Západočeská univerzita v Plzni

Fakulta aplikovaných věd

Katedra matematiky

DIPLOMOVÁ PRÁCE

Plzeň, 2013 Jonáš Volek

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Parciální diferenciální rovnice na semidiskrétních oblastech

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UNIVERSITY OF WEST BOHEMIA

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS

DIPLOMA THESIS

Pilsen, 2013 Jonáš Volek

University of West Bohemia

Faculty of Applied Sciences

Department of Mathematics

DIPLOMA THESIS

Partial Differential Equations on Semidiscrete Domains

Pilsen, 2013

Jonáš Volek

to Radka

Declaration

I do hereby declare that the entire diploma thesis is my original work and that I have used only the cited sources.

Pilsen, 20th May 2013

Jonáš Volek

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I thank my supervisor Petr Stehlík for his devoted guidance and support. I also would like to thank Pavel Řehák for his helpful advices during my stage at the Institute of Mathematics of Czech Academy of Sciences in Brno.

Abstrakt

Diplomová práce se zabývá transportní rovnicí na semidiskrétních oblastech. V první části se věnujeme lineární rovnici, kde nejdříve představíme základní vlastnosti klasické transportní parciální diferenciální rovnice, potom zkoumáme semidiskrétní případ s diskrétním prostorem a spojitým časem a poté opačný problém s diskrétním časem a spojitým prostorem. Nakonec studujeme transportní diferenční rovnici. U těchto lineárních úloh se zaměříme na zachování znaménka, sumy a integrálu a jejich souvislosti v teorii pravděpodobnosti. Dále se zde věnujeme periodicitě řešení a směru šíření extrémů. V druhé části analyzujeme nelineární semidiskrétní transportní rovnici s diskrétním prostorem a spojitým časem. Zde zkoumáme existenci a jednoznačnost řešení a odvozujeme principy maxima a minima s jejich důsledky.

Klíčová slova

transportní rovnice, semidiskrétní oblasti, diferenční rovnice, diferenciální rovnice, nelineární rovnice, zachování znaménka, zachování integrálu, zachování sumy, periodicita, existence, jednoznačnost, principy maxima

Abstract

This diploma thesis deals with the transport equation on semidiscrete domains. In the first part we focus on the linear equation. We present basic properties of the classical transport partial differential equation, then we study the semidiscrete case with discrete space and continuous time and then the opposite problem with discrete time and continuous space. Finally, we deal with the transport difference equation. In these linear problems we are concerned with sign, sum and integral preservation and their consequences to the probability theory. Further, we analyze the periodicity of solution and the direction of extremum propagation. In the second part we study the nonlinear semidiscrete transport equation with discrete space and continuous time. We concentrate on the existence and uniqueness results and we derive the maximum and minimum principles with their applications.

Key words

Transport Equation, Semidiscrete Domains, Difference Equations, Differential Equations, Nonlinear Equations, Sign Preservation, Integral Preservation, Sum Preservation, Periodicity, Existence, Uniqueness, Maximum Principles

Contents

Preface

Mathematical models are often expressed by differential equations. In case of more complicated processes we generally need partial differential equations. These models are mainly continuous but the world around us is principally discrete. We can mention problems from economics and biology. For example the compound interest in economics leads to the simple difference equation which is solved by geometric sequence or population problems in biology have obviously discrete structure.

The continuous approach is simpler and more intuitive but we can ask what happens if we use discrete models. Moreover, we deal with problems that combine both cases, we study semidiscrete equations. If we understand problems of basic partial equations on semidiscrete domains we can use these models better and more effectively.

Why could be the discrete structure interesting and important? We live in the world where the capabilities of IT are higher and higher. Therefore, the usage of numerical methods is permanently more powerful. In these numerical methods the discretization is the main tool. As the second example we can mention the probability theory. Discrete random variables are structural elements of this field of study.

In numerical methods we often consider continuous variables as discrete and in models we use the difference instead the derivative. But the differential equation and equivalent difference equation can have completely different behavior. For example we can mention the logistic equation (see Elaydi [7]). This is the first-order equation which can be simply solved in the form of differential equation. But in the discrete form its behavior is more complicated and can lead up to chaos. This is the reason why we have to appreciate the differences between the continuous and discrete models. The study of semidiscrete case could help.

We study the transport equation on various domains. The transport equation describes advective transport of fluid and it also forms the base for the study of wave equation (see Drábek, Holubová [3]). Semidiscrete equations appear in special numerical methods that uses the discretization only for some variables and others are considered continuous (e.g. Galerkin or Rothe method, see Rektorys [15]).

In this diploma thesis we extend the problems that are studied in the bachelor thesis Volek [21]. In the first section we present some results from the theory of classical transport partial differential equation. The following three chapters study the linear transport equation with discrete variables. Section 2 studies the semidiscrete equation with discrete space and continuous time, Section 3 deals with the opposite case with discrete time and continuous space and Section 4 presents the transport difference equation with discrete space and discrete time. Finally, Section 5 studies the nonlinear semidiscrete transport equation with discrete space and continuous time.

In sections that deal with linear equations we focus on the sign preservation and on the sum and integral preservation. These results have interesting consequences to the probability theory. Further, we study the periodicity of solution and the direction of extremum propagation. In the final section which deals with the nonlinear transport equation we are concerned with the existence and uniqueness results and with maximum and minimum principles and their applications.

We hope that the reader finds our text, assertions and results interesting. Knowledge of mathematical analysis, theory of ordinary differential and difference equations and basic knowledge of theory of partial differential equations are expected.

1 Transport Partial Differential Equation

In the first section we present basic facts about the classical transport partial differential equation. Transport equation is the first-order hyperbolic equation which describes the signal propagation. The basic idea which is common for hyperbolic equations is the idea of characteristics. If we assume one dimensional space these are curves in the xt-plane along that the signal is propagated. The idea of characteristics helps us to reduce the partial equation to a simpler form, e.g. to an ordinary equation that we can solve. We focus on initial value problems in the one space variable and their classical solution. We mention basic properties of linear and nonlinear problems. More details could be found in Logan [10].

1.1 Linear Transport Equation with Constant Coefficients

In this subsection we study the following initial value problem

$$\begin{cases} u_t(x,t) + k u_x(x,t) = 0, \quad x \in \mathbb{R}, \quad t \in (0, +\infty), \quad k \in \mathbb{R} \setminus \{0\}, \\ u(x,0) = \phi(x), \quad \phi \in \mathcal{C}^1(\mathbb{R}). \end{cases} \tag{1.1}$$

THEOREM 1.1. *The solution of* (1.1) *is given by*

$$u(x,t) = \phi(x - kt). \tag{1.2}$$

Proof. We apply a basic fact from elementary mathematical analysis that if we consider a function $u(x,t)$ and a smooth curve C in the xt-plane defined by $x = x(t)$ then the total derivative of $u(x,t)$ along the curve C is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x(t),t) = u_t(x(t),t) + u_x(x(t),t) \frac{\mathrm{d}x(t)}{\mathrm{d}t}.$$

Hence, the left-hand side of equation in (1.1) is the total derivative of $u(x,t)$ along the curves defined by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = k.$$

Therefore, $u(x,t)$ is constant along straight lines

$$x - kt = c$$

when $c \in \mathbb{R}$ is arbitrary. Finally, because $u(x,t)$ is constant along these lines we get by application of the initial condition

$$u(x,t) = u(c,0) = \phi(c) = \phi(x - kt).$$

□

REMARK 1.2. *Curves C mentioned in Proof of Theorem 1.1 are called characteristics.*

Theorem 1.3. *The initial value problem* (1.1) *possesses unique solution which is given by* (1.2).

Proof. We prove the statement by contradiction. Let us assume there exist two distinct solutions $u(x,t)$ and $v(x,t)$ of (1.1). We set $h(x,t) = u(x,t) - v(x,t)$ and thus, $h(x,t)$ solves (1.1) with the vanishing initial condition $\phi(x)$,

$$\phi(x) = 0 \quad \text{for all } x \in \mathbb{R}. \tag{1.3}$$

Problem (1.1) + (1.3) is solved only by the trivial solution $h(x,t) = 0$ for all $x \in \mathbb{R}$ and $t \in [0, +\infty)$.

Indeed, if we assume that $h(x,t)$ solves (1.1) + (1.3) and that there exist $x_0 \in \mathbb{R}$ and $t_0 \in [0, +\infty)$ such that

$$h(x_0, t_0) \neq 0$$

then from the idea of characteristics and from the fact that $h(x,t)$ is constant along characteristics there exists $c \in \mathbb{R}$ such that

$$h(x_0, t_0) = h(c, 0) = \phi(c) \neq 0.$$

This contradicts (1.3).

Consequently, $h(x,t) = 0$ and then $u(x,t) = v(x,t)$ which is the final contradiction. □

Example 1.4. Let us consider problem (1.1) for $k = 1$ when

$$\phi(x) = x^2 e^{-x}.$$

By application of Theorem 1.1 we get solution

$$u(x,t) = (x-t)^2 e^{t-x}$$

and characteristics

$$x - t + c = 0, \quad c \in \mathbb{R}.$$

Some solution cuts and characteristic for $c = 0$ are shown on Figure 1. ■

Now we present few properties of solution of (1.1) that are interesting and important for later comparison to semidiscrete models. The following assertions deal with the sign and integral preservation.

Proposition 1.5. *Let $u(x,t)$ be the solution of* (1.1) *with the initial condition $\phi(x)$ that satisfies*

$$\phi(x) \geq 0 \quad \textit{for all } x \in \mathbb{R}.$$

Then

$$u(x,t) \geq 0$$

holds for all $x \in \mathbb{R}$ and $t \in [0, +\infty)$.

