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Non-uniform quaternion spline interpolation in vehicle kinematics

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Interpolation plays an important role in nowadays world. By interpolating data, we save time and money in general. The main areas where interpolation is applied are robotics, automotive, medicine, biology etc. One of the possible basis splines for interpolation are B-splines, which are also used in Computer Aided Geometric Design (CAGD) due to their smoothness and locality properties [5]. To fit a curve to a given set of points, B-spline can be used either in interpolation or in approximation [5]. In this work we consider the application of B-splines (cumulative) for the non-uniform interpolation of quaternions. This requires to overcome some difficulties. Firstly it is necessary to compute control points (sometimes called de Boor points [4]) to fulfil the basic interpolation property. Second problem is hidden in non-uniformity of data points as formulas available for quaternion spline interpolation generally consider uniformly distributed points [4]. The last problem lies in discretization: to achieve desired maximum error of the interpolation we have to choose the proper density of interpolation points.

B-spline is a spline function driven by an independent parameter that will be denoted here by t which usually varies from $t = T_0$ to $t = T_n$, with $\mathbf{T}$ the knot vector (T_0,..., T_n) and n positive integers. The associated B-splines B_i^k of order k (degree = $k-1$) are defined by [4]

$$B_i^1(t) \quad = \begin{cases} 1 & T_i \le t \le T_{i+1}, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for $k = 1$ and

$$B_i^k(t) \quad = \tfrac{t-T_i}{T_{i+k-1}-T_i} B_i^{k-1}(t) + \tfrac{T_{i+k}-t}{T_{i+k}-t_{i+1}} B_{i+1}^{k-1}(t), \quad k > 1 \text{ and } i = 0, 1, ..., n. \tag{2}$$

We can now define B-spline curve [3]. With a given list of control points (also called de Boor points), $p_i \in R^m$ ($m \ge 1$), $0 \le i \le n$, and a knot vector $\mathbf{T}$, then B-spline interpolation of order k (degree $k-1$) is defined by

$$P(t) = \sum_{i=0}^{n} p_i B_i^k(t) \qquad \text{for } T_0 \le t < T_n. \tag{3}$$

With a given list of data points $P_i \in R^m$, $0 \le i \le n$, corresponding with the knot vector $\mathbf{T}$, we can proceed with the B-spline interpolation of order k. First we need to compute control points so that we ensure

$$P(T_i) = P_i, \tag{4}$$

which can be rewritten into the form

$$p_0 B_0^k(T_i) + p_1 B_1^k(T_i) + ... + p_n B_n^k(T_i) = P_i, \qquad 0 \le i \le n. \tag{5}$$

This can be rewritten [5] into condensed matrix form for all rows

$$\mathbf{Ap} = \mathbf{P}, \tag{6}$$

where

$$\mathbf{A} = \begin{pmatrix} B_0^k(T_0) & B_1^k(T_0) & \dots & B_n^k(T_0) \\ B_0^k(T_1) & B_1^k(T_1) & \dots & B_n^k(T_1) \\ \vdots & \vdots & \vdots & \vdots \\ B_0^k(T_n) & B_1^k(T_n) & \dots & B_n^k(T_n) \end{pmatrix}, \text{ where } \begin{cases} \mathbf{p} = [p_0, p_1, ..., p_n]^T, \\ \mathbf{P} = [P_0, P_1, ..., P_n]^T. \end{cases} \tag{7}$$

With respect to the previous definitions we can define B-spline quaternion interpolation of order 4 ($k = 4$, degree $= 3$), so that we achieve C^2 continuity. We consider unit quaternion definition given by [2] in this work. With a given sequence of data points (data unit quaternions) Q_i ($i = 0, 1,...,n$), the interpolation can be proceeded by constructing the B-spline quaternion curve $Q(t)$ which interpolates a given sequence of unit quaternions Q_i ($i = 0, 1,...,n$). The B-spline quaternion curve is defined as

$$Q(t) = q_{-1}^{\tilde{B}_0(t)} \prod_{i=0}^{n+1} (q_{i-1}^{-1} q_i)^{\tilde{B}_i(t)}, \qquad \tilde{B}_i(t) = \sum_{j=i}^{n+1} B_i(t), \tag{8}$$

where q_i are control points ('control quaternions') and $B_i(t) \equiv B_i^4(t)$. To compute the control points from non-uniform knot vector, we have to start with the condition $Q(T_i) = Q_i$, so we get

$$\begin{aligned} &q_0^{\tilde{B}_0(T_i)} (q_0^{-1} q_1)^{\tilde{B}_1(T_i)} (q_1^{-1} q_2)^{\tilde{B}_2(T_i)} ... (q_{i-2}^{-1} q_{i-1})^{\tilde{B}_{i-1}(T_i)} ... \\ &(q_{i-1}^{-1} q_i)^{\tilde{B}_i(T_i)} (q_i^{-1} q_{i+1})^{\tilde{B}_{i+1}(T_i)} (q_{i+1}^{-1} q_{i+2})^{\tilde{B}_{i+2}(T_i)} ... = Q_i. \end{aligned} \tag{9}$$

