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On algebraic flux corrections for finite element approximation of transport phenomena

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1. Introduction

In this paper we shall focus on numerical discretization of a general hyperbolic system written for the vector of conservative variables $u = u(x, t)$ in the form

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f}(u) = 0 \qquad \text{in } \Omega \times I_T, \tag{1}$$

where $\boldsymbol{f} = \boldsymbol{f}(u)$ denotes the array of the inviscid fluxes. Eq. (1) is equipped with the initial condition $u(x, 0) = u_0(x)$ for $x \in \Omega$.

For the hyperbolic system (1) the aspects of spatial discretization by the finite element method is discussed. For such a systems there are several physical constraints for the solution as boundedness of physical quantities (e.g., positive density) which needs to be guaranteed also for the numerical solution. Moreover, the solution of such system needs to satisfy the entropy inequality, see, e.g., [1, 6]. The numerical analysis of the approximate method usually focus on the consistency, stability and convergence. The mostly used techniques as finite volumes or discontinuous-Galerkin finite elements moreover satisfy the conservativity naturally by the construction of the scheme. However, for the higher-order finite element method the conservativity needs to be discussed, see, e.g., [2].

Moreover, the construction of the numerical scheme should also avoid the occurrence of nonphysical states. To this end many modern high-resolution schemes use limiters to ensure preservation of local bounds or at least positivity for scalar quantities of interest. In the context of finite element approximations, such schemes can be constructed using the framework of algebraic flux correction (AFC) and its extensions to hyperbolic systems, see, e.g., [2, 3]. However, the bound-preserving schemes for nonlinear hyperbolic problems are usually not entropy stable and vice versa, the entropy stable schemes are usually not bound-preserving. In [4], the AFC scheme is extended for continuous finite element discretization of a scalar conservation law using a bound-preserving flux limiter and a semi-discrete entropy fix based on Tadmor's condition.

In this paper the construction of the AFC for continuous linear finite element method is discussed, its application realized on a simple linear scalar problem. The attention is paid on the realization of the Dirichlet boundary conditions. Numerical results are shown.

2. FE discretization and its properties

We consider a scalar transport equation in the form

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f}(u) = s \tag{2}$$

equipped with a suitable initial and boundary conditions. We consider the fluxes in the form $\boldsymbol{f} = \boldsymbol{v}u - \varepsilon \nabla u$, where $\boldsymbol{v}$ is the velocity and $\varepsilon \geq 0$ denotes the diffusion.

2.1 Linear FE method

For the purpose of the application of FE method, the weak formulation of (2) is derived in the form

$$\int_\Omega w \frac{\partial u}{\partial t} + w\nabla \cdot (\boldsymbol{v}u) + \nabla w \cdot (\varepsilon \nabla u) dx = \int_\Omega w s dx + \int_{\Gamma_N} w(\varepsilon \nabla u) \cdot \boldsymbol{n} dS. \tag{3}$$

Consider the FE space V_h with base ϕ_j and approximate the solution u and the velocity $\boldsymbol{v}$ by the FE approximations u_h and $\boldsymbol{v}_h$ as $u \approx u_h(x,t) = \sum_j u_j(t)\phi_j(x)$, $\boldsymbol{v} \approx \boldsymbol{v}_h = \sum_j \boldsymbol{v}_j \phi_j$. For the multiplication $u\boldsymbol{v}$ the approximation the *group FE formulation* is used $\boldsymbol{v}u \approx (\boldsymbol{v}u)_h = \sum_j (u_j \boldsymbol{v}_j)\phi_j$. This leads to system of ODE written as

$$\sum_j \left(m_{ij} \frac{du_i}{dt} \right) + \sum_j \left((c_{ij} + d_{ij}) u_j \right) = r_i \tag{4}$$

where $\mathbb{M} = (m_{ij})$ is the mass matrix, $\mathbb{C} = (c_{ij})$ represents the convective terms, $\mathbb{D} = (d_{ij})$ corresponds to the diffusion terms (proportional to ε) and (r_i) are the source terms. For the inviscid limit $\varepsilon \to 0_+$ with zero source terms we get the system in the form $\mathbb{M}\dot{\vec{u}} = \mathbb{K}\vec{u}$ with $\mathbb{K} = -\mathbb{C}$, or in the discrete form