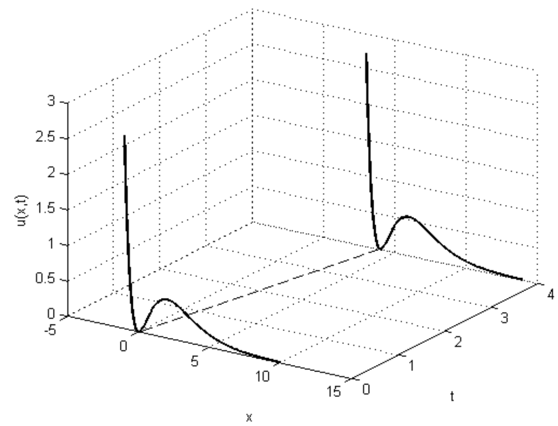

Figure 1: *Some solution cuts and characteristic $x - t = 0$ from Example 1.4.*

Proof. The statement is the direct consequence of Theorem 1.1. □

THEOREM 1.6. *Let $u(x,t)$ be the solution of* (1.1) *with the initial condition $\phi(x)$ that satisfies*

$$\int_{-\infty}^{+\infty} \phi(x)\mathrm{d}x = K, \quad K \in \mathbb{R}. \tag{1.4}$$

Then for all $t \in [0, +\infty)$ the following holds

$$\int_{-\infty}^{+\infty} u(x,t)\mathrm{d}x = K. \tag{1.5}$$

Proof. The statement can be proved by direct computation of (1.5) with help of substitution,

$$\begin{aligned} \int_{-\infty}^{+\infty} u(x,t)\mathrm{d}x &= \int_{-\infty}^{+\infty} \phi(x - kt)\mathrm{d}x = \left| \begin{array}{c} s = x - kt, \\ \mathrm{d}s = \mathrm{d}x, \end{array} \right| \\ &= \int_{-\infty}^{+\infty} \phi(s)\mathrm{d}s \overset{(1.4)}{=} K. \end{aligned}$$

□

THEOREM 1.7. *Let $u(x,t)$ be the solution of* (1.1) *and let $x \in \mathbb{R}$ be arbitrary and fixed. Then for $k > 0$ the following holds*

$$\int_{0}^{+\infty} u(x,t)\mathrm{d}t = \frac{1}{k} \int_{-\infty}^{x} \phi(s)\mathrm{d}s$$

and for $k < 0$ there is

$$\int_0^{+\infty} u(x,t)\mathrm{d}t = -\frac{1}{k}\int_x^{+\infty} \phi(s)\mathrm{d}s.$$

Proof. We use the substitution again and we can compute

$$\begin{aligned}
\int_0^{+\infty} u(x,t)\mathrm{d}t &= \int_0^{+\infty} \phi(x-kt)\mathrm{d}t = \left| \begin{array}{c} s = x-kt, \\ \mathrm{d}s = -k\mathrm{d}t, \end{array} \right| \\
&= \begin{cases} -\dfrac{1}{k}\displaystyle\int_x^{-\infty} \phi(s)\mathrm{d}s = \dfrac{1}{k}\displaystyle\int_{-\infty}^{x} \phi(s)\mathrm{d}s, & k > 0, \\ -\dfrac{1}{k}\displaystyle\int_x^{+\infty} \phi(s)\mathrm{d}s, & k < 0. \end{cases}
\end{aligned}$$

□

REMARK 1.8. *We see that integral in the space variable x is preserved in general. Integral in the time variable t is preserved in the following sense. If there is $k > 0$ and $\phi(x) = 0$ for all $x \geq x_0$ (or for $k < 0$ there is $\phi(x) = 0$ for all $x \leq x_0$) then the integral in the time variable t is preserved for $x \geq x_0$ (or for $x \leq x_0$).*

We later compare these properties with results from semidiscrete and discrete models.

The following two paragraphs show basics about the linear transport equation with nonconstant coefficients and about the nonlinear equation. We concentrate only on the solution there because these problems are not included in our comparison later.

1.2 Linear Transport Equation with Nonconstant Coefficients

This is the second paragraph about the linear transport equation. We extend here the statement of Theorem 1.1 for equation with nonconstant coefficients. We consider the following initial value problem

$$\begin{cases} u_t(x,t) + k(x,t)u_x(x,t) = 0, \quad x \in \mathbb{R}, \quad t \in (0,+\infty), \quad k \in \mathcal{C}(\mathbb{R} \times (0,+\infty)), \\ u(x,0) = \phi(x), \quad \phi \in \mathcal{C}^1(\mathbb{R}). \end{cases} \tag{1.6}$$

Again the left-hand side of equation in (1.6) is the total derivative of $u(x,t)$ along the curves defined by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = k(x,t).$$

Let us mention that for nonconstant $k(x,t)$ these are not straight lines. Along these curves there is

$$\frac{\mathrm{d}u(x(t))}{\mathrm{d}t} = u_t(x(t),t) + u_x(x(t),t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u_t(x(t),t) + k(x,t)u_x(x(t),t) = 0.$$

In other words, $u(x,t)$ is constant along these curves. Therefore, we finally get

$$u(x,t) \text{ is constant} \quad \text{on} \quad \frac{\mathrm{d}u(x(t))}{\mathrm{d}t} = k(x,t).$$

EXAMPLE 1.9. We consider the following initial value problem

$$\begin{cases} u_t(x,t) - xtu_x(x,t) = 0, \\ u(x,0) = \phi(x). \end{cases} \tag{1.7}$$

The characteristics are described by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t)t$$

which is the ordinary differential equation with solution (by separation of variables)

$$x(t) = ce^{-\frac{t^2}{2}}, \quad c \in \mathbb{R}.$$

Hence, with help of idea of characteristics there is

$$u(x,t) = u(c,0) = \phi(c) = \phi\left(xe^{\frac{t^2}{2}}\right).$$

Figure 2 shows some solution cuts and characteristics for the initial value problem (1.7) with the initial condition

$$\phi(x) = e^{-x^2}. \tag{1.8}$$

■

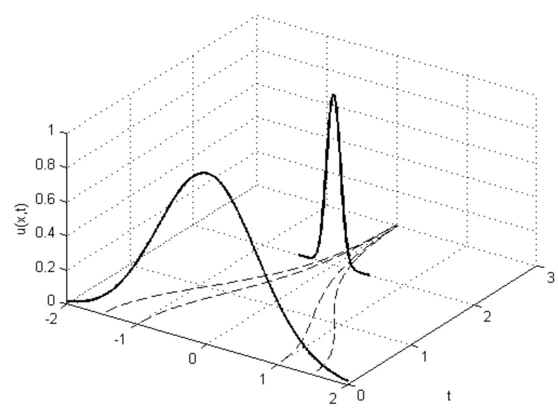

Figure 2: *Some solution cuts and characteristics from Example 1.9 for initial condition* $\phi(x) = e^{-x^2}$.

REMARK 1.10. *In the analogous way we can solve even nonhomogeneous initial value problem*

$$\begin{cases} u_t(x,t) + k(x,t)u_x(x,t) = f(x,t), \\ u(x,0) = \phi(x) \end{cases}$$

when $f \in \mathcal{C}(\mathbb{R} \times (0,+\infty))$. In this case $u(x,t)$ is not constant along characteristics but $u(x,t)$ satisfies the following ordinary differential equation along them,

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = f(x,t) \quad \textit{on} \quad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = k(x,t).$$

1.3 Nonlinear Transport Equation

Final paragraph of this section is about the following nonlinear initial value problem

$$\begin{cases} u_t(x,t) + F(u(x,t))u_x(x,t) = 0, \quad x \in \mathbb{R}, \quad t \in (0,+\infty), \quad F \in \mathcal{C}^1(\mathbb{R}), \\ u(x,0) = \phi(x), \quad \phi \in \mathcal{C}^1(\mathbb{R}). \end{cases} \tag{1.9}$$

For analysis of (1.9) we assume that there exists unique solution for all $t > 0$. By the same process as in the previous linear part we define characteristics by differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(u(x,t)). \tag{1.10}$$

But we do not know the right-hand side of (1.10) a priory and hence, we cannot determine characteristics in advance. But along curves given by (1.10) we have

$$u_t(x,t) + F(u(x,t))u_x(x,t) = u_t(x,t) + u_x(x,t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = 0,$$

i.e. $u(x,t)$ is constant. Because $u(x,t)$ is constant along characteristics we get

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}F(u(x(t),t))}{\mathrm{d}t} = F'(u(x(t),t))\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = 0$$

and therefore, the characteristics are straight lines. Consequently, we can construct them from an arbitrary point $(x,t) \in \mathbb{R} \times (0,+\infty)$ to a point $(c,0) \in \mathbb{R} \times (0,+\infty)$ on axis x. The equation of characteristics is

$$x - c = F(\phi(c))t, \quad c \in \mathbb{R},$$

and it follows that

$$u(x,t) = u(c,0) = \phi(c).$$

Finally, we derive the solution of (1.9) in the implicit form by following parametric equations

$$\begin{aligned} x - c &= F(\phi(c))t, \\ u(x,t) &= \phi(c), \quad c \in \mathbb{R}. \end{aligned} \tag{1.11}$$

We have to appreciate that the previous process we do under the assumption that there exits an unique solution of (1.9) for all $t > 0$. The proof of the following Theorem 1.11 is more technical and can be found in Logan [10].

THEOREM 1.11 ([10], THEOREM, P. 70). *Let functions F and ϕ are both $\mathcal{C}^1(\mathbb{R})$ and both decreasing or increasing on $\mathbb{R}$ (not necessarily strictly). Then the initial value problem* (1.9) *has unique solution defined implicitly by* (1.11).

EXAMPLE 1.12. We consider the following initial value problem

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) = 0, \\ \\ u(x,0) = \begin{cases} 0, & x \le 0, \\ e^{-\frac{1}{x}}, & x > 0. \end{cases} \end{cases}$$

In this problem the functions F and ϕ are both $\mathcal{C}^1(\mathbb{R})$ and increasing on $\mathbb{R}$. The characteristic that begins at a point $(c, 0)$ is the straight line defined by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(\phi(c)) = \phi(c) = \begin{cases} 0, & c \le 0, \\ e^{-\frac{1}{c}}, & c > 0, \end{cases}$$

and by application of Theorem 1.11 we get the unique solution given implicitly by

$$u(x,t) = \begin{cases} 0, & x \le 0, \\ e^{-\frac{1}{c}} \quad \text{where} \quad x - c = te^{-\frac{1}{c}} & \text{for} \quad x > 0. \end{cases}$$

Some solution cuts and characteristics are shown on Figure 3. ■

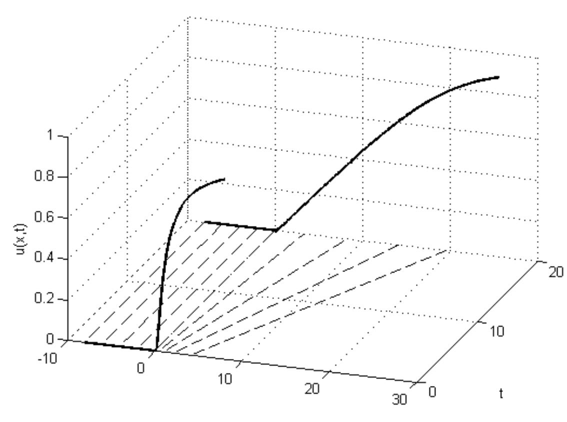

Figure 3: *Some solution cuts and characteristics from Example 1.12.*

2 Linear Transport Equation with Discrete Space and Continuous Time

In this section we consider the semidiscrete domain with discrete space and continuous time. We focus on the initial-boundary value problem. Main problems that we study are the preservation of sum in the space variable, the preservation of integral in the time variable, the sign preservation and their consequences to the probability theory. We derive that under certain assumptions the solution forms the probability distribution in both variables. We also study the direction of extremum propagation for special initial conditions.

2.1 Problem

In this section we concentrate on the following initial-boundary value problem

$$\begin{cases} u_t(x,t) + k\nabla_x u(x,t) = 0, \quad x \in \mathbb{Z}, x > x_0, \quad t \in (0,+\infty), \quad k > 0, \\ \\ u(x,0) = \phi_x, \quad \phi_x \in \mathbb{R} \quad \text{for all} \quad x \in \mathbb{Z}, x > x_0, \\ u(x_0,t) = 0, \quad t \in [0,+\infty). \end{cases} \tag{2.1}$$

2.2 Solution

We provide an explicit solution of (2.1) and show its behavior on some examples in this paragraph.

