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On comparison of suitable interpolations for finite element meshes respecting physical laws

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1. Introduction

This paper is devoted to the interpolation between two computational finite element meshes. Such interpolation of FE solution onto a new mesh is needed in many applications like material cutting, casting, welding, etc., or in the numerical simulation of fluid-structure interaction with large displacements, where a computational flow mesh quality can significantly deteriorate. In this talk we are interested in the interpolation with restrictions as introduced by authors Pont & Codina [3]. They proposed to combine a computationally cheap interpolation method together with constraints in the form of Lagrange multipliers which enforce conservation of desired quantities, like e.g. total mass, kinetic energy or potential energy. This approach respects physical laws and it is efficient, on the other hand its disadvantage is only a global conservation of physical quantities not the local one. The numerical results consist of comparison of the Lagrange interpolation and the natural neighbour as representatives of cheap interpolation methods on a few test cases.

2. Interpolation with restrictions

Let us assume for sake of simplicity a two-dimensional Ω of $\mathbb{R}^2$ which is covered by triangulations $\mathcal{T}^o$ and $\mathcal{T}^n$ representing the old donor and the new target FE mesh, respectively. Further we suppose that boundary vertices of $\mathcal{T}^o$ and $\mathcal{T}^n$ are identical. Generalizations of these assumptions are quite straightforward, see e.g. [3].

Next, we denote a FE function from FE space $\mathcal{V}_h^o$ built over the FE mesh $\mathcal{T}^o$ by u^o and similarly FE function $u^n(x) \in \mathcal{V}_h^n$ connected with the given triangulation $\mathcal{T}^n$. Function u^o can be expressed as linear combination of FE basis functions $\psi_j^o(x)$, i.e., $u^o(x) = \sum_j U_j^o\, \psi_j^o(x)$; correspondingly for function u^n.

The interpolation with restrictions (IwR) as introduced in [3] consists of two steps. First, the solution from the old mesh is projected on the new mesh. There are many possibilities, the preferred one is Lagrange projection, see [3]. Our aim is to compare performance of the Lagrange projection and the Natural neighbour (NN) interpolation as two different ingredients of the IwR.

The second step consists of application of appropriate restrictions as a correction step. The idea of imposing additional restrictions is a key how to improve some bad behaviour of presented interpolations. One of the biggest interpolation problems is the violation of physical nature of interpolated variable. The advantage of using restrictions is a generality of the algorithm which can be potentially used in many different scenarios. The disadvantage is that restriction (i.e., conservation) is still valid only in global and not local sense.

Natural Neighbour. Natural neighbor (NN) interpolation, introduced in [4], is based on Voronoi tessellation of given points, i.e., vertices of a considered mesh. The interpolant is a continuously differentiable function everywhere except at locations of the donor vertices, nevertheless values of the interpolant coincides with the input data here, see [1]. The computation of interpolating function G at a query point X is following. Point X is added to the given Voronoi tessellation leading to the creation of a new containing polygon (also called neighboring polygons) and it's associated sample points x_i are the natural neighbors of the point X. Then $G(X)$ is evaluated as

$$G(X) = \sum_{i=1}^{N} w_i \, f(x_i), \tag{1}$$

where function f denotes known input data at N points x_i. The weights w_i are defined as $w_i = \sum_{i=1}^{N} \frac{S_i}{S}$, where S_i are area of intersection of i-th original polygon and the newly inserted polygon having total surface $S = \sum_{i=1}^{N} S_i$, see [4].

3. Application to fluid flow problem

The previous general concept is now applied on incompressible flow velocity $\mathbf{v}^o \in \mathbf{V}_h^o = \mathcal{V}_h^o \times \mathcal{V}_h^o$. By $\widetilde{\mathbf{v}}^n$ is denoted the result of projection $\mathcal{P}$ used in first step of IwR $\widetilde{\mathbf{v}}^n = \mathcal{P}(\mathbf{v}^o)$ and $\mathbf{v}^n$ is the final result of the IwR on the target mesh $\mathcal{T}^n$. We require the conservation of following quantities: 1) mass (more precisely only velocity divergence), 2) both linear momenta and 3) kinetic energy. This leads to following problem: Find

$$[\mathbf{v}^n, \boldsymbol{\lambda}] = \arg \inf_{\mathbf{u}^n \in \mathbf{V}_h^n} \sup_{\boldsymbol{\mu} \in \mathbb{R}^4} L(\mathbf{u}^n, \boldsymbol{\mu}), \tag{2}$$

where $\boldsymbol{\mu}$ are Lagrangian multipliers and $L(\mathbf{u}^n, \boldsymbol{\mu})$ is Lagrangian function defined as

$$\begin{aligned} L(\mathbf{u}^n, \boldsymbol{\mu}) = \frac{1}{2} \int_\Omega \left(\sum_k (U_k^n - \widetilde{U}_k^n) \, \boldsymbol{\psi}_k^n \right)^2 \mathrm{d}x - \mu_1 \int_\Omega \nabla \cdot \left(\sum_k U_k^n \, \boldsymbol{\psi}_k^n - \sum_j U_j^o \, \boldsymbol{\psi}_j^o \right) \mathrm{d}x \\ - \sum_{l=1}^{2} \mu_l \int_\Omega \left(\sum_k U_{k,l}^n \, \boldsymbol{\psi}_k^n - \sum_j U_{j,l}^o \, \boldsymbol{\psi}_j^o \right) \mathrm{d}x - \mu_4 \int_\Omega \left(\sum_k U_k^n \, \boldsymbol{\psi}_k^n \right)^2 - \left(\sum_j U_j^o \, \boldsymbol{\psi}_j^o \right)^2 \mathrm{d}x. \end{aligned} \tag{3}$$

The sought solution needs to have first derivatives with respect to all variables equal to zero. Equations obtained by this differentiating can be written in matrix form as

$$\begin{pmatrix} \mathbb{M}^n & -R_1^T & -R_{2:3}^T & -2\mathbb{M}^n \mathbf{U}^n \\ R_1 & 0 & 0 & 0 \\ R_{2:3} & 0 & 0 & 0 \\ (\mathbb{M}^n \mathbf{U}^n)^T & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}^n \\ \mu_1 \\ \mu_{2:3} \\ \mu_4 \end{pmatrix} = \begin{pmatrix} \mathbb{M}^n \widetilde{\mathbf{U}}^n \\ R_1^o \mathbf{U}^o \\ R_{2:3}^o \mathbf{U}^o \\ (\mathbf{U}^o)^T \mathbb{M}^o \mathbf{U}^o \end{pmatrix}, \tag{4}$$

where $\mathbb{M}^n$ denotes mass matrix with components $m_{ij}^n = \int_\Omega \boldsymbol{\psi}_j^n \boldsymbol{\psi}_i^n \, \mathrm{d}x$, $\mathbb{M}^o$ is the mass matrix defined on the old mesh $\mathcal{T}^o$ and vectors R_1, R_2, R_3 have components

$$(R_1)_i = \int_\Omega \nabla \cdot \boldsymbol{\psi}_i^n \, \mathrm{d}x, \qquad (R_2)_i = \int_\Omega \boldsymbol{\psi}_{i,x}^n \, \mathrm{d}x, \qquad (R_3)_i = \int_\Omega \boldsymbol{\psi}_{i,y}^n \, \mathrm{d}x. \tag{5}$$

Vectors $R_{1:3}^o$ are defined similarly on the old mesh. Nonlinear problem (4) is solved with the Newton-Rhapson method.

4. Numerical results

The Lagrange (Lag) and NN interpolations alone and also as part of the IwR are compared in two tests. Further, by IwR are denoted the results based on the Lagrange interpolation.

First, academic interpolation test. The interpolation test of [2, 3] consists of 20 pairs of interpolations between the donor and target triangular meshes covering domain $\langle 0, 1\rangle^2$. Both unstructured meshes has characteristic length $h = 0.025$ and the inner vertices of target mesh are shifted by $h/2$ to the right. The considered divergence-free velocity $\mathbf{F}(x, y)$ has given components $f_1(x, y) = 2x^2(x-1)^2y(y-1)(2y-1)$, $f_2(x, y) = -2y^2(y-1)^2x(x-1)(2x-1)$.

Fig. 1 shows error distributions after all interpolations and Table 1 quantitatively summarizes the results. The NN interpolation alone is even more diffusive and produces bigger error than the Lagrange projection, e.g. compare kinetic energy E_{kin}. Nevertheless method IwR-NN substantially improves the NN results and it even provides slightly lower error than the IwR with approximately similar time of interpolation computations.

Table 1. Comparison of interpolation results of the first test

method	max $\lvert\mathbf{F}\rvert$	E_{kin}	L_2 error	L_∞ error	approx. time [s]
exact	1.200×10^{-2}	6.013×10^{-5}	–	–	–
Lag	1.142×10^{-2}	5.126×10^{-5}	5.760×10^{-7}	1.698×10^{-3}	2.5
IwR	1.237×10^{-2}	6.013×10^{-5}	2.405×10^{-7}	1.137×10^{-3}	215.8
NN	1.087×10^{-2}	4.748×10^{-5}	9.082×10^{-7}	1.869×10^{-3}	1.8
IwR-NN	1.223×10^{-2}	6.013×10^{-5}	1.823×10^{-7}	1.036×10^{-3}	247.2

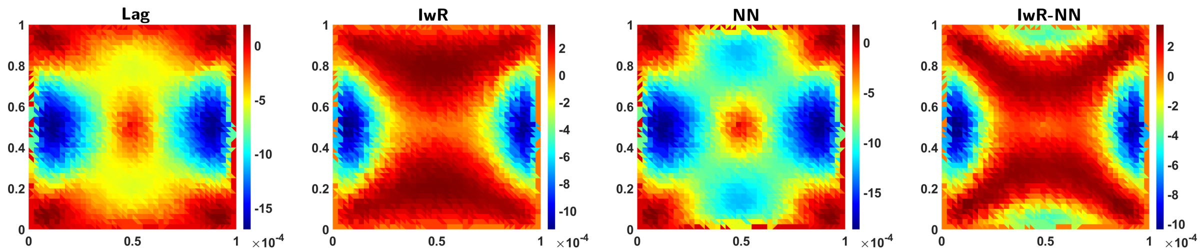

Fig. 1. Error magnitude of interpolated vector field on structured FE mesh after 20 runs. Mind the different scales of colorbars for each result

Second interpolation test – real data. The velocity field obtained during a simulation motivated by human phonation is used in the second test, see [5]. The second test performs one pair of interpolations from the donor to the target mesh and back where both meshes differ in the middle part representing a channel constriction.

Fig. 2 illustrates relative error distributions after one pair of interpolation runs. The NN results are again significantly worse than for Lag projection however in this case the application of IwR-NN does not improve results substantially. This is obvious from quantitatively viewpoint – see Table 2, as well as qualitatively from the extent of domain with relative maximal error bigger than 1% (Fig. 2). The IwR results surpasses the Lag one here only slightly with a considerable increase of computational time on the other hand the IwR conserves total kinetic energy (and linear momenta and divergence) and, thus, its application once per ten time step is still favourable.

Special attention should be paid also to the target mesh. A relative high interpolation error, particularly in the boundary layer, is caused by a coarseness of the target mesh. In the case with similarly dense target mesh the interpolation error would be significantly lower as in the first test.

Table 2. Comparison of interpolation results of the second test

method	max $\|\mathbf{F}\|$	E_{kin}	L_2 error	L_∞ error	approx. time [s]
exact	114.200	9.644×10^{-2}	–	–	–
Lag	114.081	9.580×10^{-2}	9.914×10^{-5}	35.115	1.643
IwR	114.459	9.644×10^{-2}	9.838×10^{-5}	34.985	81.092
NN	114.039	9.473×10^{-2}	2.617×10^{-4}	55.723	0.242
IwR-NN	115.062	9.644×10^{-2}	2.564×10^{-4}	55.426	78.537

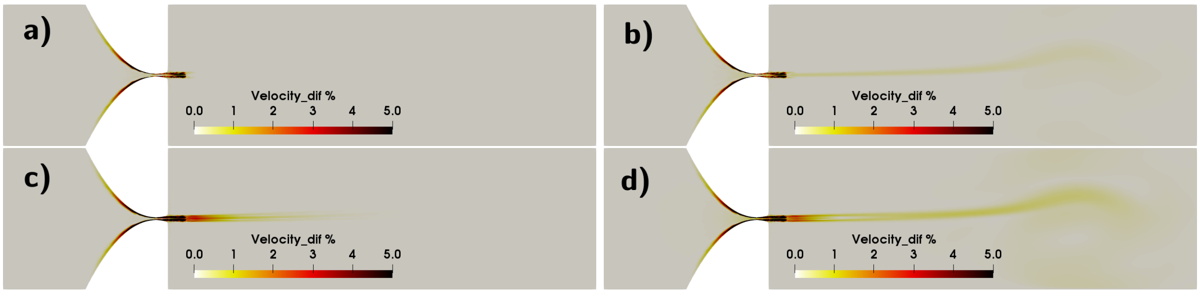

Fig. 2. Distributions of relative error magnitudes after one pair of interpolations. Results: a) Lag (top left) (max $E_\infty = 31.1\%$), b) IwR (top right) (max $E_\infty = 31\%$), c) NN (bottom left) (max $E_\infty = 49\%$) and d) IwR-NN (bottom right) (max $E_\infty = 48.8\%$). The maximal error (out of presented colorbar scale) is located for all methods similarly in a few elements of boundary layer inside constriction

5. Conclusion

The paper describes method called interpolation with restrictions (IwR) based on [3]. This general interpolation method between FE meshes improves performance of classical interpolation techniques by additional requirement of conservation of arbitrary chosen (physical) quantities. Such approach offers an interesting mix of a relatively computationally cheap method which moreover conserves (total) physical key quantities of the given problem.

Here, the natural neighbour (NN) interpolation is described and used as the first step of the IwR. The four different interpolations are compared in two tests. The NN interpolation provides surprisingly more diffusive results than the Lagrange interpolation though it theoretically provides smoother results. In connection with IwR it performs better on a relatively dense mesh.

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