Line Scratch Detection on Digital Images: An Energy Based Model

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ABSTRACT

In this paper, a model for automatic detection of line scratches on digital images is proposed. It consists in a generalization of the Kokaram's model using width and height features of the line profile along with Weber's law, to test scratches visibility on the image. Experimental results show that the algorithm works better than the existing models in detecting true scratches with a lower computing time.

Keywords: Digital Film Restoration, Scratch Detection, Weber's Law

1 INTRODUCTION

Line scratches are common defects of old film sequences, caused by the abrasion of the film material during film transport or the developing process. They consist of long, vertical lines of bright or dark intensity, oriented more or less vertically over much of the image, with width from about 3 to 10 pixels [Kokar96], [Kokar98], [Schal99], [Joyeu99], [Joyeu00].

The detection phase results difficult because lines can represent natural components in the scene and occupy the same or nearly the same location in several subsequent frames. Thus such defects cannot be detected as temporal discontinuities of image brightness. According to this fact, each frame is to be analysed separately by means of an algorithm with a low computing time and almost completely automatic, such as reducing or entirely removing operator's interaction.

However some observations on the typical line-profile can be made, so that thickness, contrast and vertical extend of the line can be used to define its features as a defect.

The only spatial model for detection has been proposed by Kokaram [Kokar96],[Kokar98]. His technique consists in two parts: the first one employs deterministic methods based on the geometric technique of the Hough Transform; the second part uses a Bayesian refinement based strategy which rejects or accepts candidates from the previous stage. Though an interesting formulation describing scratches, it has some drawbacks such as a quite huge computational cost along with a number of thresholds to be tuned by the user. Moreover it detects only lines extending over most of the image without distinguishing image defects (scratches) from image features (*false alarms*).

In the following we will show that the elimination of the hypothesis of additivity, i.e. considering a scratch as a partial lack of the image information, along with some considerations about visibility of the scratch, allows us to considerably improve the

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model performances. In other words, our model consists in a generalization of the Kokaram's one that allows us to obtain an algorithm that

- is completely automatic except for the binary choice of the color defect (black or white)
- is able to detect almost all kinds of scratches (see below to better understand this point)
- has a low computing time (linear on the number of the columns except for the cross-section computation)
- proposes an approach to distinguish image features (*false alarms* as the rope in Sitdown image) from image defects.

As we have mentioned before, both models detect scratches using one frame at a time. In [Joyeu99], [Joyeu00] is presented another model for the detection, based on temporal features of a scratch. Thus, considering the whole corrupted sequence, true scratches are detected by means of a Bayesian estimation technique, the Kalman filter. This method presents the same previously mentioned drawbacks.

The organization of the paper is as follows. Section 2 presents Kokaram's model. In Section 3 we propose our model whose experimental results are presented in Section 4. Some concluding remarks constitute the topic of Section 5.

2 KOKARAM'S MODEL

Let's define the sequence

$$x(n) \in \ell^2(R), \quad n = 1, \ldots, N$$

as the horizontal luminance cross-section of the image and $\Omega = [M \times N] \in \ell^2(R)$, the image domain. $x(n)$ is obtained by taking the vertical mean of columns after correcting for a space varying local mean image intensity [Kokar96], [Kokar98].

Observing $x(n)$'s shape, we can assert that in correspondence of the defect position it presents a peak (maximum for white scratches, minimum for the black ones) and it is similar to a damped sinusoid, i.e.:

$$L_n^{(p)}(i,j) = b_p k_p^{|i-(m_p j+c_p)|} \cos\left(\frac{3\pi|i-(m_p j+c_p)|}{2\omega_p}\right)$$

where (i,j) are respectively the horizontal and vertical abscissa, m_p, c_p are the slope and intercept with the horizontal edge of the image, b_p is the brightness of the central portion of the scratch, k_p is the decay of the line profile and w_p is defined as the line width.