THEOREM 2.1. *The function*

$$u(x,t) = \sum_{i=x_0+1}^{x} \phi_i \frac{k^{x-i}}{(x-i)!} t^{x-i} e^{-kt}, \quad x > x_0. \tag{2.2}$$

is a solution of (2.1).

REMARK 2.2. *We deal with the uniqueness of* (2.2) *in Section 5.*

Proof. We prove the statement by induction on $x \in \mathbb{Z}, x > x_0$.

1. For $x = x_0 + 1$ we apply the boundary condition $u(x_0,t) = 0$ to (2.1). We get the homogeneous equation

   $$\begin{cases} u_t(x_0+1,t) + ku(x_0+1,t) = 0, \\ u(x_0+1,0) = \phi_{x_0+1} \end{cases}$$

   with the solution

   $$u(x_0+1,t) = \phi_{x_0+1} e^{-kt}.$$

2. We perform the inductive step. Let us suppose that (2.2) holds for all $\overline{x} < x$ when $x > x_0 + 1$. We plug the induction hypothesis into (2.1) and the problem has the form

$$\begin{cases} u_t(x,t) + ku(x,t) = k \sum_{i=x_0+1}^{x-1} \phi_i \frac{k^{x-1-i}}{(x-1-i)!} t^{x-1-i} e^{-kt}, \\ u(x,0) = \phi_x. \end{cases} \tag{2.3}$$

This is the linear nonhomogeneous ordinary differential equation. The solution of the homogeneous equation is

$$u_H(x,t) = C_x e^{-kt}, \quad C_x \in \mathbb{R}.$$

We search for the particular solution in the form

$$u_P(x,t) = te^{-kt} \sum_{j=0}^{x-x_0-2} D_j t^j.$$

Now we specify real constants D_j. We plug $u_P(x,t)$ into the equation in (2.3) and get

$$e^{-kt} \sum_{j=0}^{x-x_0-2} D_j t^j - kte^{-kt} \sum_{j=0}^{x-x_0-2} D_j t^j + te^{-kt} \sum_{j=1}^{x-x_0-2} jD_j t^{j-1}$$

$$+kte^{-kt} \sum_{j=0}^{x-x_0-2} D_j t^j = ke^{-kt} \sum_{j=x_0+1}^{x-1} \phi_j \frac{k^{x-1-j}}{(x-1-j)!} t^{x-1-j}$$

and after the simplification

$$e^{-kt} \sum_{j=0}^{x-x_0-2} (j+1) D_j t^j = ke^{-kt} \sum_{j=x_0+1}^{x-1} \phi_j \frac{k^{x-1-j}}{(x-1-j)!} t^{x-1-j}. \tag{2.4}$$

We compare the powers of t in (2.4) for the computation of D_j.

$$\begin{aligned} D_0 &= k\phi_{x-1}, && \text{i.e.} && D_0 = \phi_{x-1} k, \\ 2D_1 &= k\phi_{x-2} \frac{k}{1!}, && \text{i.e.} && D_1 = \phi_{x-2} \frac{k^2}{2!}, \\ 3D_2 &= k\phi_{x-3} \frac{k^2}{2!}, && \text{i.e.} && D_2 = \phi_{x-3} \frac{k^3}{3!}, \\ &\vdots \end{aligned}$$

We have

$$D_j = \phi_{x-(j+1)} \frac{k^{j+1}}{(j+1)!}$$

in general. Thus, we know the particular solution $u_P(x,t)$ and the solution of (2.3) is

$$u(x,t) = u_H(x,t) + u_P(x,t) = C_x e^{-kt} + \sum_{j=0}^{x-x_0-2} \phi_{x-(j+1)} \frac{k^{j+1}}{(j+1)!} t^{j+1} e^{-kt}.$$

After the application of the initial condition $u(x,0) = \phi_x$ we get the required relation

$$u(x,t) = \sum_{j=0}^{x-x_0-1} \phi_{x-j} \frac{k^j}{j!} t^j e^{-kt} \overset{(i=x-j)}{=} \sum_{i=x_0+1}^{x} \phi_i \frac{k^{x-i}}{(x-i)!} t^{x-i} e^{-kt}.$$

□

In the following three examples we suppose the vanishing boundary condition in $x_0 = -1$, i.e.

$$u(-1,t) = 0, \quad t \in [0,+\infty). \tag{2.5}$$

EXAMPLE 2.3. First, we study the solution of (2.1), $x_0 = -1$, with one-point initial condition ϕ_x, i.e. ϕ is defined as

$$\phi_x = \begin{cases} A, & x = 0, \\ 0, & x \in \mathbb{N}. \end{cases} \tag{2.6}$$

Then the solution is

$$u(x,t) = A \frac{k^x}{x!} t^x e^{-kt},$$

for $x \in \mathbb{N} \cup \{0\}$ and $t \in (0,+\infty)$. The computation follows directly from Theorem 2.1.

We try to find a curve χ along which maximums are propagated. We search the local maximum of functions

$$f(t) = t^x e^{-kt}.$$

The first derivative

$$f'(t) = x t^{x-1} e^{-kt} - k t^x e^{-kt} = t^{x-1} e^{-kt}(x - kt)$$

gives us stationary points $t_1 = 0$ and $t_2 = \frac{x}{k}$. The point t_1 lies on the boundary and is not interesting. For the point t_2 the derivative $f'(t)$ changes the sign from positive to negative. Thus, the point t_2 is the local maximum.

We see that the mentioned curve χ is the straight line

$$t = \frac{x}{k} \tag{2.7}$$

in this case. The solution $u(x,t)$ of (2.1) with the initial condition (2.6) when $k = 1$, $A = 1$ and the curve χ is shown on Figure 4. ■

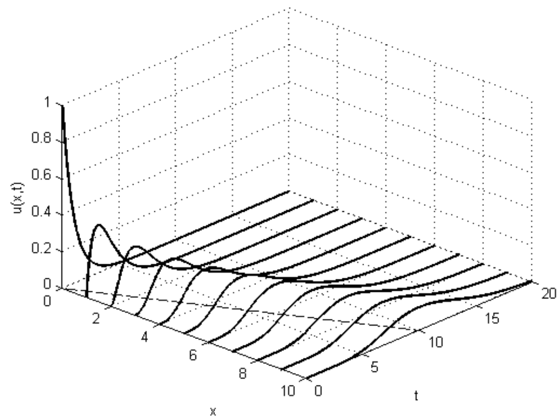

Figure 4: *The solution $u(x,t)$ of the initial-boundary value problem* (2.1) *when $k = 1$ with the initial condition given by* (2.6). *The straight line χ* (2.7).

The following example studies the constant two-point initial condition.

EXAMPLE 2.4. Let us suppose the problem (2.1), $x_0 = -1$, with constant two-point initial condition ϕ. Therefore, ϕ_x is given as follows

$$\phi_x = \begin{cases} 1, & x = 0 \text{ or } x = 1, \\ 0, & x = 2, 3, 4, \ldots \end{cases} \tag{2.8}$$

The solution is

$$u(x,t) = \begin{cases} \dfrac{k^x}{x!} t^x e^{-kt} + \dfrac{k^{x-1}}{(x-1)!} t^{x-1} e^{-kt}, & x \geq 1, \\ \\ e^{-kt}, & x = 0, \end{cases}$$

directly from Theorem 2.1.

Now we try to find a curve χ along which maximums are propagated again. We limit ourselves to the case $x \geq 1$. First, we compute the derivative

$$u_t(x,t) = \begin{cases} \dfrac{k^x}{(x-1)!} t^{x-1} e^{-kt} - \dfrac{k^{x+1}}{x!} t^x e^{-kt} + \dfrac{k^{x-1}}{(x-2)!} t^{x-2} e^{-kt} - \dfrac{k^x}{(x-1)!} t^{x-1} e^{-kt}, & x \geq 2, \\ \\ ke^{-kt} - k^2 te^{-kt} - ke^{-kt} = -k^2 te^{-kt}, \quad x = 1. \end{cases}$$

We see that for $x = 1$ there is the trivial solution $t = 0$. For $x \geq 2$ we put $u_t(x,t) = 0$ and assume $t \neq 0$,

$$\frac{k^x}{(x-1)!} t^{x-1} e^{-kt} - \frac{k^{x+1}}{x!} t^x e^{-kt} + \frac{k^{x-1}}{(x-2)!} t^{x-2} e^{-kt} - \frac{k^x}{(x-1)!} t^{x-1} e^{-kt} = 0. \tag{2.9}$$

We multiply (2.9) by the term $(x!)$ and divide by $(k^{x-1} t^{x-2} e^{-kt})$ and we get

$$kxt - k^2t^2 + x(x-1) - kxt = 0$$

and after the conversion

$$\left(x - \frac{1}{2}\right)^2 - k^2t^2 = \frac{1}{4}. \tag{2.10}$$

This is the hyperbola with the center $(\frac{1}{2}, 0)$. Hence, the curve χ is the curve of the second order.

It is clear from the preceding computation that, in general, for the n-point initial condition we get the curve of n^{th} order.

The solution of (2.1) with the initial condition (2.8) for $k = 1$ and the hyperbola (2.10) are on Figure 5 and Figure 6. ■

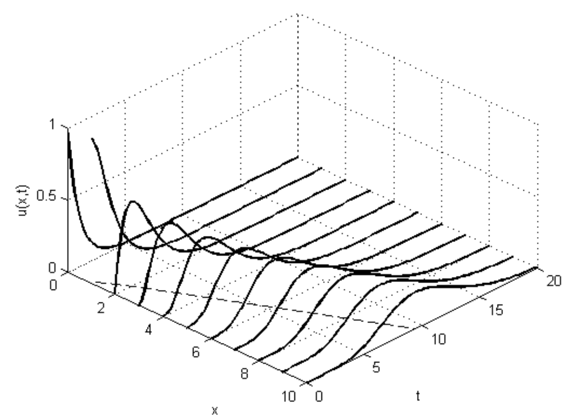

Figure 5: *The solution $u(x,t)$ of the initial-boundary value problem* (2.1) *when $k = 1$ with the initial condition* (2.8)*. The hyperbola* (2.10)*.*

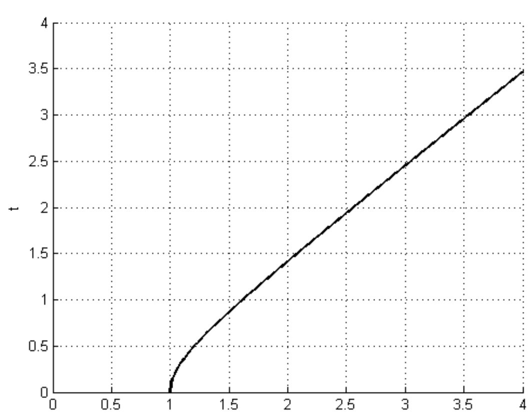

Figure 6: *The hyperbola* (2.10) *in xt-plane.*

As the last example we investigate the nonconstant two-point initial condition.

