for Gao beam, see in [4]. Finally, the identification problem is formulated as follows

$$\begin{cases} \text{Find } p^* \in U_{ad} \text{ such that} \\ J(w(p^*)) \ = \ \min\limits_{p \in U_{ad}} J(w(p)), \\ \text{where } w(p) \text{ solves the contact problem} (\mathcal{P}_R) \text{ or } (\mathcal{P}_D). \end{cases} \quad (10)$$

Numerical solution of the minimization problem (10) is based on using the standard finite element method and discretization is composed of two parts. The first part is the discretization of the problems $(\mathcal{P}_R)$ and $(\mathcal{P}_D)$ that are solved by using the penalized method and nonsmooth Newton method. The second part concerns the discretization of the cost functional that is given by

$$\mathbf{J}(\mathbf{p}) = \frac{1}{2}\|\mathbf{S}\mathbf{w}(\mathbf{p}) - \mathbf{z}\|^2,$$

where $\mathbf{S}$ is the matrix representing the restriction mapping, $\mathbf{z}$ denotes the vector of given measured data and $\|\cdot\|$ is $L^2((0,L))$-norm. Finally, we obtain the nonlinear programming problem

$$\begin{cases} \text{Find vector } \ \mathbf{p}^* \in U_{ad} \ \text{ such that} \\ \mathbf{J}(\mathbf{p}^*) = \min\limits_{\mathbf{p} \in U_{ad}} \mathbf{J}(\mathbf{p}), \\ \text{where } \ \mathbf{w}(\mathbf{p}) \ \text{ solves the discretized problems } (\mathcal{P}_R) \text{ or } (\mathcal{P}_D). \end{cases} \quad (11)$$

The nonlinear problem (11) can be solved by the nonlinear conjugate gradient method and the gradient is computed by using so called adjoint state problem. For more information we refer to [5] and [4]. Finally, we state a numerical scheme for the nonlinear problem (11):

- Compute $\mathbf{w}^0(\mathbf{p}^0)$ for given $\mathbf{p}^0 := (\mathbf{E}^0, \boldsymbol{\nu}^0)$.
- Compute gradient $\mathbf{g}^0 = \mathbf{g}(\mathbf{p}^0)$ of $\mathbf{J}(\mathbf{p}^0)$ by using adjoint state problem.
- Let $\mathbf{d}^0 = -\mathbf{g}^0$.

For $k = 0, 1, 2, \cdots$ (until the stopping criterion is fulfilled)

1. Determine step length $\alpha^k > 0$ by using Wolfe conditions.
2. Set $\mathbf{p}^{k+1} = \mathbf{p}^k + \alpha^k \mathbf{d}^k$ and compute $\mathbf{w}^{k+1}(\mathbf{p}^{k+1})$.
3. Compute $\mathbf{g}^{k+1} = \mathbf{g}(\mathbf{p}^{k+1})$ by using adjoint state problem.
4. Compute $\beta^{k+1} = \dfrac{(\mathbf{g}^{k+1})^\top \mathbf{g}^{k+1}}{(\mathbf{g}^k)^\top \mathbf{g}^k}$.
5. Let $\mathbf{d}^{k+1} = -\mathbf{g}^{k+1} + \beta^{k+1} \mathbf{d}^k$.
6. Set $k = k + 1$.

Numerical computations were realized by using the mathematical software Matlab.

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