Sectional Discrete Curvature Estimation Based on the Parabola

Hyoungseok Kim[1] and Hosook Kim[2]

[1] Dept. of Multimedia Engineering, Dongeui University, KOREA

[2] Division of Internet Information & Multimedia, Dongeui Institute of Technology, KOREA

Abstract

The local geometric properties such as curvatures and normal vectors play important roles in analyzing the local shape of objects. The result of the geometric operations such as mesh simplification and mesh smoothing is dependent on how to compute the curvature of vertices, because there is no its exact definition in meshes. In this paper, we indicate the fatal error in computing discrete sectional-curvatures by the previous discrete curvature estimations. Moreover, we present a new discrete sectional-curvature estimation to overcome the error, which is based on the parabolic interpolation and the geometric properties of Bezier curve.

Keywords: Discrete Curvature and Parabolic Interpolation

1. Introduction

The problem of estimating the geometric properties such as normal vectors and curvatures in triangular meshes plays important role in many applications such as surface segmentation and anisotropic remeshing. A lot of efforts have been devoted to this problem, but there is no consensus on the most appropriate way [1,3,4,5,8,9]. Popular methods typically consider some definition of curvature that can be extended to the polyhedral setting. Taubin presented a method to estimate the tensor of curvature of a surface at vertices of a mesh [6]. Watanabe proposed a simple method of estimating the principal curvatures of a discrete surface [7]. Meyer et. al proposed a discrete analog of the Laplace-Beltrami operator to estimate the discrete curvature[2].

Most of these methods compute directly the sectional curvatures for each adjacent edge of a vertex. They assume that the normal curve interpolates both the given vertex and an adjacent vertex and the curve is represented by Taylor series. However, they make the same mistake that they adopt the distance between the given vertex and its adjacent neighbor vertex as the parameter of the series. There are several polygons of different interior angles, all of which are circumscribed by circles of the same radius. The discrete curvatures of all vertices estimated by those methods are the same as that of the circle although they have different interior angle. It is quite alien to universal concepts.

2. Parabola-Based Discrete Curvature

We adopt a quadratic Bezier curve as an interpolating curve. Let A, B, C be three consecutive vertices. The general form of the quadratic Bezier curve satisfying P(1/2) = B is as follows:

$$P(t) = A\,B_0^2(t) + (4B-A-C)/2\;B_1^2(t) + C\,B_2^2(t),$$

where $B_i^n(t) = n!/((n-i)!\,i!)\,(1-t)^{(n-i)}\,t^i$ are the Bernstein polynomials of degree n. The curvature of P(t) at t=1/2 is

$$\kappa_P(1/2) = \| P''(1/2) \times P'(1/2) \| / \|P'(1/2)\|^3 = \| 4(A-2B+C) \times (C-A) \| / \|C-A\|^3.$$

Hence, we can define a new Parabola-based discrete curvature of the given vertex B as follows:

$$\kappa_P(B) \equiv \|4(A-2B+C)\times(C-A)\|/\|C-A\|^3.$$

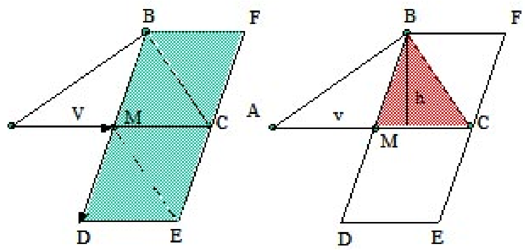

Figure 1. The geometric property

First of all, we find out the geometric properties of the P-discrete curvature formula. Let V=(C-A)/2 and G = A-2B+C. The P-discrete curvature formula is

$$\kappa_P(B) = \| 4G \times 2V \| / \| 2V \|^3 = \| G \times V \| / \|V\|^3 = (\|G\|\,\|V\| \sin\theta) / \|V\|^3, \qquad (1)$$

where θ is the in-between angle of the vectors G and V. The numerator of Equation (1) is the area of the parallelogram BDEF and is four times as much as the area of the triangle BCF as shown in Figure 1. Therefore, the P-discrete curvature formula is $\kappa_P(B) = 2h/v^2$, where, h and v are the height and the width of the triangle BCF, respectively.

3. Experimental Results

In order to verify the propriety of the P-discrete curvature, we regularly sample n points on a circle of radius 1 and compute their P-discrete curvature. Let $p_i=(\cos((2\pi\ i)/n), \sin((2\pi\ i)/n))$, i = 0, … , n-1, be the vertices of a n-gon on the circle. By trigonometry, we can compute the values of v and h as follows:

$$v = r \sin((2\pi\ i)/n), \quad h = r(1 - \cos((2\pi\ i)/n))$$.

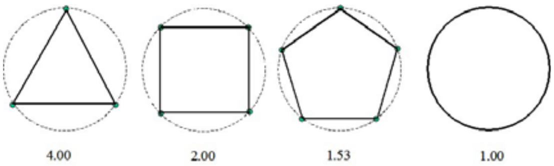

Figure 2. Polygons with the different p-curvature values

Therefore, as the number of sampling points increases to the infinity, the value of curvature at a vertex of the n-gon becomes that of the circle.

$$\mathrm{Lim}_{n\to\infty} (2(1-\cos((2\pi i)/n))/r\sin^2((2\pi i)/n) = 1/r.$$

Figure 2 shows the several polygons on a circle of radius 1 and their P-discrete curvature values. The result is an excellent contrast to that of circular based (C-type) discrete curvature estimation. That estimation wishes that the curvature at the sampled vertices may become that of a circle. The method puts emphasis on the point of view that the vertices are on a circle. However, it goes against the concept of curvatures. It loses the information on the local shape. On the other hand, our method recognizes the vertices of polygons to have a sharper angle, not to be on a circle. That is, the P-discrete curvature of vertices of a triangle is 4 and that of rectangle is 2.0 (see Figure 2). More the number of vertices increases to the infinity, less the curvature value decreases to 1. Table 1 shows the differences of C-type discrete curvature and P-type discrete curvature

4. Conclusion

The analysis on the local properties of 3D meshes plays an important role in the applications such as morphing, simplification, smoothing. In special, the curvature at a point on a surface may represent the shape of its neighborhood. However, there is an exact definition of the curvature at a vertex. So, one has to approximate the value as a discrete curvature. The common previous methods compute directly the sectional curvatures for each one-ring neighbor, and then derive the Gaussian curvature and the mean curvature using the sectional curvatures. All of them utilize the circle-based discrete curvature to compute the sectional curvature. In this paper, we find out a fatal mistake and propose the parabola-based discrete curvature estimation to resolve the problem. Our method may be the basis of normal vector estimation and segmentation of meshes.

Table 1. P-type curvature vs. C-type curvature

	C-type Discrete urvature	P-type Discrete Curvature
Formula	$\kappa_C(B) = (2\ N\cdot BA) / \|\| AB\|\|^2$	$\kappa_{SP}(B) = (2\ N\cdot BA) /\|\| BA - (N\cdot BA)N\|\|^2$
Parameter	Distance	Horizontal Distance
Range	$\kappa_C(B) \le 2$ if $\|\|AB\|\| \ge 1$	$\kappa_{SP}(B) < \infty$
Trajectory	circle	parabola
Magnitude	$\kappa_C(B) \le \kappa_{SP}(B)$	

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