EXAMPLE 2.5. We study the problem (2.1), $x_0 = -1$, with the following initial condition

$$\phi_x = \begin{cases} A, & x = 0, \\ B, & x = 1, \\ 0, & x = 2, 3, 4, \ldots \end{cases} \tag{2.11}$$

The solution has the form

$$u(x,t) = \begin{cases} A\dfrac{k^x}{x!}t^x e^{-kt} + B\dfrac{k^{x-1}}{(x-1)!}t^{x-1}e^{-kt}, & x \geq 1, \\ \\ Ae^{-kt}, & x = 0, \end{cases}$$

by application of Theorem 2.1.

For finding a curve χ we can make analogous steps as in Example 2.4. We get the hyperbola again

$$B\left(x - \frac{1}{2}\right)^2 - Ak^2t^2 + (A-B)kxt = \frac{B}{4}. \tag{2.12}$$

Figures 7, 8 show the solution of the problem (2.1) with the nonconstant two point initial condition (2.11) and hyperbola (2.12). ■

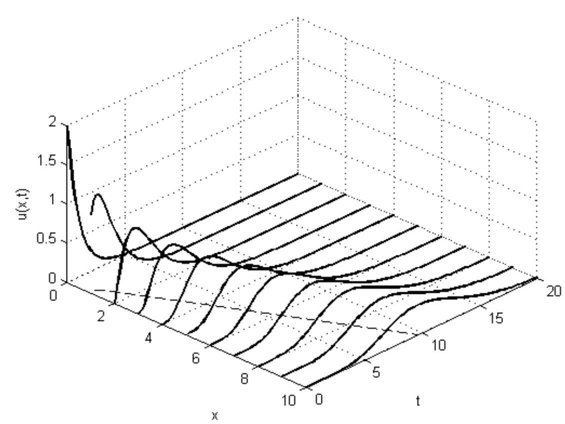

Figure 7: *The solution* $u(x,t)$ *of the initial-boundary value problem* (2.1) *when* $k = 1$ *with the initial condition* (2.11) *for* $A = 2$, $B = 1$. *The hyperbola* (2.12).

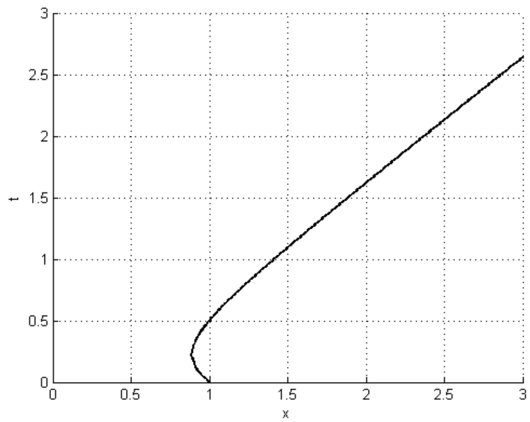

Figure 8: *The hyperbola* (2.12) *for* $A = 2$, $B = 1$ *in xt-plane.*

2.3 Sum and Integral Preservation

We study the sum preservation in the discrete variable and the integral preservation in the continuous variable. Before we provide these two results we deal with the sign preservation at first.

PROPOSITION 2.6. *Let* $u(x,t)$ *be the solution of* (2.1) *given by* (2.2) *with the assumption*

$$\phi_x \geq 0 \quad \textit{for all} \quad x \in \mathbb{Z}, x > x_0.$$

Then

$$u(x,t) \geq 0$$

holds for all $x \in \mathbb{Z}, x > x_0$ *, and for all* $t \in (0, +\infty)$.

Proof. The statement is the direct consequence of Theorem 2.1. □

The following two results give us the interesting relation with the theory of probability distributions. This is the goal of this section. We study the problem from Example 2.3, i.e. the problem

$$\left\{ \begin{array}{l} u_t(x,t) + k\nabla_x u(x,t) = 0, \quad k > 0, \\ \\ u(x,0) = \left\{ \begin{array}{ll} A, & x = 0, \\ 0, & x \in \mathbb{N}, \end{array} \right. \\ u(-1,t) = 0, \quad t \in [0,+\infty). \end{array} \right. \tag{2.13}$$

The solution of (2.13) is

$$u(x,t) = A\frac{k^x}{x!}t^x e^{-kt}. \tag{2.14}$$

THEOREM 2.7. *Let $u(x,t)$ be the solution of* (2.13) *given by* (2.14). *Then*

$$\sum_{x=0}^{+\infty} u(x,t) = A$$

holds for all $t \in (0,+\infty)$.

Proof. We compute the mentioned sum,

$$\sum_{x=0}^{+\infty} u(x,t) = \sum_{x=0}^{+\infty} A\frac{k^x}{x!}t^x e^{-kt}.$$

We use the Taylor series definition of the exponential function $e^y = \sum_{n=0}^{+\infty} \frac{y^n}{n!}$ and get

$$\sum_{x=0}^{+\infty} u(x,t) = \sum_{x=0}^{+\infty} A\frac{k^x}{x!}t^x e^{-kt} = Ae^{-kt} \underbrace{\sum_{x=0}^{+\infty} \frac{(kt)^x}{x!}}_{=e^{kt}} = A.$$

□

THEOREM 2.8. *Let $u(x,t)$ be the solution of* (2.13) *given by* (2.14). *Then*

$$\int_0^{+\infty} u(x,t)\mathrm{d}t = \frac{A}{k}.$$

holds for all $x \in \mathbb{N} \cup \{0\}$.

Proof. We prove the statement by induction on $x \in \mathbb{N} \cup \{0\}$.

1. For $x = 0$ we have

$$\int_0^{+\infty} u(0,t)\mathrm{d}t = \int_0^{+\infty} Ae^{-kt}\mathrm{d}t = A\left[-\frac{1}{k}e^{-kt}\right]_0^{+\infty} = \frac{A}{k}.$$

2. We assume that the assertion holds for all $\overline{x} < x$, i.e.

$$\int_0^{+\infty} u(\overline{x},t)\mathrm{d}t = \int_0^{+\infty} A\frac{k^{\overline{x}}}{(\overline{x})!}t^{\overline{x}}e^{-kt}\mathrm{d}t = \frac{A}{k}. \tag{2.15}$$

We use integration by parts and the preceding induction hypothesis to compute the integral

$$\int_0^{+\infty} u(x,t)\mathrm{d}t$$

for $x \geq 1$. Hence, we get

$$
\begin{aligned}
\int\limits_0^{+\infty} u(x,t)\mathrm{d}t &= \int\limits_0^{+\infty} A\frac{k^x}{x!}t^x e^{-kt}\mathrm{d}t = \left|\begin{array}{cc} \mathrm{D} & \mathrm{I} \\ t^x & e^{-kt} \\ xt^{x-1} & -\frac{1}{k}e^{-kt} \end{array}\right| \\
&= A\frac{k^x}{x!}\underbrace{\left[-\frac{1}{k}t^x e^{-kt}\right]_0^{+\infty}}_{=0} + A\frac{k^x}{x!}\int\limits_0^{+\infty}\frac{1}{k}xt^{x-1}e^{-kt}\mathrm{d}t \\
&= \int\limits_0^{+\infty} A\frac{k^{x-1}}{(x-1)!}t^{x-1}e^{-kt}\mathrm{d}t \overset{(2.15)}{=} \frac{A}{k}.
\end{aligned}
$$

□

Therefore, if we assume the solution $u(x,t)$ given by (2.14) of the problem (2.13) for $A = k = 1$ then Proposition 2.6, Theorem 2.7 and Theorem 2.8 imply that $u(x,t)$ makes the probability distribution in both variables. It is the Poisson probability distribution in the discrete variable x and the Erlang distribution in the continuous variable t (see Ross [16]). Together it forms so called Poisson stochastic process.

The problem (2.13) can be generalized on time scales. More informations about this generalization and its consequences to the probability theory with several examples can be found in Stehlík, Volek [20].

3 Linear Transport Equation with Discrete Time and Continuous Space

In this section we suppose the opposite semidiscrete domain with the discrete time and continuous space. First, we deal with the relationship to the previous section with discrete space and continuous time. Then we consider the initial value problem. We study the same properties, i.e. integral and sign preservation. We come to the conclusion that for this structure of variables these problems are much more complicated. For example the sign preservation does not hold in general in this case. Further, we focus on the periodicity again.

3.1 Problem

In this section we consider the following initial value problem

$$\begin{cases} \Delta_t u(x,t) + k u_x(x,t) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{N} \cup \{0\}, \quad k \in \mathbb{R} \setminus \{0\}, \\ u(x,0) = \phi(x) \end{cases} \tag{3.1}$$

when $\phi \in \mathcal{C}^\infty(\mathbb{R})$.

3.2 Relationship to Problem with Discrete Space and Continuous Time

Before we solve the problem presented above we motivate why we choose the right difference

$$\Delta_t u(x,t) = u(x,t+1) - u(x,t). \tag{3.2}$$

Let us consider the left difference

$$\nabla_t u(x,t) = u(x,t) - u(x,t-1)$$

and the boundary value problem

$$\begin{cases} \nabla_t u(x,t) + \frac{1}{k} u_x(x,t) = 0, \\ u(0,t) = \begin{cases} A, & t = 0, \\ 0, & t \neq 0, \end{cases} \end{cases} \tag{3.3}$$

for $x \in \mathbb{R}, t \in \mathbb{Z}$. Then we substitute

$$\tilde{k} = \frac{1}{k}, \quad \tilde{x} = t, \quad \tilde{t} = x$$

and the problem (3.3) has the following form

$$\begin{cases} \tilde{k} \nabla_{\tilde{x}} u(\tilde{t},\tilde{x}) + u_{\tilde{t}}(\tilde{t},\tilde{x}) = 0, \\ u(0,\tilde{x}) = \begin{cases} A, & \tilde{x} = 0, \\ 0, & \tilde{x} \neq 0, \end{cases} \end{cases}$$

that is the essential problem of Section 2. Therefore, let us study the problem (3.1) with the right difference (3.2).

3.3 Solution

This paragraph deals with the solution of (3.1) and its uniqueness.

THEOREM 3.1. *The solution* $u(x,t)$ *of* (3.1) *is given by*

$$u(x,t) = \sum_{i=0}^{t} (-1)^i \binom{t}{i} k^i \phi^{(i)}(x). \tag{3.4}$$

Proof. We prove the statement by induction on $t \in \mathbb{N} \cup \{0\}$.

1. First, we place $t = 0$ to (3.4). We have

$$u(x,0) = \phi(x)$$

   what is the initial condition.