Considering scratch as an information that hides the original image one, Kokaram proposes a purely additive model that can be written as follows:

$$I(\vec{x}) = G(\vec{x}) + \sum_{p=0}^{P-1} L_n^{(p)}(\vec{x}) + e(\vec{x}) \quad (1)$$

where I is the degraded image brightness, G is the original image brightness, $\vec{x} = [i,j]$, P is the number of scratches on the degraded image, $e(\vec{x})$ is an additive gaussian noise $\sim \mathcal{N}(0, \sigma_\epsilon^2)$.

According to the fact that the scratch vertically occupies almost all the image and its line profile has noticeable horizontal impulsive features, the degraded image is firstly subsampled after using a low-pass filter [Schaf81] and then horizontally filtered by a median filter. If I_s is the vertically subsampled image and M_s the horizontal one, the signal $e(i,j) = I_s(i,j) - M_s(i,j)$ is then thresholded to obtain $B(i,j)$ defined as follows:

$$B(i,j) = \begin{cases} 1 & \text{if } e(i,j) > e_t(i,j) \\ 0 & \text{otherwise} \end{cases}$$

where e_t is the threshold. The geometric technique of the Hough Transform [Jain89][Canto94] is then used to identify the lines in $B(i,j)$. This transform maps lines in (i,j) space onto points in (m,c) space (the Hough space), where m, c are the gradient and intercept on the j-axis respectively. Strictly speaking, $A(c,m)$ is the accumulator array which samples the Hough space,

$$\forall\ (i,j)\ :\ B(i,j) = 1,$$

$$A(c,m) = A(c,m) + 1 \quad if \quad i = mj + c$$

$\forall\ (c,m)$ belonging to a suitable range considering that scratches can be slightly curved (up to 5% of the width of the film [Schal99]). Peaks in the accumulator array represent lines in $B(i,j)$. We have to underline that at this stage the operator has to tune thresholds relative to the length of a scratch, (i.e. the height of a selected peak) and to the maximum number N_1 of scratches to be detected. Moreover, to avoid aliasing in the Hough space, a 2-D Gaussian filter has to be employed. Consequently the first N_1 peaks exceeding their height threshold are selected, that is candidate lines in $B(i,j)$ are selected and all pixels set to 1 in $B(i,j)$ and which are within $\omega_t/2$ pixels from the candidate line in $B(i,j)$, are taken off the accumulator array (ω_t is the greatest line width). A Bayesian refinement technique is now performed to detect real scratches. This strategy is based on the examination of the marginal distribution for the scratch brightness, $p(b_p)$. Then the *line feature is assumed to be significant if*

$$p(b_p \leq 1.0) < Rp(b_p > 1.0).$$

$p(b_p)$ is computed performing the Gibbs sampling technique on the parameters vector $P = [k_p, b_p, \omega_p, m_p, c_p, \sigma_\epsilon^2]$ [Kokar96][Kokar98].

3 THE PROPOSED MODEL

Before showing our model it is worth spending some words about some kinds of scratches. We have seen that a scratch is usually described to extend on almost all the frame ([Kokar96], [Kokar98], [Joyeu99], [Joyeu00], [Schal99]). Nonetheless, looking at various degraded images, two kinds of scratches can be identified so that a more detailed classification can be made:

- *Principal scratches*, extending vertically on more than 95% of the image. An example is at column no.222 of Sitdown image (Fig. 1);
- *Secondary scratches*, very short, usually not higher than half image. More precisely they have a length up to 95% of image height. See for instance column no.257 of Sitdown image (Fig. 1).

We have to point out that all scratches (principal or secondary) can present another scratch at a distance of few pixels and having sidelobes influencing each other (Fig. 2). Consequently a scratch can be *alone* (see column no.50 in Fig. 8) or *not alone* (see column no.165 in Fig. 8).

