Opponents report on dissertation thesis of Mgr. Jakub Janoušek

"Existence and bifurcation of periodic solutions in models of suspension bridges"

The dissertation thesis is based on three joint papers by the student and his advisor. That is reflected in the structure of this thesis.

The chapter "Introduction and model overview" starts with the introduction of the topic of suspension bridges and the model of A. C. Lazer and P. J. McKenna with damping and constant stiffness of suspension cables is introduced. Then the author shows PDE models without damping, which can be under certain assumptions transformed into ODEs. The connection of the non-uniqueness and the bifurcation of periodic solutions with the value of the constant of stiffness of the cables in relation to the eigenvectors of corresponding linear operator emerges. The end of this short overview is devoted to the equations with damping and there are comments about previous results. The chapter ends with the summary, motivation for and the outline of the thesis.

In the following three chapters "Models with damping", "Weighted non-damped PDE model" and "Strictly inverse-positive operators" the author describes in more detail the new results he achieved with his supervisor. He comments on them and puts them in the context of previously known and published results.

The last chapter "Appendices" contains the mentioned publications.

Let's turn our attention to the new results. The first of them is about one-dimensional model of suspension bridge with damping. The authors were able to generalize and extend the results of G. Tajčová from the paper "Mathematical models of suspension bridges", where she proved that there exists unique solution for a given model for values of stiffness of suspension cables in certain interval. The authors were able to double the length of this interval of "admissibility".

The second result concerns the models of bridges without damping with non-constant stiffness of suspension bridges (which depends on a space variable). They achieved new results showing for which values of stiffness there is a unique solution and when there is exactly one positive stationary solution. They have furthermore shown when the bifurcation appears and the relation to the spectrum of Fučík.

The last of the presented results contains sufficient conditions for the positivity of the solution (and its "boundary" derivatives) to a certain linear boundary value problem of 4th order with variable coefficients. It turns out that this relates to the bifurcations of periodic solutions of the models of bridges with non-constant stiffness of the cables. The authors were able to significantly extend the known limitations for the variable coefficients which guarantees that every solution to the corresponding boundary value problem with non-negative right hand side has required properties.

The models of suspension bridges are one of the important part of the research of the supervising department. At the same time it is a very current topic, which is shown (among other things) by the number of publications on the topic of models of suspension bridges.

Mgr. Jakub Janoušek was able to continue in this research and achieve new, non-trivial and interesting results important for the further study of the qualitative properties of this models. He has not only published his results in good journals, but he has written them down in a comprehensive way into this thesis. The dissertation is nicely written, well-structured and I don't have any reservations.

For those reasons I **recommend** this thesis for the defence and after its successful completion I recommend to grant the student the title "doctor" (Ph.D.).

In Orlová 28.11.2021

Prof. RNDr. Jiří Bouchala, Ph.D.

Report on the doctoral thesis
"Existence and bifurcation of periodic solutions in models of suspension bridges"
by J. Janoušek

The thesis deals with various ODEs and PDEs which arise in the modeling of structures of beam and suspension bridge type. After an Introduction dedicated to a review of the investigated models, with comments about how they settle in the existing literature and about the main motivations for their study, the contents are organized as follows.

Chapter 2 is devoted to the study of the damped and forced PDE

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ = h(x,t),$$

where the asymmetric restoring force of Lazer-McKenna type models, in a bridge perspective, the slackening of the hangers. The ODE counterpart of such an equation is discussed, as well. The first concern is here the (weak) solvability of the corresponding problem with Navier boundary conditions on $[0,\pi]$ in dependence on the stiffness parameter k. The usual degree approach, based on a priori estimates of the solutions, guarantees here the existence for $k > -\alpha^2$; as for the uniqueness of the solution, the use of the Banach contraction theorem in the standard abstract framework provides a unique solution for $|k|$ less than the distance of the spectrum of the associated linear operator L from the origin. It is in this respect that the original contribution comes. On the one hand, a visual interpretation of the uniqueness condition is presented, based on seeing the eigenvalues of L as intersections between suitable lines and parabolas in the complex plane (depending on the damping parameter β and on α); this allows one to understand that there is room to improve the original estimates of the uniqueness range by suitably shifting the operator through a small parameter ϵ. Precise estimates of the best choices of the shift in terms of α and β are provided, turning into an enlargement of the uniqueness interval; an algorithm for the computation of such an interval is also presented. Hence, the abstract technique ensuring uniqueness is still relying on the Banach contraction theorem, while the improvement consists in finer estimates of its range of applications in terms of the parameters, through a sharp interpretation of the uniqueness conditions.