2. Let us assume that (3.4) holds for all $\bar{t} < t, t \in \mathbb{N}$. We know from (3.1) that there is

$$u(x,t) = u(x,t-1) - ku_x(x,t-1). \tag{3.5}$$

   From the induction hypothesis we get the relation

$$u(x,t-1) = \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} k^i \phi^{(i)}(x). \tag{3.6}$$

   Hence, (3.5) and (3.6) imply

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} k^i \phi^{(i)}(x) - k \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} k^i \phi^{(i+1)}(x) \\ &= \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} k^i \phi^{(i)}(x) + \sum_{i=1}^{t} (-1)^i \binom{t-1}{i-1} k^i \phi^{(i)}(x) \\ &= \phi(x) + \sum_{i=1}^{t-1} (-1)^i \underbrace{\left[\binom{t-1}{i} + \binom{t-1}{i-1}\right]}_{=\binom{t}{i}} k^i \phi^{(i)}(x) + (-1)^t \binom{t}{t} k^t \phi^{(t)}(x) \\ &= \sum_{i=0}^{t} (-1)^i \binom{t}{i} k^i \phi^{(i)}(x). \end{aligned}$$

□

Theorem 3.2. *The initial value problem* (3.1) *possesses unique solution which is given by* (3.4).

Proof. We prove the statement by contradiction. Let us consider that $u(x,t)$ and $v(x,t)$ are solutions of (3.1) and $u(x,t) \neq v(x,t)$. We define the function $h(x,t) = u(x,t) - v(x,t)$. This function is the solution of the problem

$$\begin{cases} \Delta_t h(x,t) + k h_x(x,t) = 0, \\ h(x,0) = 0 \end{cases} \tag{3.7}$$

because

1.
$$\begin{aligned} \Delta_t h + k h_x &= \Delta_t(u - v) + k(u - v)_x = \Delta_t u - \Delta_t v + k u_x - k v_x \\ &= \Delta_t u + k u_x - (\Delta_t v + k v_x) = 0, \end{aligned}$$

2.
$$h(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) = 0.$$

We prove also by contradiction that the problem (3.7) is solved only by trivial solution.

Let $h(x,t) \neq 0$, i.e. there exist $x_0 \in \mathbb{R}$ and $t_0 \in \mathbb{N} \cup \{0\}$ such that $h(x_0,t_0) \neq 0$. If there is $t_0 = 0$ then we have contradiction with the vanishing initial condition. Hence, we suppose that $t_0 > 0$. Directly from (3.1) we get

$$h(x_0, t_0 - 1) + k h_x(x_0, t_0 - 1) \neq 0.$$

Now we have two possibilities.

1. If there is $h(x_0, t_0 - 1) = 0$ then $h_x(x_0, t_0 - 1) \neq 0$ holds. From continuity of $h(x, t_0 - 1)$ in the variable x there exists a punctured neighbourhood $P_\delta(x_0)$ with the radius $\delta > 0$ such that $h(x, t_0 - 1) \neq 0$ for all $x \in P_\delta(x_0)$. We choose arbitrary $x_P \in P_\delta(x_0)$ and define x_1 and t_1 as follows
$$x_1 := x_P \quad \text{and} \quad t_1 := t_0 - 1.$$
2. On the other hand if $h(x_0, t_0 - 1) \neq 0$ we can define x_1 and t_1 directly,
$$x_1 := x_0 \quad \text{and} \quad t_1 := t_0 - 1.$$

Consequently, we have $h(x_1, t_1) \neq 0$ and then we can continue iteratively and we end after $m = t_0$ steps with $t_m = 0$ and we have either $h_x(x_m, 0) \neq 0$ or $h(x_m, 0) \neq 0$. This is the contradiction with $h(x,0) = 0$ for all $x \in \mathbb{R}$. Hence, the problem (3.7) has only trivial solution and therefore $u = v$ what is the final contradiction. □

We assume in (3.1) that the initial condition is a $\mathcal{C}^\infty$ function. We can ask what happens if $\phi \notin \mathcal{C}^\infty(\mathbb{R})$. The following observation is clear.

OBSERVATION 3.3. 1. *If $\phi \notin \mathcal{C}^\infty(\mathbb{R})$ then there does not exist a solution of* (3.1) *for all $t \in \mathbb{N} \cup \{0\}$.*

2. *If the initial condition $\phi \in \mathcal{C}^m(\mathbb{R}), m \in \mathbb{N} \cup \{0\}$, then there exists a solution of* (3.1) *only for $t \leq m$. The solution has the form* (3.4).

Proof. Both statements are immediate consequences of Theorem 3.1 and Theorem 3.2. □

3.4 Piecewise Smooth Initial Condition

In this section we are interested in solution of the problem (3.1) in the case that the initial condition is not a $\mathcal{C}^\infty$ function but it is a continuous and piecewise $\mathcal{C}^\infty$ function.

DEFINITION 3.4. *If a solution $u(x,t)$ of* (3.1) *exists for almost all $x \in \mathbb{R}$ and for all $t \in \mathbb{N} \cup \{0\}$ we call it the generalized solution.*

PROPOSITION 3.5. *Assume that $\phi \in \mathcal{C}(\mathbb{R})$ and $\phi \in \mathcal{C}^\infty(M_j), j \in \mathbb{N}$, where M_j are open and disjoined sets and $\overline{\bigcup_{j\in\mathbb{N}} M_j} = \mathbb{R}$. Then there exists the generalized solution $u(x,t)$ of* (3.1)*. It is defined for all $x \in M_j$ and is given by*

$$u(x,t) = \sum_{i=0}^{t} (-1)^i \binom{t}{i} k^i \phi^{(i)}(x).$$

Proof. The assertion follows directly from the Proof of Theorem 3.1. □

REMARK 3.6. *If we suppose the problem from Proposition 3.5 then for some $t_L \in \mathbb{N}$ the generalized solution $u(x,t_L)$ can lose the "continuity" in the following sense*

$$\lim_{x\to x_0+} u(x,t_L) \neq \lim_{x\to x_0-} u(x,t_L) \quad \textit{for } x_0 \notin M_j, j \in \mathbb{N}.$$

We can see this phenomenon in the following example.

EXAMPLE 3.7. We study the initial condition $\phi(x)$ which is piecewise linear,

$$\phi(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } x \in (0,1), \\ 1 & \text{for } x \geq 1. \end{cases}$$

From Proposition 3.5 we have the generalized solution

$$u(x,t) = \begin{cases} 0 & \text{for } x < 0, t \in \mathbb{N}, \\ x - kt & \text{for } x \in (0,1), t \in \mathbb{N}, \\ 1 & \text{for } x > 1, t \in \mathbb{N}. \end{cases} \tag{3.8}$$

The solution does not exist for $x = 0$ and for $x = 1$ and in these points we have

$$\lim_{x\to 0+} u(x,t) = -kt, \quad \lim_{x\to 0-} u(x,t) = 0,$$

$$\lim_{x\to 1+} u(x,t) = 1, \qquad \lim_{x\to 1-} u(x,t) = 1 - kt.$$

The situation for $k = 1$ is shown on Figure 9 . ■

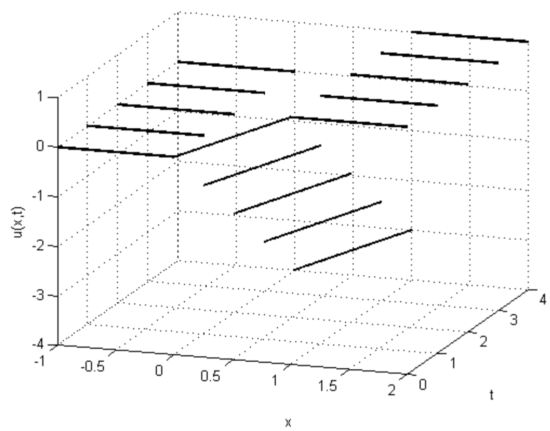

Figure 9: *The solution $u(x,t)$ given by* (3.8) *from Example 3.7.*

3.5 Periodicity of Solution

We prove in this section that if we have a periodic initial condition $\phi(x)$ then the solution of (3.1) is also periodic in the space variable for all $t \in \mathbb{N}$ with the same period.

THEOREM 3.8. *Let $u(x,t)$ be the solution of* (3.1) *when the initial condition $\phi(x)$ is periodic with the period $p > 0$, i.e.*

$$\phi(x) = \phi(x + mp) \quad \textit{for all } x \in \mathbb{R} \textit{ and } m \in \mathbb{Z} \tag{3.9}$$

and p is the smallest one. Then $u(x,t)$ is periodic in the space variable x for all $t \in \mathbb{N}$ with the period p, i.e.

$$u(x,t) = u(x + mp, t) \quad \textit{for all } x \in \mathbb{R}, t \in \mathbb{N} \textit{ and } m \in \mathbb{Z}.$$

Proof. First, we show that if the function $\phi(x)$ is periodic with the period $p > 0$ and if the derivative $\phi'(x)$ exists for all $x \in \mathbb{R}$ then the derivative is also periodic with the same period.

Therefore, let $\phi(x)$ be periodic with the period p and the derivative $\phi'(x)$ exists for all $x \in \mathbb{R}$. Thus, we know that (3.9) holds. Now we can find out the value of $\phi'(x + mp)$ when $x \in \mathbb{R}$ and $m \in \mathbb{Z}$ are arbitrary

$$\begin{aligned}
\phi'(x+mp) &= \lim_{h \to 0} \frac{\phi((x+mp)+h) - \phi(x+mp)}{h} = \lim_{h \to 0} \frac{\phi((x+h)+mp) - \phi(x+mp)}{h} \\
&\overset{(3.9)}{=} \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = \phi'(x).
\end{aligned}$$

We have even $\phi \in \mathcal{C}^\infty(\mathbb{R})$. Thus, every derivative $\phi^{(i)}(x), i \in \mathbb{N}$, is periodic with the period p.

The solution of (3.1) is given by (3.4) and for all $t \in \mathbb{N}$ it is the linear combination of derivatives of $\phi(x)$.

We prove that the linear combination of periodic functions with the same period p is also periodic function with the period p. Let $c_1, c_2, \dots, c_n, n \in \mathbb{N}$, be real constants and $f_1(x), f_2(x), \dots,$ $f_n(x)$ periodic functions with the same period $p > 0$. Then the function

$$h(x) = c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x)$$

satisfies

$$\begin{aligned} h(x+mp) &= c_1 f_1(x+mp) + c_2 f_2(x+mp) + \ldots + c_n f_n(x+mp) \\ &= c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = h(x) \end{aligned}$$

when $m \in \mathbb{Z}$ is arbitrary.

Consequently, our solution $u(x,t)$ is periodic with the period p in the space variable x. □

3.6 Integral Preservation

We can prove only the integral preservation in the continuous variable x in this case.

Theorem 3.9. *Let $u(x,t)$ be the solution of* (3.1) *when the initial condition $\phi(x)$ satisfies*

1. *$\phi(x) \geq 0$ for all $x \in \mathbb{R}$,*
2. *limits $\lim\limits_{x\to-\infty} \phi(x)$ and $\lim\limits_{x\to+\infty} \phi(x)$ exist,*
3. $\int\limits_{-\infty}^{+\infty} \phi(x)\mathrm{d}x = K \quad$ *for* $\quad K \geq 0.$

Then

$$\int\limits_{-\infty}^{+\infty} u(x,t)\mathrm{d}x = K$$

holds for all $t \in \mathbb{N}$.