Coming back to Kokaram's algorithm, in the detection phase, it treats the scratch as an information lying on the original image one. As a matter of fact, according to the action of the projector, scratch must be considered as a region of partially missing data. Thus Eq. 1 can be generalized as follows:

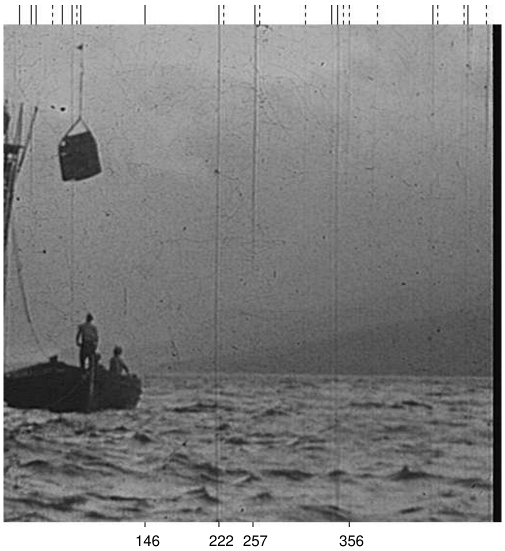

Figure 1: 512 × 512 × 8 bits Sitdown image: Top) detected (solid lines) and not detected (dashed lines) scratches by the proposed algorithm.

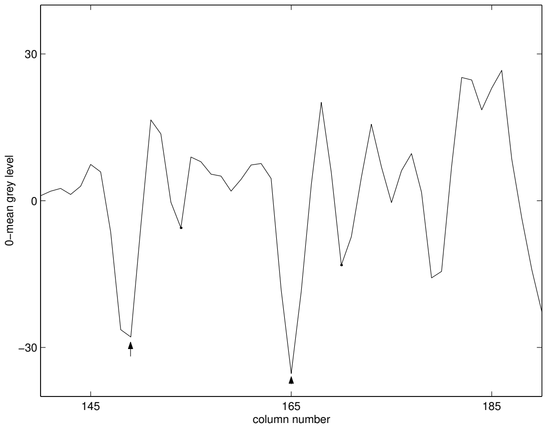

Figure 2: Cross-section profile of two scratches that are *not alone* (see arrows)

$$\begin{aligned} I(i) = & \\ & (1-(1-\gamma)\, k_p^{|i-c_p|}) \quad G^{'}(i) + \\ & +(1-\gamma) \quad k_p^{|i-c_p|} \quad L_n^{(p)'}(i). \end{aligned} \tag{2}$$

We outline that we use only the horizontal abscissa i because we are focusing on the cross section profile. Moreover

$$G' = G + e$$

because we are interested only in discriminating scratches on the image. Finally $L_n^{(p)'}$ is a simple sinusoid, i.e.:

$$L_n^{(p)'} = b_p \cos(\frac{3\pi|i-c_p|}{2\omega_p}).$$

Eq. 2 is confirmed by the careful observation of the cross-section profile nearby a line scratch. Strictly speaking, observing the line profile along its vertical extension (as shown in Fig. 3), we can note that the intensity of its luminance is considerable, compared to the rest of the signal which is almost smooth. Consequently we can assert that in correspondence of that position we have only information about the line scratch. We express this fact using the coefficient k_p.

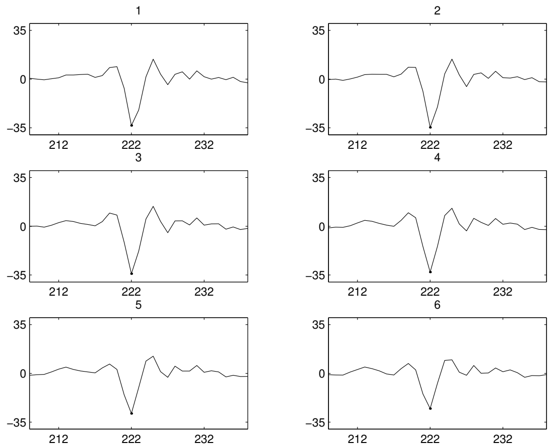

Figure 3: Cross-section profiles of an evident part of the scratch at column no. 222 (top of Sitdown image). 50×30 size regions have been considered along the vertical extension of the scratch.