It holds that $\tilde{B}_x(T_i) = 1$, $0 \le x \le i-1$ and $\tilde{B}_y(T_i) = 0$, $i+2 \le y \le n+1$, following this rule we get

$$q_0 (q_0^{-1} q_1)(q_1^{-1} q_2) ... (q_{i-2}^{-1} q_{i-1})(q_{i-1}^{-1} q_i)^{\tilde{B}_i(T_i)} (q_i^{-1} q_{i+1})^{\tilde{B}_{i+1}(T_i)} \cdot 1 = Q_i, \tag{10}$$

which can be rewritten as

$$q_{i-1} (q_{i-1}^{-1} q_i)^{\tilde{B}_i(T_i)} (q_i^{-1} q_{i+1})^{\tilde{B}_{i+1}(T_i)} = Q_i, \quad \text{for } i = 0, 1, 2, ..., n, \tag{11}$$

Note that for uniformly distributed knot vector $\mathbf{T}$, we can easily write

$$q_{i-1} (q_{i-1}^{-1} q_i)^{\frac{5}{6}} (q_i^{-1} q_{i+1})^{\frac{1}{6}} = Q_i. \tag{12}$$

Eq. (11) forms $n+1$ equations. Since the dominant term on the left side is $(q_{i-1}^{-1} q_i)^{\tilde{B}_i(t_i)}$ we can use the iterative refinement procedure for the solution from [3] as

$$(q_{i-1}^{-1} q_i)^{\tilde{B}_i(T_i)} = q_{i-1}^{-1} Q_i (q_i^{-1} q_{i+1})^{-\tilde{B}_{i+1}(T_i)}, \tag{13}$$

$$q_i^* = q_{i-1} (q_{i-1}^{-1} Q_i (q_i^{-1} q_{i+1})^{-\tilde{B}_{i+1}(T_i)})^{\frac{1}{\tilde{B}_i(T_i)}}, \tag{14}$$

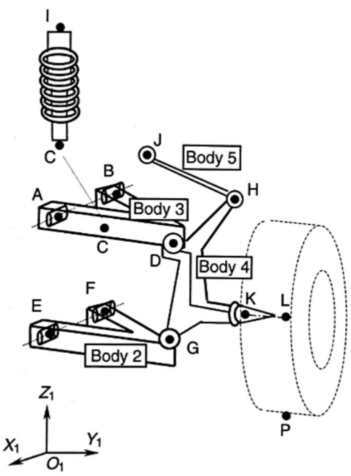

Fig. 1. Double wishbone suspension example [1]

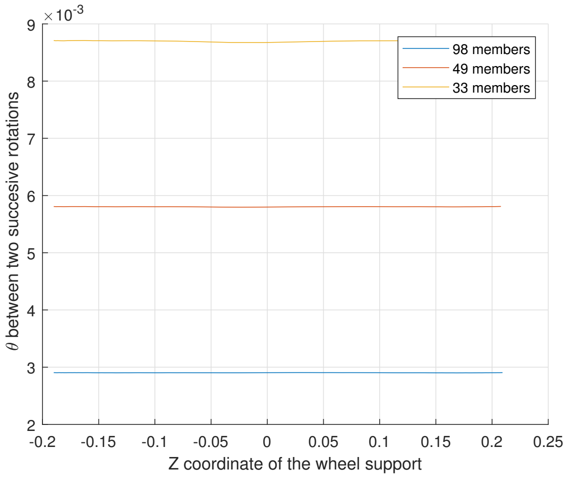

Fig. 2. Angle θ between two successive data points (Q_i and Q_{i+1})

which is the iterative solution of the system of equations, assuming q_i^* is a next iteration step of q_i. Already known formula for uniform B-splines is analogically given by [3]

$$q_i^* = q_{i-1}(q_{i-1}^{-1}Q_i(q_i^{-1}q_{i+1})^{-\frac{1}{6}})^{\frac{6}{5}}. \tag{15}$$

Because there are $n+1$ equations for $n+3$ unknowns q_{-1}, q_0,..., q_{n+1}, two boundary conditions are needed. The end conditions for natural spline are [3]

$$Q''(0) = 0 \quad \text{and} \quad Q''(n) = 0. \tag{16}$$

When these two boundary conditions are applied to previous equation, we obtain non-linear system of equations

$$\begin{aligned} q_{-1} &= Q_0(Q_0^{-1}Q_1)^{-1}, && (17)\\ q_{i-1}(q_{i-1}^{-1}q_i)^{\tilde{B}_i(T_i)}(q_i^{-1}q_{i+1})^{\tilde{B}_{i+1}(T_i)} &= Q_i, \quad \text{for } i = 0, 1, 2, ..., n, && (18)\\ q_{n+1} &= Q_n(Q_{n-1}^{-1}Q_n). && (19) \end{aligned}$$

As far as there is no other known method to compute the exact solution, the proposed iterative method, Eq. (14), is utilized to solve this system. For the initial guess, $q_i = Q_i$ is considered. However, due to the non-linearity of the problem, there are some restrictions for the input values of Q_i, but this is not the topic of this conference paper.

We interpolated the orientations of classical double wishbone suspension support (Fig. 1) in terms of vertical coordinate and we compared it with exact results. We chose the data so that we ensure almost constant angle θ (Fig. 2) between two successive data points (Q_i and Q_{i+1}), considering θ as an angle of axis-angle representation between two orientations. For the interpolation we used 3 different look-up tables which differed in number of data points:$n = 33$,$n = 49$ and $n = 98$ (see Fig. 2), these data were compared with the data set of 500 members, so that we can observe the interpolation error trend. The interpolation relative error was computed as the relative angle between the exact and the interpolated orientation and divided by the constant angle distance value (Fig. 2) and was denoted by θ_e. The results are shown in Fig. 3 and 4, the latter being a zoom of Fig. 3 on the vertical axis.

Figs. 3 and 4 show that we successfully achieved interpolation property, i.e., zero error at data points. The figures show that the highest interpolation error is at the beginning and at the

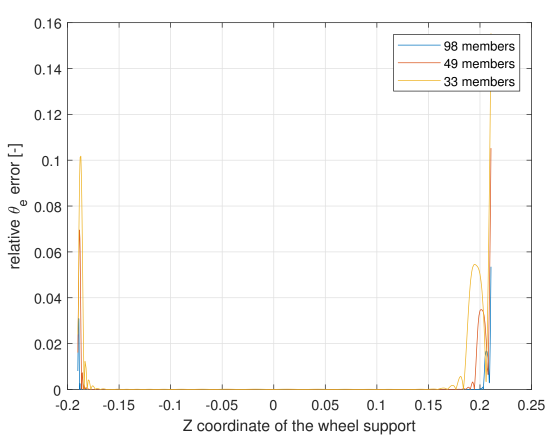

Fig. 3. Interpolation error θ [rad]

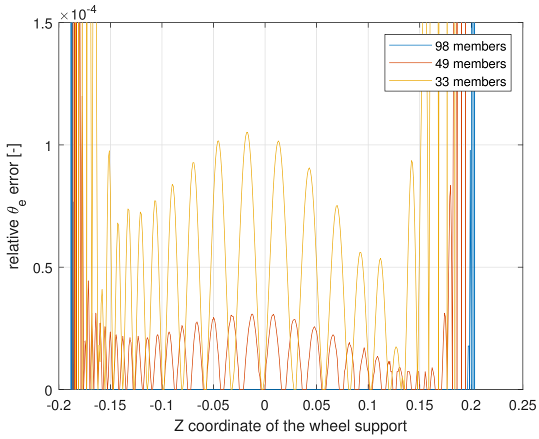

Fig. 4. Interpolation error – zoomed

end of the interpolated interval in our case. Whatever the number of interpolation points the highest interpolation error intervals are always given by the first and last k data points ($k = 4$ is the order of the interpolation). In case of high demand for low interpolation error at the beginning or at the end of the interpolated interval it is possible to add more data points at the beginning and at the end of the look-up table so that higher density of data points is ensured and the high interpolation error interval becomes smaller, or it is possible to change the boundary conditions. However, in practise it is not usual to achieve these end positions.

We can also see that the maximum interpolation error θ_e remains approximately constant inside the interpolated interval, which means that the main impact on the interpolation error has the development of angle θ between the two data points and the precision is not dependent on the relative axis development, considering the axis-angle representation. We can see that for 33 data points (members of look-up table) we achieved $\theta_e \approx 10^{-4}$, for 49 data points $\theta_e = 3 \times 10^{-5}$ and for 98 data points θ_e is equal to zero inside the interpolated interval. As far as the proposed interpolation fulfils C^2 continuity, we found formulas for the first and second derivative to be used for dynamics simulation of the suspension.

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