Proof. First, we prove by contradiction that

$$\lim_{x\to+\infty} \phi(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} \phi(x) = 0 \tag{3.10}$$

and also for all $i \in \mathbb{N}$

$$\lim_{x\to+\infty} \phi^{(i)}(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} \phi^{(i)}(x) = 0. \tag{3.11}$$

Let us assume that it is not true. Thus, we assume for example

$$\lim_{x\to+\infty} \phi(x) = L \in (0, +\infty].$$

Then there exist $x_0 \in \mathbb{R}$ and $c > 0$ that for all $x > x_0$ is

$$\phi(x) \geq c,$$

certainly. Now we get from the comparison

$$\int_{x_0}^{+\infty} \phi(x)\mathrm{d}x \geq \int_{x_0}^{+\infty} c \, \mathrm{d}x = +\infty$$

a contradiction with the assumptions on $\phi(x)$. The case for $x \to -\infty$ is analogical.

Equalities (3.11) are the consequence of (3.10).

Now the solution $u(x,t)$ is given by (3.4) and for all $t \in \mathbb{N}$ we get

$$\begin{aligned}
\int_{-\infty}^{+\infty} u(x,t)\mathrm{d}x &= \int_{-\infty}^{+\infty} \sum_{i=0}^{t} (-k)^i \binom{t}{i} \phi^{(i)}(x)\mathrm{d}x = \sum_{i=0}^{t} (-k)^i \binom{t}{i} \int_{-\infty}^{+\infty} \phi^{(i)}(x)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \phi(x)\mathrm{d}x + \sum_{i=1}^{t} (-k)^i \binom{t}{i} \int_{-\infty}^{+\infty} \phi^{(i)}(x)\mathrm{d}x \\
&= \underbrace{\int_{-\infty}^{+\infty} \phi(x)\mathrm{d}x}_{=K} + \sum_{i=1}^{t} (-k)^i \binom{t}{i} \underbrace{\left(\lim_{x\to+\infty} \phi^{(i-1)}(x) - \lim_{x\to-\infty} \phi^{(i-1)}(x) \right)}_{=0} = K.
\end{aligned}$$

□

3.7 Solution Oscillations

In this paragraph we are concerned with the sign preservation. We do not present the final result about preservation but we give two lemmas that deal with solution oscillations. These can show that the sign preservation is more complicated than in the case of the discrete space and continuous time.

Lemma 3.10. *Let $u(x,t)$ be the solution of* (3.1) *in the fixed $t \in \mathbb{N} \cup \{0\}$. Let $u(x,t)$ have the local extremum in the variable x for $x = x_0$. Then for $u(x,t+1)$ there is*

$$u(x_0, t+1) = u(x_0, t).$$

Proof. The function $u(x,t)$ has the local extremum in the variable x for $x = x_0$. The necessary condition for the local extremum gives us

$$u_x(x_0, t) = 0.$$

We differentiate the relation

$$u(x, t+1) = u(x,t) - k u_x(x,t)$$

from (3.1) with respect to the variable x and we get

(3.12)
$$u_x(x,t+1) = u_x(x,t) - ku_{xx}(x,t).$$

We can do this because $u(x,t)$ is the solution (3.4) of the problem (3.1) with the initial condition $\phi \in \mathcal{C}^\infty(\mathbb{R})$. Hence, $u(x,t)$ is the $\mathcal{C}^\infty(\mathbb{R})$ function in the variable x for all $t \in \mathbb{N}$.

Now we integrate both sides of (3.12) from x_0 to x with respect to the first variable.

$$\begin{aligned}
\int_{x_0}^{x} u_s(s,t+1)\mathrm{d}s &= \int_{x_0}^{x} \left[u_s(s,t) - ku_{ss}(s,t)\right]\mathrm{d}s, \\
\left[u(s,t+1)\right]_{x_0}^{x} &= \left[u(s,t) - ku_s(s,t)\right]_{x_0}^{x}, \\
u(x,t+1) - u(x_0,t+1) &= u(x,t) - ku_x(x,t) - \left[u(x_0,t) - ku_x(x_0,t)\right], \\
\underbrace{u(x,t+1) - u(x,t) + ku_x(x,t)}_{=0} + u(x_0,t) &= u(x_0,t+1) + k\underbrace{u_x(x_0,t)}_{=0}, \\
u(x_0,t+1) &= u(x_0,t).
\end{aligned}$$

□

Now we use stronger assumptions and get the result about oscillations mentioned on the beginning of this subsection.

LEMMA 3.11. *Let $u(x,t)$ be the solution of* (3.1) *in the fixed $t \in \mathbb{N} \cup \{0\}$. Let $u(x,t)$ have the strict local extremum in the variable x for $x = x_0$. Furthermore, assume that*

(3.13)
$$u_{xx}(x_0,t) \neq 0.$$

Then

$$u(x_0,t+1) = u(x_0,t)$$

holds and the function $u(x,t+1)$ is strictly monotone in $x = x_0$.

Proof. The first part is Lemma 3.10. Therefore, we concentrate on the monotonicity of $u(x,t+1)$ in $x = x_0$. We use the relation (3.12) again and for $x = x_0$ we get

(3.14)
$$\begin{aligned}
u_x(x_0,t+1) &= \underbrace{u_x(x_0,t)}_{=0} - ku_{xx}(x_0,t), \\
u_x(x_0,t+1) &= -ku_{xx}(x_0,t).
\end{aligned}$$

We go through the case $k > 0$ in details. The proof of situation for $k < 0$ is analogous. Hence, let $k > 0$, we have two possibilities.

1. If the function $u(x,t)$ has the local maximum in $x = x_0$ then from the assumption (3.13) we get

$$u_{xx}(x_0,t) < 0. \tag{3.15}$$

We apply (3.15) to (3.14) and get

$$u_x(x_0, t+1) = -ku_{xx}(x_0,t) > 0.$$

Consequently, the function $u(x, t+1)$ is strictly increasing in $x = x_0$.

2. On the other hand, if $u(x,t)$ has the local minimum in $x = x_0$ then we get

$$\begin{aligned} u_{xx}(x_0,t) &> 0, \\ u_x(x_0,t+1) &= -ku_{xx}(x_0,t) < 0. \end{aligned}$$

The function $u(x,t+1)$ is strictly decreasing in $x = x_0$.

If the parameter satisfies $k < 0$ the proof has same structure. Only the cases when $u(x, t+1)$ is increasing and decreasing in $x = x_0$ interchange. □

REMARK 3.12. *We see that the function $u(x,t+1)$ has higher values than $u(x,t)$ has in local maximums and lower values than $u(x,t)$ has in local minimums. Therefore, the sign is not preserved in general. For example, if we assume $\phi(x) = \sin x + 1$ which is nonnegative and its minimums are equal to zero then the function $u(x,1)$ has already negative values.*

We illustrate the behavior presented in Lemmas 3.10 and 3.11 in the following example.

EXAMPLE 3.13. We study the problem (3.1) when the initial condition is

$$\phi(x) = \cos x, \quad \phi \in \mathcal{C}^\infty(\mathbb{R}).$$

For $i \in \mathbb{N} \cup \{0\}$ we get

$$\begin{aligned} \phi^{(2i)}(x) &= (-1)^i \cos x, \\ \phi^{(2i+1)}(x) &= (-1)^{i+1} \sin x. \end{aligned}$$

We apply Theorem 3.1 and the solution $u(x,t)$ is

$$u(x,t) = \sum_{i=0}^{t}(-1)^i\binom{t}{i}k^i\phi^{(i)}(x)$$

$$= \begin{cases} \sum_{i=0}^{\frac{t-1}{2}}(-1)^i\binom{t}{2i}k^{2i}\cos x + \sum_{i=0}^{\frac{t-1}{2}}(-1)^i\binom{t}{2i+1}k^{2i+1}\sin x, & \text{when } t \text{ is odd,} \\ \\ \sum_{i=0}^{\frac{t}{2}}(-1)^i\binom{t}{2i}k^{2i}\cos x + \sum_{i=0}^{\frac{t}{2}-1}(-1)^i\binom{t}{2i+1}k^{2i+1}\sin x, & \text{when } t \text{ is even.} \end{cases}$$

For $k = -1$ we can compute

$$\begin{aligned} u(x,0) &= \cos x, \\ u(x,1) &= \cos x + (-\sin x) = \cos x - \sin x, \\ u(x,2) &= \cos x + 2(-\sin x) + (-\cos x) = -2\sin x, \\ u(x,3) &= \cos x + 3(-\sin x) + 3(-\cos x) + \sin x = -2(\cos x + \sin x), \\ u(x,4) &= \cos x + 4(-\sin x) + 6(-\cos x) + 4\sin x + \cos x = -4\cos x. \\ &\vdots \end{aligned}$$

The solution is shown on Figures 10, 11. ■

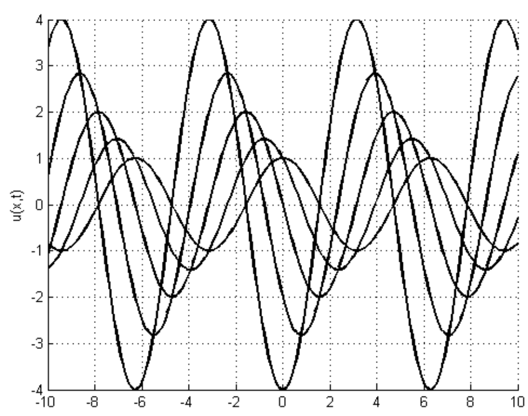

Figure 10: *The solution $u(x,t)$ from Example 3.13.*

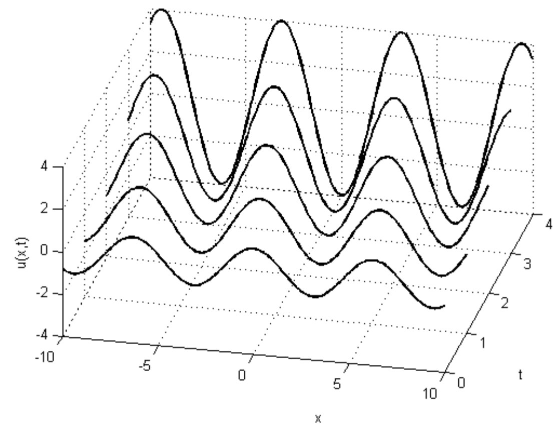

Figure 11: *The solution $u(x,t)$ from Example 3.13 in $\mathbb{R}^3$ space.*

4 Linear Transport Difference Equation

For the compact survey about linear transport equation with discrete variables we study the transport difference equation in this section. We deal with the special initial value problem which is analogous to problems from previous sections. For this special initial condition we study the sum and sign preservation again. For this purely discrete case we get interesting consequences to the probability distributions. We need some special techniques from the theory of difference equations here (see e.g. Elaydi [7] or Kelley, Peterson [8]). In the end of this section we compare results about linear transport equations on distinct domains.

Fundamentals about partial difference equations can be found e.g. in Cheng [2].