Chapter 3 presents some bifurcation results for the PDE

$$u_{tt} + u_{xxxx} + kr(x)u^+ = h(x,t),$$

where the damping β is set equal to 0, but the restoring force is assumed to be non-homogeneous, modeling the fact that the hangers are placed in a discrete way along the span of the considered beam. The main concern is here to understand the role of the nonhomogeneity in the estimates of the values of the stiffness parameter k for which additional solutions appear, giving rise to the potentially dangerous phenomenon of buckling. Rephrasing the abstract results for constant weights in terms of the weighted spectrum of the operator, it is shown that the nonhomogeneity improves the uniqueness range for the stiffness parameter, because it suitably shifts the eigenvalues. Nevertheless, bifurcation of solutions for a time-independent right-hand side still appears, as is proved by applying the celebrated Rabinowitz bifurcation theorem; in this respect, it appears fundamental to be able to write the general solution u as a perturbation of a positive stationary solution (the information on the sign is crucial in order to properly deal with the nonlinearity u^+). Some complementary remarks about bifurcation from infinity are presented, in which respect the main obstacle seems to be the fact that the Fučik spectrum of the PDE operator is still not determined. Hence, the main novelty represents here the model which is investigated, which is not an artificial variation upon the existing literature, but represents a reasonable way of modeling the behavior of realistic structures; the results are particularly interesting in this respect. The model is properly framed into the corresponding abstract setting, and the techniques used, which are well established in literature, are suitably employed and presented. The concluding discussion about possible bifurcations from infinity in correspondence of the Fučík spectrum could be the starting point for several progresses, especially from a purely mathematical point of view.

The previous requirement of positivity for the stationary solution naturally leads to the investigation of the inverse positivity property for the weighted operator L defined by $Lu = u'''' + kr(x)u$, which is the object of Chapter 4. Here some classical results coming from inverse positive operator theory, rephrased and adapted in a nontrivial way to the weighted framework, are employed to conclude strict inverse positivity of the operator as long as the (positive) stiffness parameter k does not reach the principal eigenvalue of the associated $(3,1)$ conjugate boundary value problem with weight r. Also the negative indefinite case, significant mainly from a mathematical point of view, is taken into account; at the end of the chapter, the better estimates obtained by framing the problem in the weighted setting are compared to previous estimates in literature, showing the effectiveness of the authors' approach. Along the chapter, the main novelty is thus again represented by the weighted model under consideration and the development of the related theoretical scheme; this turns into an effective improvement of the bounds for the intervals of strict positivity in terms of the stiffness. The discussion has a relevant part which becomes of interest mainly from a mathematical point of view, since both the indefinite case and the shape of some of the weights shown as examples may appear less in line with the usual modeling of real structures. In fact, in some points the relationship with beams and suspension bridge models seems more hidden in this section; maybe it could be good to stress again what are the consequences of the results of the chapter in this respect.

From the point of view of the contents, the thesis is significant, since it presents some new results about realistic models for structures of beam type with nonhomogeneous restoring force, providing original and quite sharp information about bifurcations and positivity of solutions. This might be the starting point to approach more complete models for suspension bridges, where also the torsional oscillations are taken into account. The investigation is inspired by a clear guideline, so as to reflect a structured organization of the doctoral research activity; the knowledge of the model has been enriched by tackling questions arisen step by step along the investigation, as is usual in research. Therefore, the thesis is not a collection of independent and separate works, but has a quite clear thread (this is explicitly remarked along the text, though maybe even too many times). I think this is appreciable.

The articles representing the core of the contents appear as joint works, so the contribution to the field by the candidate appears sufficient for the title of Ph.D. The quality of the contents is good and the works have been published mostly on renowned journals in the field, generally of good rate. The techniques used are well established ones in the field of Mathematical Analysis and Differential Equations, employed with an appropriate order of precision and clarity and, mostly, suitably introduced.