$$\sum_j \left(m_{ij} \frac{du_j}{dt} \right) = \sum_j k_{ij} u_j. \tag{5}$$

2.2 Discrete operator properties

For the discrete operators the partition of unity(PU) valid for finite elements is important. Taking the partition of unity propery, i.e., $\sum_j \phi_j = 1$, we by differentiation get $\sum_j \nabla \phi_j = 0$. This has direct influence on the properties of the discrete (e.g., mass, Laplace) operators. The mass matrix is symmetric, positive definite and with the PU we have $\sum_i \sum_j m_{ij} = |\Omega|$. The diffusion matrix $\mathbb{D} = (d_{ij})$ is symmetric with zero row and column sums. The discrete gradient/divergence operator $\overline{\mathbb{C}} = (\overline{\boldsymbol{c}}_{ij})$ is nonsymmetric with zero row sums whereas the column sums does not have to be always zero as it is influenced by the mesh properties and boundary fluxes. For the interior nodes $\overline{\boldsymbol{c}}_{ii} = 0$ and $\overline{\boldsymbol{c}}_{ij} = -\overline{\boldsymbol{c}}_{ji}$.

2.3 Algebraic flux limiting technique

The description of the main idea is shown for the finite element approximation of the problem with zero viscosity and zero sources, which leads to the discrete equations in the form $\mathbb{M}\dot{\vec{u}} = \mathbb{K}\vec{u}$, where $\vec{u} = (u_i)$ is vector of the nodal values, $\mathbb{M}$ is the consistent mass matrix and $\mathbb{K}$ is the discrete transport operator. In order to obtain scheme both without undershoots/overshoots as well as not too diffusive, we need to switch between linear "upwind-like" approximations and the original scheme. In the finite element context the idea of algebraic flux corrections reads: replace the consistent mass matrix M_C by lumped mass matrix M_L, and add an artificial diffusion operator D to operator K to eliminate all negative off-diagonal coefficients of K. The linear local extremum diminishing scheme then reads $M_L\dot{\vec{u}} = L\vec{u}$, $L = K + D$. The artificial diffusion operator D can be rewritten as

$$(D\vec{u})_i = -\sum_{j\neq i} f_{ij}^d, f_{ij}^d = d_{ij}(u_i - u_j) = -f_{ji}^d.$$

The original scheme can be then recovered $M_L\dot{\vec{u}} = L\vec{u} - D\vec{u} + (M_L - M_C)\vec{u}$, or component by component

$$m_i\dot{u}_i = \sum_j l_{ij}U_j + \sum_{j\neq i} f_{ij}, \qquad f_{ij} = f_{ij}^d + m_{ij}(\dot{u}_i - \dot{u}_j) = -f_{ji}, \tag{6}$$

where m_i are coefficients of the lumped mass matrix. In order to prevent the oscillations of the solution, the fluxes f_{ij} are multiplied by suitable correction factors

$$f_{ij}^* = \alpha_{ij}f_{ij}, \text{ where } 0 \leq \alpha_{ij} \leq 1.$$

Inserting these fluxes into (6) we get the nonlinear combination of the low order scheme ($\alpha_{ij} = 0$) and the original higher order scheme ($\alpha_{ij} = 1$). Following the detailed description from [2] the positive and negative edge contribution to fluxes are accounted for separately

$$\begin{aligned} P_i &= P_i^+ + P_i^-, & P_i^\pm &= \textstyle\sum_{j\neq i} \min\{0, k_{ij}\}\, min_{max}\{0, u_j - u_i\}, \\ Q_i &= Q_i^+ + Q_i^-, & Q_i^\pm &= \textstyle\sum_{j\neq i} \min\{0, k_{ij}\}\, max_{min}\{0, u_j - u_i\} \end{aligned}$$

and use in order to limit positive or negative antidiffusive fluxes. The nodal corrections factors are computed by $R_i^\pm = \min\{1, Q_i^\pm/P_i^\pm\}$ which determine the percentage of $P_i^\pm$ that can be accepted to node i without violating the LED constraint for row i of the modified transport operator K_{mod}. The corrections α_{ij} are then computed using a suitable limiter, see [2], by

$$\alpha_{ij} = \begin{cases} R_i^+ d_{ij}(u_i - u_j) & \text{if } u_i \geq u_j, \\ R_i^- d_{ij}(u_i - u_j) & \text{if } u_i < u_j. \end{cases} \tag{7}$$

3. Algebraic flux corrections for hyperbolic systems

As an example we can consider Euler equations in the conservative form $\frac{\partial W}{\partial t} + \nabla \cdot \mathbb{F}(W) = 0$ where $W = (\rho, \rho\boldsymbol{u}, E)^T$ and flux is given as $\mathbb{F}(W) = (\rho\boldsymbol{u}, \rho\boldsymbol{u} \otimes \boldsymbol{u} + p\mathbb{I}, E\boldsymbol{u} + p\boldsymbol{u})^T$ Here, the total energy E is given as sum of internal energy and kinetic energy, i.e. as $E = \rho e + \frac{1}{2}\rho|\boldsymbol{u}|^2$ and the pressure is then computed using additional equation of state, for ideal gas expressed as $p = (\gamma - 1)\rho e$. In this case the application of AFC needs to take into account the properties of the hyperbolic system, see [3].

4. Numerical results

First, the developed flux corrected transport scheme was tested for finite element implementation in 1d and 2d (see Figs. 1–2). The exact solution in this case is just transported with a constant velocity $\boldsymbol{v}$ and no diffusive fluxes are used, i.e., $\varepsilon = 0$. In Fig. 1 the convection of rectangle (left) and semiellipse (right) is shown, where the dashed line shows the initial condition, the dotted line shows the exact solution, and the solid line shows the numerical approximation. In this case the transport velocity for 1D case was chosen as $\boldsymbol{v} = 1$ (1D vector) and the computations was performed for time period of $T = 0.5$. For the two dimensional case, the convection of the block with the rotational flow velocity $\boldsymbol{v}(x, y) = (-(y - 1/2), (x - 1/2))$ around the origin is approximated. Here, the time period of $T = 2\pi$, so that the initial condition is the exact solution at time instant $t = T$. Fig. 2 shows the exact solution (left) and the numerical

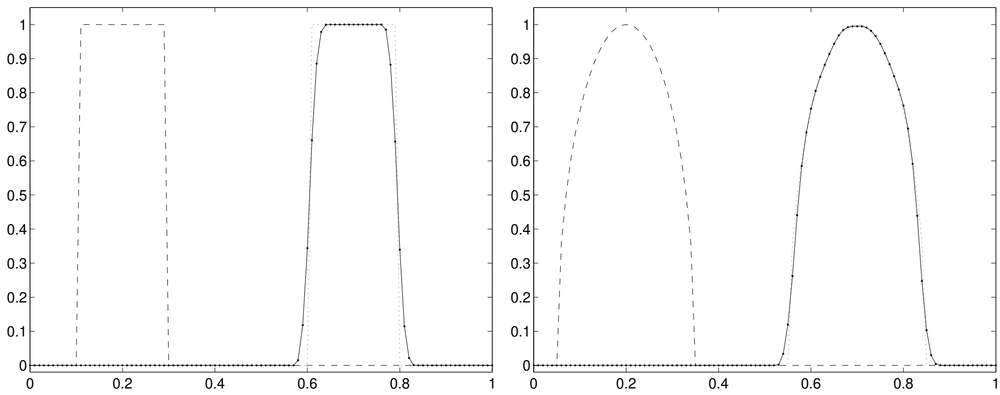

Fig. 1. Algebraic flux corrections (1d)

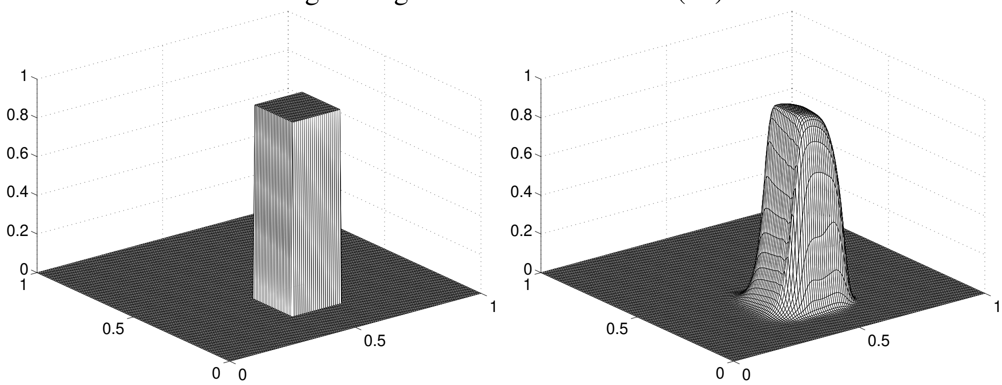

Fig. 2. Algebraic flux corrections (2d)

solution(right). Although the solution is slightly smeared, no undershoots or overshoots were detected.

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