4.1 Auxiliary Assertions

Before we define the problem and before we solve it we have to expose some necessary results from the theory of difference equations. We recommend Elaydi [7] and Kelley, Peterson [8] for their further study.

LEMMA 4.1 ([7], (1.2.4), P. 3). *The initial value problem*

$$\begin{cases} y(n+1) = a(n)y(n) + b(n), \quad n \in \mathbb{Z}, \quad n \geq n_0 \geq 0, \quad n_0 \in \mathbb{Z}, \\ y(n_0) = y_0, \quad y_0 \in \mathbb{R}, \end{cases}$$

has the solution

$$y(n) = \left[\prod_{i=n_0}^{n-1} a(i)\right] y_0 + \sum_{r=n_0}^{n-1} \left[\prod_{i=r+1}^{n-1} a(i)\right] b(r).$$

DEFINITION 4.2. *Assume $n \in \mathbb{Z}$ and $r \in \mathbb{N} \cup \{0\}$. The falling factorial $n^{\underline{r}}$ is defined as follows*

$$\begin{aligned} n^{\underline{r}} &= n(n-1)\dots(n-r+1), \qquad r = 1,2,3,\dots, \\ n^{\underline{r}} &= 1, \qquad r = 0. \end{aligned}$$

More detailed description of the falling factorial can be found in Kelley, Peterson [8].

LEMMA 4.3 ([8], THEOREM 2.5). *Assume $n \in \mathbb{Z}$ and $r \in \mathbb{N} \cup \{0\}$. Then*

$$\sum\nolimits_n n^{\underline{r}} = \frac{n^{\underline{r+1}}}{r+1} + C \tag{4.1}$$

holds when the symbol $\sum_n y(n)$ denotes the indefinite sum (see Kelley, Peterson [8]) and $C \in \mathbb{R}$.

THEOREM 4.4 ([8], THEOREM 2.7). *If $z(i)$ is the indefinite sum of the function $y(i)$ then*

$$\sum_{i=m}^{n-1} y(i) = [z(i)]_m^n = z(n) - z(m) \tag{4.2}$$

holds.

4.2 Problem

We study the linear transport difference equation. We assume that both variables are discrete and each variable has its own discretization. Let us set

$$x = m\mu_x \quad \text{when} \quad m \in \mathbb{Z}, \quad \mu_x \in \mathbb{R}, \quad \mu_x > 0,$$

$$t = n\mu_t \quad \text{when} \quad n \in \mathbb{N} \cup \{0\}, \quad \mu_t \in \mathbb{R}, \quad \mu_t > 0.$$

Furthermore, we suppose the left relative difference in the space variable x, we denote it

$$\nabla_x^{(\mu_x)} u(x,t) = \frac{u(x,t) - u(x-\mu_x,t)}{\mu_x}$$

and the right relative difference

$$\Delta_t^{(\mu_t)} u(x,t) = \frac{u(x,t+\mu_t) - u(x,t)}{\mu_t}$$

in the time variable t.

Finally, we study the following initial value problem

$$\begin{cases} \Delta_t^{(\mu_t)} u(x,t) + k\nabla_x^{(\mu_x)} u(x,t) = 0, \quad k > 0, \\ u(x,0) = \begin{cases} A > 0, & x = 0, \\ 0, & x \neq 0. \end{cases} \end{cases} \tag{4.3}$$

4.3 Solution

Now we can start solving the problem (4.3). We modify the equation in (4.3) to the form

$$u(x,t+\mu_t) = \left(1 - \frac{k\mu_t}{\mu_x}\right) u(x,t) + \frac{k\mu_t}{\mu_x} u(x-\mu_x,t) \tag{4.4}$$

and we denote

$$L = \frac{k\mu_t}{\mu_x}. \tag{4.5}$$

The relation (4.4) changes to

$$u(x,t+\mu_t) = (1-L)u(x,t) + Lu(x-\mu_x,t). \tag{4.6}$$

The following theorem gives us the solution of (4.3).

THEOREM 4.5. *The solution of* (4.3) *is given by*

$$u(m\mu_x, n\mu_t) = A\left(1 - \frac{k\mu_t}{\mu_x}\right)^{n-m} \left(\frac{k\mu_t}{\mu_x}\right)^m \sum_{r_m=0}^{n-1} \dots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1. \tag{4.7}$$

Proof. We assume the right partial difference in x and hence, for all $m < 0$ we immediately get

$$u(m\mu_x, t) = 0 \tag{4.8}$$

which satisfies (4.7). For $m \geq 0$ we prove the statement by induction on $m \in \mathbb{N} \cup \{0\}$.

1. For $m = 0$ we put (4.8) to the equation (4.6)

$$u\left(0, (n+1)\mu_t\right) = (1-L)u(0, n\mu_t).$$

   Thanks to Lemma 4.1 applied on $n \in \mathbb{N} \cup \{0\}$ we have

$$u(0, n\mu_t) = (1-L)^n u(0,0) = A(1-L)^n$$

   that also satisfies (4.7).

2. We assume that (4.7) holds for $m \in \mathbb{N} \cup \{0\}$, i.e.

$$u(m\mu_x, n\mu_t) = A(1-L)^{n-m} L^m \sum_{r_m=0}^{n-1} \dots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1. \tag{4.9}$$

   Now we prove that (4.7) holds for $m+1$. We plug (4.9) into (4.6) and then we have

$$\begin{aligned} & u\left((m+1)\mu_x, (n+1)\mu_t\right) \\ & \quad = (1-L)u((m+1)\mu_x, n\mu_t) + A(1-L)^{n-m} L^{m+1} \sum_{r_m=0}^{n-1} \dots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1. \end{aligned}$$

   We use Lemma 4.1 again that gives us

$$\begin{aligned} & u\left((m+1)\mu_x, n\mu_t\right) \\ = \;& (1-L)^n \underbrace{u((m+1)\mu_x, 0)}_{=0} \\ & + \sum_{r_{m+1}=0}^{n-1} (1-L)^{n-1-r_{m+1}} A(1-L)^{r_{m+1}-m} L^{m+1} \sum_{r_m=0}^{r_{m+1}-1} \dots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1 \\ = \;& A(1-L)^{n-(m+1)} L^{m+1} \sum_{r_{m+1}=0}^{n-1} \dots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1. \end{aligned}$$

   If we apply the definition (4.5) of L we get (4.7). □

We use the auxiliary assertions from previous subsections to simplify the solution form in Theorem 4.5.

LEMMA 4.6. *For $m, n \in \mathbb{N} \cup \{0\}$ the following holds*

$$\sum_{r_m=0}^{n-1} \cdots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1 = \frac{n^{\underline{m}}}{m!}.$$

Proof. We prove the statement by induction on $m \in \mathbb{N} \cup \{0\}$.

1. For $m = 0$ we have

$$1 = \frac{n^{\underline{0}}}{0!} = 1.$$

2. We assume that

$$\sum_{r_m=0}^{n-1} \cdots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1 = \frac{n^{\underline{m}}}{m!}. \tag{4.10}$$

Then we use Lemma 4.3 and Theorem 4.4 and we get

$$\begin{aligned} \sum_{r_{m+1}=0}^{n-1} \sum_{r_m=0}^{r_{m+1}-1} \cdots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1 &\overset{(4.10)}{=} \sum_{r_{m+1}=0}^{n-1} \frac{r_{m+1}^{\underline{m}}}{m!} \\ &= \frac{1}{m!} \sum_{r_{m+1}=0}^{n-1} r_{m+1}^{\underline{m}} \\ &\overset{(4.1),(4.2)}{=} \frac{1}{m!} \left[\frac{r_{m+1}^{\underline{m+1}}}{m+1} \right]_0^n = \frac{n^{\underline{m+1}}}{(m+1)!}. \end{aligned}$$

□

LEMMA 4.7. *Consider $n, m \in \mathbb{N} \cup \{0\}$ such that $n \geq m$. Then*

$$\frac{n^{\underline{m}}}{m!} = \binom{n}{m}$$

holds.

Proof. We use the Definition 4.2 of the falling factorial

$$\begin{aligned} \frac{n^{\underline{m}}}{m!} &= \frac{n(n-1)\dots(n-m+1)}{m!} \\ &= \frac{n(n-1)\dots(n-m+1)(n-m)(n-m-1)\cdot\ldots\cdot 3\cdot 2\cdot 1}{m!(n-m)(n-m-1)\cdot\ldots\cdot 3\cdot 2\cdot 1} \\ &= \frac{n!}{m!(n-m)!} = \binom{n}{m}. \end{aligned}$$

□

Now we present the final theorem that gives us the general form of the solution.

THEOREM 4.8. *The solution of* (4.3) *is given by*

$$u(m\mu_x, n\mu_t) = \begin{cases} A\binom{n}{m}\left(1 - \frac{k\mu_t}{\mu_x}\right)^{n-m}\left(\frac{k\mu_t}{\mu_x}\right)^m, & n \geq m, \\ 0, & n < m. \end{cases} \tag{4.11}$$

Proof. The statement is the direct consequence of Theorem 4.5 and Lemmas 4.6 and 4.7. □

Some solution cuts are shown on Figures 12, 13.

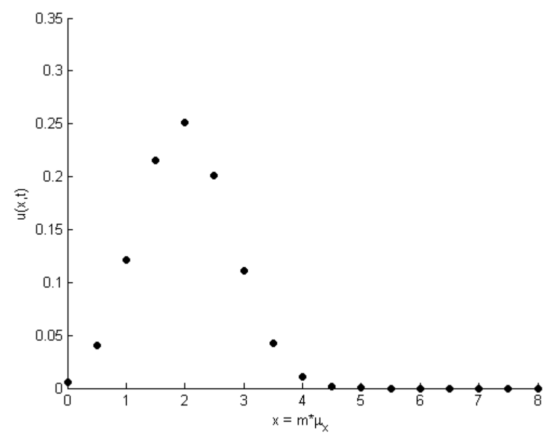

Figure 12: *The time cut* $u(x, 2)$ *of the solution* $u(x, t)$ *of the problem* (4.3) *for* $\mu_x = 0.5$, $\mu_t = 0.2$, $A = 1$, $k = 1$.

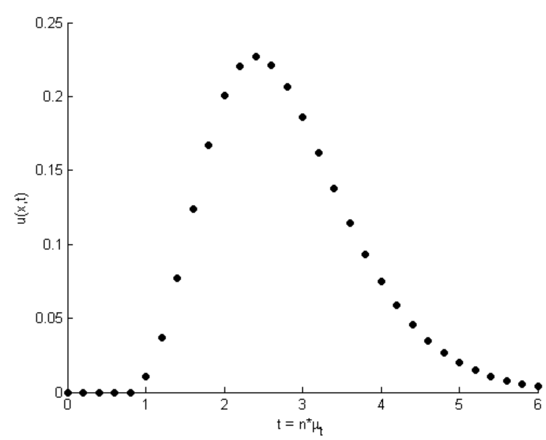

Figure 13: *The space cut* $u(2.5, t)$ *of the solution* $u(x, t)$ *of the problem* (4.3) *for* $\mu_x = 0.5$, $\mu_t = 0.2$, $A = 1$, $k = 1$.