As for the style of the manuscript, the way the contents are presented is acceptable, though one notices some evident differences between the style of the attached articles, more professional and formal, and the one of the thesis, quite informal and closer to a narration. This may for sure be pleasant in some circumstances, but sometimes I find this novel-like style not completely appropriate, concerning both isolated expressions and entire paragraphs. However, the readability of the contents is not penalized by this aspect. Another point is related to the completeness of the monograph: even if it is not meant to be a self-contained text, I find a quite clear difference between Chapter 4 and the rest of the manuscript, in that the last chapter is not particularly easy to read and is written in a quicker style, sometimes too sketchy. For instance, it appears unbalanced with respect to Chapter 2, where the abstract scheme leading to existence is presented in a detailed way, even if already well established in literature. As an overall assessment of form and style, I would say that the form could be improved (also the English in some points) and perhaps it would be preferable to use a more article-like style, anyway the mathematical content is mainly presented in a sufficiently suitable way.

I list here some comments and questions, following the order the topics are presented along the thesis:

- in Chapter 2, I think that the requirement that the solutions are periodic in time should be commented more extensively. What happens if we remove such an assumption?
- as far as I know, the usual definition of weak solution found in literature imposes to consider H^2-solutions satisfying the integral inequality where, roughly speaking, half of the spatial derivatives are transferred on the test function and half of them remain on the function u. What are the advantages/the differences in considering Definition 1 (formula (2.1))?
- p. 11, fourth line: $u = \|r\|$ is not appropriate;
- in formula (2.15), the constant appearing in front of the positive part of u is tk. Hence, it would be better to say that, in view of Lemma 5 and Remark 4, tk should be $> -\alpha^2$, and then it could be remarked that this is implied by $k > -\alpha^2$; the same on p. 15 (regarding Theorem 11);
- it seems to me that (2.22) is not necessary to reach (2.25);
- p. 17: it would be far better to clearly point out the results from [39] which are improved, with a precise reference (in the form "see [39, Theorem ...]");
- several precise references are missing also elsewhere in the thesis: for instance, on p. 33, which result from [21] coincides with Proposition 26? The same holds for Lemma 31, Lemma 32 and Theorem 33;
- the end of p. 18 (last two lines and first two lines of p.19) is not clear, it should be reformulated;
- in Figure 3.2, the case of a piecewise constant function h is presented, however Proposition 22 assumes $h \in C([0,1])$; I see that this is essentially required in order to obtain the regularity of u_{st}, but it would be better to present Figure 3.2 for a continuous function h (which may be obtained regularizing the currently used piecewise constant function), since it is preceded by references to Proposition 22;
- is it possible to extend some results of Section 3 to the damped case, at least for a small damping (like ϵu_t)?
- in suspension bridge models, it is quite common to assume that the external forcing term actually depends on time, since it models the aerodynamic forces (the wind). What about this case in the perspective of Chapter 3? Are there any hints about what to be expected?
- the reduction procedure on p. 36 is not so clearly explained;
- some more proofs in Section 4 would be of help for the reader;
- on p. 41, it would be preferable to list the definitions of cone, generating cone and total cone separately, numbering each one of them;
- does the candidate envisage some parts of the whole discussion which could fit another choice of the nonlinear restoring force, such as the usual cubic nonlinearity ku^3 or a cubic positive part $k(u^+)^3$?

I finally point out some misprints (of course, this is not meant to be a complete list):

p. 4, below (1.8): descriprion; p. 6, section 1.4: bahaviour; bifucartions; p. 14, formula (2.21), in the first inequality it is $(1/2)[(y')^2]_0^{2\pi}$; p.25, line -3, considerd; p. 35, first line: led us to study; p. 41, line 2: To show that the eigenfunction (I think the comma should be canceled; some commas appear misplaced along the thesis); p. 42, before Lemma 38: "...a provides a more..."; p. 43, two lines before Section 4.6: assymetrical; p. 49, the second reference misses the journal.

In view of the above comments, taking into account the appropriate number of the presented results, their good quality and their relevance in relation to the existing literature, the organic structure of the thesis and the contribution given by the candidate to the field, in my opinion the manuscript and its contents meet the requirements for the Ph.D. degree. Hence, I recommend the thesis for defence.

November 15th, 2021

Dr. Maurizion Garrione (Politecnico di Milano)