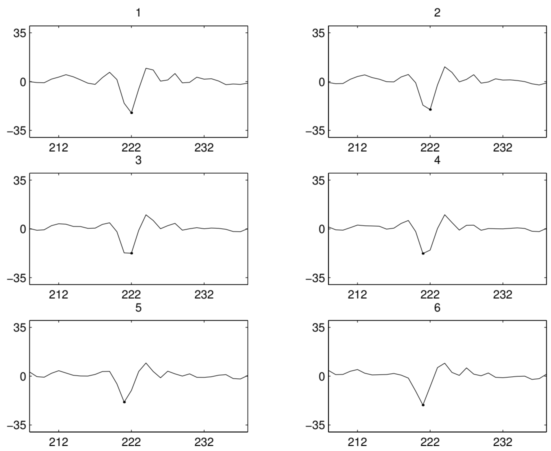

Figure 4: Cross-section profiles, as in Fig. 3, of a bit less noticeable part of the scratch at column no. 222 (bottom of Sitdown image).

If we keep on following line profile (Fig. 4), we note that when it assumes weak intensity values (scratch isn't too much visible on the image), it changes its position and the neighbouring pixels assume similar grey levels. Then we deduce that the signal is strongly influenced by the visibility of the scratch and the model has to include the measure of the ratio between the height of the signal at the scratch peak location and the rest of the signal. For that reason we introduce the coefficient γ defined as follows:

$$\gamma = \frac{E^{+}[x(n)]}{\alpha b_p}$$

where $E^{+}[x(n)]$ is the average of the difference of adjacent extrema of the cross section $x(n)$ (see 1^{st} in Fig. 5), and α is a normalization factor such that γ assumes values in the range [0,1]. In our experimental results we have set α to 10. It is not the only value α can assume. In fact, since it is tied to all the possible ratios between the height of the signal and the scratch one, we have empirically found that all values in the range [6,10] are good.

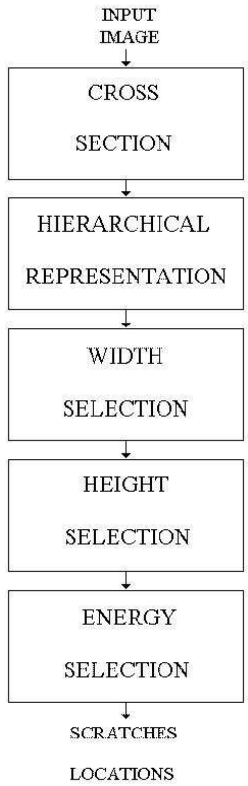

Figure 5: Block scheme of the proposed algorithm

According to the fact that a scratch is represented by a peak on the $x(n)$'s profile, we introduce a hierarchical representation (see 2^{nd} block in Fig. 5) focusing on the sequence $s(k)$ which is composed by $x(n)$'s local extrema.

In order to simplify the description of the algorithm, from now on we will focus only on black scratches, since for the white ones it is symmetric. It turns out that the candidates are the $x(n)$'s minima.

In order to noticeably reduce the number of candidates, we select minima having adjacent maxima distance in the range of [3,10] pixels (according to [Kokar96][Kokar98] and our empirical results) (see 3^{rd} block in Fig. 5). These minima are the $s_1(k)$ elements. To put in evidence the horizontal impulsive feature of the line scratch profile we keep minima resulting greater than $E^+[s_1(k)]$(see 4^{th} block in Fig. 5).

The next step of our algorithm needs more attention. According to Weber's law [Jain89], we are interested in computing the least value of brightness $\overline{b_p}$ that a minimum has to assume to be perceivable on the image. At this stage we have also to consider that the line scratch profile isn't symmetric (like sinusoid is); it turns out that the sidelobes values on the cross section are affected by an error due to the underlying original image information. So we have to include this error in the computation of a scratch visibility threshold.

As regards the value $\overline{b_p}$, the Weber's law states that an object of brightness f_0 is noticeable on a surround of brightness f_s if

$$\frac{|f_s - f_0|}{f_0} \geq .02.$$

Then, if $E[b_p]$ is the average of all minima of $x(n)$, we have $\overline{b_p} = \frac{E[b_p]}{.98}$.

Now we are ready to compute the visibility threshold (see 5^{th} block in Fig. 5). For each candidate we define as energy the quantity

$$E_n(b_p) = \int_{c_p-m}^{c_p+m} |L_n^{(p)}(i) - a| \, di \quad (3)$$

where a and m are the sidelobe value and its position respectively; they are defined as follows:

$$a = \frac{x(m_l) + x(m_r)}{2}$$
$$m = \lfloor \frac{m_r - m_l}{2} \rfloor$$

where m_l and m_r are respectively the left and right sidelobe position on the $x(n)$ profile.

So the least energy value relative to a just noticeable scratch on the image is $E_n(\overline{b_p})$. From the Eq. 3, we have that the energy measure depends on b_p, a and m values, i.e. on m_l, m_r, $x(m_l)$, $x(m_r)$. These measures, as we said before, are affected by an error due to the cross-section profile asymmetry; then we have to spread them over $E_n(b_p)$ measure using the well-known formula [Brand70]

$$\Delta f = \sqrt{\sum_{h=1}^{q} f_h'^2 \, \Delta_h^2} \quad (4)$$

where f is the indirect measure, f_h' is the partial derivative of f with respect to the h-th direct measure and Δ_h is its error. In order to simplify calculation we can approximate $E_n(b_p)$ with $E_r(b_p) = |b_p - a|m$, (as shown in Fig. 6), that is the area inside the triangle whose vertices are (c_p, b_p), $(c_p - m, a)$ and $(c_p + m, a)$. In fact in Appendix A we prove [Vitul01] that

$$\forall \, b_p : |b_p| \geq |a| \quad \text{if} \quad E_r(\overline{b_p}) \leq E_r(b_p)$$

then

$$E_n(\overline{b_p}) \leq E_n(b_p). \quad (5)$$

Then we can use Eq. 4 where $f = E_r(b_p)$.

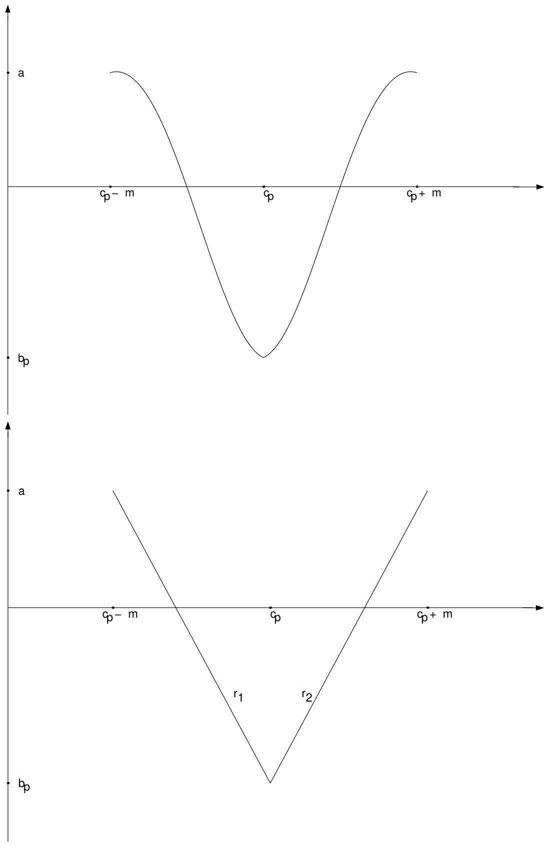

Figure 6: **Top)** Damped sinusoid; **Bottom)** its approximation by straight lines.

Finally we have to compute m_l, m_r, $x(m_l)$, $x(m_r)$ and b_p 's errors. They are caused by the sum of two kinds of information: the original image and the scratches one. Considering the first one in a suitable interval $[c_p - \delta, c_p + \delta]$, according to the Central Limit Theorem, we can assume that it represents a noise with a gaussian distribution

$N(\mu, \sigma)$ and then it assumes values in the range $[\mu - 3\sigma, \mu + 3\sigma]$. If we denote with $6\sigma_l$ the greatest variation of the image signal on the left of the scratch position, according to Eq. 2, we can say that the error on $x(m_l)$ measure is

$$\Delta_{x(m_l)} \;=\; 6(1 - (1-\gamma)k_p^{|m_l - c_p|})\sigma_l.$$

We can compute $\Delta_{x(m_r)}$ in the same way while

$$\Delta_{b_p} = (1-\gamma)\frac{\sigma_l + \sigma_r}{2}$$

and $\Delta_{\overline{b_p}} = 0$ because this latter has been obtained considering the whole available sample.

As a result we have:

$$\Delta E_r(b_p) \;=\; \sqrt{E_1 + E_2 + E_3} \qquad (6)$$

$$\Delta E_r(\overline{b_p}) \;=\; \sqrt{E_1 + \overline{E}_3} \qquad (7)$$

where

$$E_1 \;=\; \tfrac{(m_r - m_l)^2}{16}(\Delta^2_{x(m_l)} + \Delta^2_{x(m_r)}),$$

$$E_2 \;=\; \tfrac{(m_r - m_l)^2}{4}\Delta^2_{b_p},$$

$$E_3 \;=\; \tfrac{1}{4}(b_p - \tfrac{x(m_l)+x(m_r)}{2})^2(\Delta^2_{m_l} + \Delta^2_{m_r}),$$

$$\overline{E}_3 \;=\; \tfrac{1}{4}(\overline{b_p} - \tfrac{x(m_l)+x(m_r)}{2})^2(\Delta^2_{m_l} + \Delta^2_{m_r}).$$

Figure 7: 256 × 256 × 8 bits Knight image. There is an only white and visible scratch on the image.

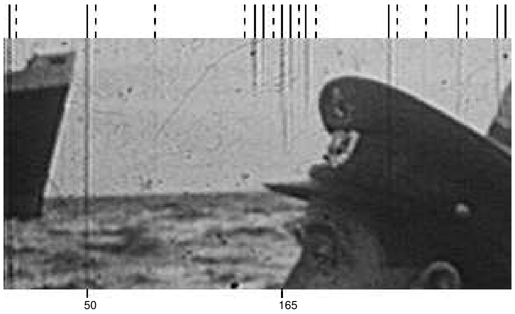

Figure 8: 150 × 300 × 8 bits Star image: Top) detected (solid lines) and not detected (dashed lines) scratches by the proposed algorithm.

4 EXPERIMENTAL RESULTS

We have tested our model on many images. In particular we have considered Kokaram's 8-bits test images (Sitdown (Fig. 1), Knight (Fig. 7) and Star (Fig. 8)) to compare our results with the Kokaram's ones.

Let's have a look at Table 1, we have compared both algorithms.

Our algorithm is more efficient due to three reasons:

- Kokaram's algorithm isn't always able to recognize secondary scratches because it depends on the threshold on the number of votes in the Hough Transform. Using a length threshold of 100 in Sitdown image [Lines], it detects the secondary line at the column position 257 as happens in our algorithm. On Star image (Fig. 8) almost all scratches are secondary; it recognizes just the principal one (col. no. 50 in Fig. 8) [Lines]; our algorithm detects almost all secondary scratches too;
- the use of the Weber's law allows us to detect only just perceivable scratches, especially if they are *alone*. See, for instance, column no. 146 of Sitdown image: at first sight it seems that the image doesn't have a defect, but looking more carefully at it, we note a less noticeable scratch; moreover, looking at the cross-section profile in correspondence of that position, it results that the signal isn't too much noisy (the scratch is *alone*);
- Kokaram's algorithm needs to know 'a priori' the number of scratches to be detected. For example [Lines], in Sitdown image it detects only 9 scratches because the max-

	SITDOWN IMAGE
True Scratches	24
Kokaram's Model	9
Proposed Model	12
	KNIGHT IMAGE
True Scratches	1
Kokaram's Model	1
Proposed Model	1
	STAR IMAGE
True Scratches	21
Kokaram's Model	1
Proposed Model	11

Table 1: Comparison between Kokaram's model and the proposed one in terms of detected scratches. True scratches are all the perceivable ones by the authors on the image.

imum number of real lines is assumed to be 9. On the contrary, our algorithm doesn't have this kind of restriction. At this stage we have to outline that our algorithm fails in detecting some scratches lying nearby the ones that are *not alone*. In fact, when the signal is particularly noisy, the error on the sinusoid lobes should be too big and then the scratch is rejected (see for instance column no.356 of Sitdown image).

We can now compare the computing time of the algorithms. The filtering phase and the Hough Transform in Kokaram's technique have an $O(MNk) + O(N^2k^2)$ complexity [Canto94][Schaf81], where k is the accumulator array size, while we cannot estimate the *burn in* period in the stochastic step because it depends on the initial conditions of the parameters. As in Kokaram's technique, our algorithm has an $O(MN)$ complexity for the cross-section computation except for an $O(N)$ for all the resting operations. Thus it results very fast.

4.1 FALSE ALARMS REJECTION

As we have mentioned in the introduction, we are able to propose a method to distinguish true scratches from scene components only if they represent secondary lines. We have considered the *rope* in Sitdown. A doubtful line can be vertically analyzed to find its end point on the image using the Weber's law. Then the vertical brightness variation is studied nearby it. According to the fact that image features have usually strong brightness variation, it follows that their end points correspond to a maximum in the brightness variation function while it doesn't happen for any true scratch.

5 CONCLUDING REMARKS AND FUTURE RESEARCH

In this paper we have described a generalized model for line scratches detection on digital images. It is based on the analysis of a monodimensional function of image brightness and produces a fast and almost completely automatic algorithm. Nevertheless, as results from Table 1, it isn't able to detect all the scratches on the image. In fact it doesn't recognize scratches that are *not alone*, whose strong mutual influence yields a wrong error calculation (see, for instance, column no. 356 of Sitdown). Our future research will consist in facing up this problem.

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REFERENCES

[Brand70] S. Brandt, *Statistical and Computational Methods in Data Analysis*, North Holland 1970.

[Canto94] V. Cantoni (ed.),*Pyramidal Architectures for Computer Vision*, Kluwer Academic Publishers, 1994.

[Kokar96] A.C. Kokaram, *Detection and Removal of Line Scratches in Degraded Motion Picture Restoration*, in Proc. of *Signal Processing VIII*, Trieste, Sept.1996, vol.1, pp.5-8, 1996.

[Kokar98] A.C. Kokaram, *Motion Picture Restoration: Digital Algorithms for Artefacts Suppression in Archived Film and Video*, Springer-Verlag, Apr. 1998.

[Jain89] Anil K. Jain, *Fundamentals of Digital Signal Processing*, Prentice Hall Inc., 1989, cap.3, cap.9.

[Joyeu99] L. Joyeux, O. Buisson, B. Besserer, S. Boukir, *Detection and Removal of Line Scratches in Motion Picture Film*, *Proc. of CVPR'99*, Fort Collins, Colorado, USA, June 1999.

[Joyeu00] L. Joyeux, S. Boukir, B. Besserer, *Film Line Scratch Removal Using Kalman Filtering and Bayesian Restoration*, *Proc. of WACV'2000*, Palm Springs, California, December 2000.