THEOREM 4.9. *The initial value problem* (4.3) *possesses unique solution which is given by* (4.11).

Proof. We prove the statement by contradiction. Let us suppose two distinct solutions $u(x,t)$ and $v(x,t)$ of (4.3). We define $h(x,t) = u(x,t) - v(x,t)$. The function $h(x,t)$ solves the problem (4.3) with the initial condition

$$u(x,0) = 0 \quad \text{for all } x = m\mu_x. \tag{4.12}$$

But we prove also by contradiction that the problem (4.3) with the vanishing initial condition is solved only by the trivial solution

$$h(x,t) = 0 \quad \text{for all } x = m\mu_x, t = n\mu_t.$$

Let us consider that $h(x,t)$ solves the problem (4.3) with (4.12) and there exist $x_0 = m_0\mu_x$ and $t_0 = n_0\mu_t$ such that

$$h(x_0,t_0) \neq 0.$$

If there is $t_0 = 0$ we have the contradiction with the vanishing initial condition. Thus, let $t_0 > 0$. The following relation holds from (4.6)

$$h(x_0,t_0) = (1-L)h(x_0,t_0-\mu_t) + Lh(x_0-\mu_x,t_0-\mu_t) \neq 0. \tag{4.13}$$

At least one of terms in (4.13) has to be nonzero. Therefore, we define

$$x_1 = x_0 \quad \text{and} \quad t_1 = t_0 - \mu_t$$

and then either

$$h(x_1,t_1) = h(x_0,t_0-\mu_t) \neq 0$$

or

$$h(x_1-\mu_x,t_1) = h(x_0-\mu_x,t_0-\mu_t) \neq 0.$$

We can continue iteratively and after n_0 steps we get

$$h(x_{n_0},0) \neq 0 \quad \text{or} \quad h(x_{n_0}-\mu_x,0) \neq 0$$

that is the contradiction with the initial condition (4.12).

Therefore, $h(x,t) = 0$ and $u(x,t) = v(x,t)$ that is the final contradiction. □

REMARK 4.10. *In the similar way as in the Proof of Theorem 4.5 we can solve the following more general initial-boundary value problem*

$$\begin{cases} \Delta_t^{(\mu_t)} u(x,t) + k\nabla_x^{(\mu_x)} u(x,t) = 0, \quad k > 0, \\ \\ u(s\mu_x,0) = \phi_s, \quad \phi_s \in \mathbb{R}, \quad s > m_0, \quad m_0 \in \mathbb{Z}, \\ u(m_0\mu_x,t) = 0 \quad \textit{for all} \quad t = n\mu_t. \end{cases} \tag{4.14}$$

One can prove that (4.14) *has the unique solution given by*

$$u\left((m_0+m)\mu_x, n\mu_t\right) = \sum_{i=0}^{m-1} \phi_{m_0+m-i} \left(1-\frac{k\mu_t}{\mu_x}\right)^{n-i} \left(\frac{k\mu_t}{\mu_x}\right)^{i} \sum_{r_i=0}^{n-1} \cdots \sum_{r_2=0}^{r_3-1} \sum_{r_1=0}^{r_2-1} 1.$$

4.4 Sign and Sum Preservation

Now we know the solution and we can study the main question of this section, the sign and sum preservation.

PROPOSITION 4.11. *Let $u(x,t)$ be the solution of* (4.3). *Let*

$$1-\frac{k\mu_t}{\mu_x} \geq 0 \tag{4.15}$$

hold. Then $u(x,t)$ satisfies

$$u(x,t) \geq 0 \quad \textit{for all} \quad x = m\mu_x, \quad t = n\mu_t.$$

Proof. The statement the direct consequence of Theorem 4.8 and assumptions $A > 0$, $k > 0$ and (4.15).

□

Finally, there are two theorems about the preservation of sums.

THEOREM 4.12. *Let $u(x,t)$ be the solution of* (4.3). *Then*

$$\sum_{m=0}^{+\infty} u(m\mu_x, n\mu_t) = A$$

holds for all $n \in \mathbb{N} \cup \{0\}$.

Proof. First, we observe that for $m > n$ is

$$u(m\mu_x, n\mu_t) = 0.$$

Then we have

$$\sum_{m=0}^{+\infty} u(m\mu_x, n\mu_t) = \sum_{m=0}^{n} u(m\mu_x, n\mu_t).$$

According to the binomial theorem we get

$$\sum_{m=0}^{+\infty} u(m\mu_x, n\mu_t) = A \sum_{m=0}^{n} \binom{n}{m} \left(1-\frac{k\mu_t}{\mu_x}\right)^{n-m} \left(\frac{k\mu_t}{\mu_x}\right)^{m} = A\left[\left(1-\frac{k\mu_t}{\mu_x}\right)+\frac{k\mu_t}{\mu_x}\right]^{n} = A.$$

□

THEOREM 4.13. *Let $u(x,t)$ be the solution of* (4.3). *If the inequality*

$$\left|1-\frac{k\mu_t}{\mu_x}\right|<1 \tag{4.16}$$

holds then there is

$$\sum_{n=0}^{+\infty} u(m\mu_x, n\mu_t) = A\frac{\mu_x}{k\mu_t} \tag{4.17}$$

for all $m \in \mathbb{N} \cup \{0\}$.

Proof. We prove the statement by induction on $m \in \mathbb{N} \cup \{0\}$.

1. We check that the result is true for $m=0$. For $m=0$ we get from (4.11)

$$\sum_{n=0}^{+\infty} u(0,n\mu_t) = \sum_{n=0}^{+\infty} A\left(1-\frac{k\mu_t}{\mu_x}\right)^n.$$

According to the assumption (4.16) it is the convergent geometric series and we get

$$\sum_{n=0}^{+\infty} A\left(1-\frac{k\mu_t}{\mu_x}\right)^n = \frac{A}{1-\left(1-\frac{k\mu_t}{\mu_x}\right)} = A\frac{\mu_x}{k\mu_t}.$$

2. Let us suppose that (4.17) holds for all $\overline{m}<m$ when $m>0$. We use the equality

$$u(x,t+\mu_t) = \left(1-\frac{k\mu_t}{\mu_x}\right)u(x,t)+\frac{k\mu_t}{\mu_x}u(x-\mu_x,t). \tag{4.18}$$

Because we know $u(m\mu_x,0)=0$ for $m>0$ from the initial condition we get

$$\sum_{n=0}^{+\infty} u(m\mu_x,(n+1)\mu_t) = \sum_{n=0}^{+\infty} u(m\mu_x,n\mu_t). \tag{4.19}$$

and

$$\begin{aligned}\sum_{n=0}^{+\infty} u(m\mu_x,n\mu_t) &\overset{(4.19)}{=} \sum_{n=0}^{+\infty} u(m\mu_x,(n+1)\mu_t)\\ &\overset{(4.18)}{=} \left(1-\frac{k\mu_t}{\mu_x}\right)\sum_{n=0}^{+\infty} u\left(m\mu_x,n\mu_t\right)+\frac{k\mu_t}{\mu_x}\sum_{n=0}^{+\infty} u\left((m-1)\mu_x,n\mu_t\right).\end{aligned}$$

Finally, we have

$$\frac{k\mu_t}{\mu_x}\sum_{n=0}^{+\infty} u(m\mu_x, n\mu_t) = \frac{k\mu_t}{\mu_x}\sum_{n=0}^{+\infty} u((m-1)\mu_x, n\mu_t),$$

$$\sum_{n=0}^{+\infty} u(m\mu_x, n\mu_t) = \sum_{n=0}^{+\infty} u((m-1)\mu_x, n\mu_t) \overset{\text{ind. h.}}{=} A\frac{\mu_x}{k\mu_t}.$$

□

These results also have consequences in the theory of probability distributions. Let us assume that $u(x,t)$ is the solution of (4.3). If we put $A = k = \mu_x = 1$ and $\mu_t = p$ then we get

$$u(m, np) = \binom{n}{m}(1-p)^{n-m}p^m.$$

Proposition 4.11, Theorem 4.12 and Theorem 4.13 give us that $u(m, np)$ forms the binomial distribution in m for all fixed $n \in \mathbb{N} \cup \{0\}$ and the product $pu(m, np)$ forms the negative binomial distribution in n for all fixed $m \in \mathbb{N} \cup \{0\}$. Together it makes so called Bernoulli stochastic process. We recommend Stehlík, Volek [20] again for the further study.

4.5 Comparison of Results about Linear Transport Equations

At the end of linear part we resume results about all cases of linear transport equations. We denote these cases as follows,

(PDE) linear transport partial differential equation with constant coefficients,

(DS) linear transport equation with discrete space and continuous time,

(DT) linear transport equation with discrete time and continuous space,

(DE) linear transport difference equation.

	(PDE)	(DS)	(DT)	(DE)
Explicit solution	$\checkmark$(Th. 1.1)	$\checkmark^{\dagger}$ (Th. 2.1)	$\checkmark$(Th. 3.1)	$\checkmark^{*}$ (Th. 4.8)
Uniqueness	$\checkmark$(Th. 1.3)	?	$\checkmark$(Th. 3.2)	$\checkmark^{*}$ (Th. 4.9)
Sign preservation	$\checkmark$(Prop. 1.5)	$\checkmark^{\dagger}$ (Prop. 2.6)	$\times$ (Ex. 3.13)	$\checkmark^{*\dagger}$ (Prop. 4.11)
Integral preservation in x	$\checkmark$(Th. 1.6)	$\checkmark^{*\dagger}$ (Th. 2.7)	$\checkmark^{\dagger}$ (Th. 3.9)	$\checkmark^{*}$ (Th. 4.12)
Integral preservation in t	$\checkmark^{\dagger}$ (Th. 1.7)	$\checkmark^{*\dagger}$ (Th. 2.8)	?	$\checkmark^{*\dagger}$ (Th. 4.13)
Stochastic process	$\checkmark^{*}$	$\checkmark^{*\dagger}$	$\times$	$\checkmark^{*\dagger}$
Periodicity in x	$\checkmark$(Th. 1.1)	?	$\checkmark$(Th. 3.8)	?

Table 1: Properties summary of linear transport equations.

$\checkmark$ - the property holds,
$\times$ - the property does not hold in general,
? - we do not know if the property holds,
* - for special initial condition ϕ,
† - with additional assumptions.

5 Nonlinear Transport Equation with Discrete Space and Continuous Time

After the survey of linear transport equations we present few results about the nonlinear case. Our main focus is on maximum and minimum principles and their applications. Maximum principles are strong tools in the theory of differential equations (see e.g. Protter, Weinberger [13] or Pucci, Serrin [14]). They have many consequences, e.g. uniqueness of solutions, a priory bounds, approximation and oscillation results. There also exists relevant literature about maximum principles for difference equations. We mention Stehlík, Thompson [18], [19] that deal with maximum principles for ordinary second order dynamic equations on time scales and their applications to uniqueness, approximation and oscillation results. In partial difference equations we can see Mawhin, