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An effective MATLAB minimizer of nonlinear energies arising from different scalar and vector problems

Alexej Moskovka[1], Jan Valdman[2]

1 Introduction

Nonlinear energy functionals appearing in the calculus of variations can be discretized by the finite element (FE) method and formulated as a sum of energy contributions from local elements. In Matonoha et al. (2022) we suggested the first version of the solver which was able to solve scalar problems and later extended to solving vector problems (Moskovka and Valdman (2022)). A fast evaluation of energy functionals containing the first order gradient terms is a central part of this contribution. We describe a vectorized implementation using the simplest linear nodal (P1) elements in which all energy contributions are evaluated all at once without the loop over triangular or tetrahedral elements. Furthermore, in connection to the first-order optimization methods, the discrete gradient of energy functional is assembled in a way that the gradient components are evaluated over all degrees of freedom all at once. The key ingredient is the vectorization of exact or approximate energy gradients over nodal patches. It leads to a time-efficient implementation at higher memory-cost. Provided codes in MATLAB related to 2D/3D hyperelasticity and 2D p-Laplacian problem are available for download and structured in a way it can be easily extended to other types of vector or scalar forms of energies.

2 Scalar and vector problems

We introduce a p-Laplace equation as the basic example of scalar problems given by

$$\begin{aligned} \Delta_p u &= f \qquad \text{in } \Omega\,, \\ u &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

for some $p > 1$. It is knows that solving (1) is equivalent to finding the minimum of the corresponding energy functional

$$J(u) = \min_{v \in V} J(v)\,, \qquad J(v) := \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega f\, v \,\mathrm{d}x\,, \tag{2}$$

where

$$V = W_g^{1,p}(\Omega) = \{v \in W^{1,p}, v = g \ \text{ on } \partial\Omega\}.$$

Fig. 1 depicts the solution of (2) for the L-shape domain and the constant right-hand side.
One of the most well-known vector problems from mechanics is the nonlinear elasticity which describes the deformation of the material loaded by a force. The corresponding energy functional is given by

$$J(\mathbf{v}) = J_{grad}(\mathbf{v}) - J_{lin}(\mathbf{v})\,,$$

[1] A. Moskovka is a Ph.D. student in the Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Technická 8, 30100 Plzeň, Czech Republic (alexmos@kma.zcu.cz)
[2] The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod vodárenskou věží 4, 18208 Praha 8, Czech Republic (jan.valdman@utia.cas.cz)

$$J_{grad}(\mathbf{v}) = \int_\Omega W\big(\mathbf{F}(\mathbf{v}(\mathbf{x}))\big)\,\mathrm{d}\mathbf{x}, \qquad J_{lin}(\mathbf{v}) = \int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} \mathbf{g}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,,$$

where $\mathbf{v} : \Omega \to \mathbb{R}^{dim}$ is a deformation mapping, $\mathbf{F} = \nabla(\mathbf{v})$, $\mathbf{f}$ is a loading, $\mathbf{g}$ is an external load on the Neumann part of the boundary and $W(\mathbf{F})$ is a density function which depends on the model choice. Fig. 2 depicts the time dependent deformation of a 3D cuboid which is twisted around the x-axis. The torsion is induced by the nonhomogenious Dirichlet boundary condition on the right wall.

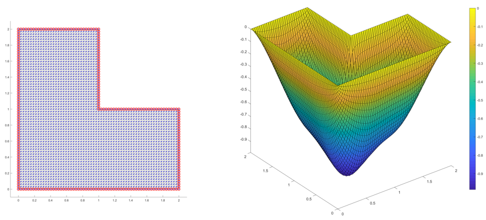
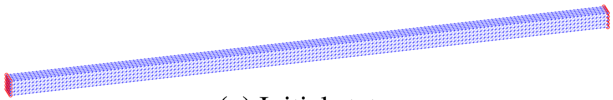
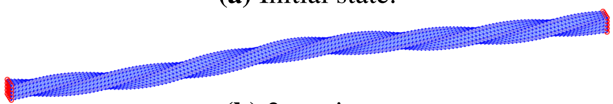


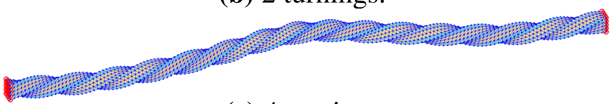
Figure 1: A triangulation of the L-shape domain (left) and the solution of p-Laplace problem for $p = 3$ and constant loading $f(\mathbf{x}) = -10$ (right).



(a) Initial state.



(b) 2 turnings.



(c) 4 turnings.

Figure 2: A triangulation of the 3D cuboid (top) and its torsional deformation around the 'x' axis.

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References

Matonoha C., Moskovka A., Valdman J. (2022) Minimization of p-Laplacian via the Finite Element Method in MATLAB, I. Lirkov and S. Margenov (eds): LSSC 2021, LNCS 13127, pp. 496–503.

Moskovka A., Valdman J. (2022) Fast MATLAB evaluation of nonlinear energies using FEM in 2D and 3D: Nodal elements (2022), Applied Mathematics and Computation 424, 127048.