[Lines] www.mee.tcd.ie/∼ack/cd/lines/lines.htm

[Schaf81] R.W. Schaffer, *Digital Signal Processing*, Prentice Hall Inc., 1981, cap.1.

[Schal99] P. Schallauer, A. Pinz, W.Haas, *Automatic Restoration Algorithms for 35mm Film*, VIDERE: Journal of Computer Vision Research, The MIT Press, vol.1, n.3, 1999.

[Vitul01] D. Vitulano, V. Bruni, P. Ciarlini, *An Energy based Model for Line Scratch Detection*, Quaderno IAC no.15/2001, IAC - Roma.

APPENDIX A: ABOUT THE ENERGY APPROXIMATION

In this section we give the proof of the Eq. 5. Since digital images produce a discrete signal, we have to consider Eq. 3 in the discrete space so that it becomes

$$E_n(b_p) \quad = \quad |b_p - a| + 2\sum_{j=1}^{m-1} |L_n^{(p)}(c_p - j) - a| \tag{8}$$

In order to completely determine this formula we have to compute the coefficients k_p and ω_p according to the greatest width of a scratch and the Sampling Theorem [Jain89]. Let's define

$$D(b_p) \quad = \quad |E_r(b_p) - E_n(b_p)| \tag{9}$$

the approximation error; in particular

$$D_1 = |E_r(\overline{b_p}) - E_n(\overline{b_p})|$$

is the one in correspondence of a scratch of brightness b_p, and

$$D_2 = |E_r(b_p) - E_n(b_p)|$$

in correspondence of a scratch of least brightness $\overline{b_p}$.

Theorem:

$$\forall \quad E_r(\overline{b_p})\,,\, E_r(b_p) \quad : \quad E_r(\overline{b_p}) \quad \leq \quad E_r(b_p)$$

one of the following relations has to be true:

1. $E_r(b_p) - D_2 \; > \; E_r(\overline{b_p}) - D_1$
2. $E_r(b_p) - D_2 \; > \; E_r(\overline{b_p}) + D_1$
3. $E_r(b_p) + D_2 \; > \; E_r(\overline{b_p}) + D_1$
4. $E_r(b_p) + D_2 \; > \; E_r(\overline{b_p}) - D_1$

(10)

Proof: Firstly we observe that

- case 1 if $|b_p| > |\overline{b_p}| > |a|$;
- case 2 if $|b_p| > |a|$ e $|\overline{b_p}| < |a|$;
- case 3 if $|b_p| < |a|$ e $|\overline{b_p}| < |a|$;
- case 4 if $|b_p| < |a|$ e $|\overline{b_p}| > |a|$.

Since error D is added if $E_n(b_p) \; > \; E_r(b_p)$ and is subtracted viceversa, Eq. 8 becomes

1. $E_r(b_p) - (E_r(b_p) - E_n(b_p)) \; > \; E_r(\overline{b_p}) - (E_r(\overline{b_p}) - E_n(\overline{b_p}))$
2. $E_r(b_p) - (E_r(b_p) - E_n(b_p)) \; > \; E_r(\overline{b_p}) + (E_n(\overline{b_p}) - E_r(\overline{b_p}))$
3. $E_r(b_p) + (E_n(b_p) - E_r(b_p)) \; > \; E_r(\overline{b_p}) + (E_n(\overline{b_p}) - E_r(\overline{b_p}))$
4. $E_r(b_p) + (E_n(b_p) - E_r(b_p)) \; > \; E_r(\overline{b_p}) - (E_r(\overline{b_p}) - E_n(\overline{b_p}))$

then in each case we obtain Eq. 5, i.e.

$$E_n(b_p) \quad > \quad E_n(\overline{b_p}).$$

Since in real conditions it can't be $|b_p| < |a|$, cases 3 and 4 have to be eliminated. It's now simple to prove case 1 and case 2 [Vitul01].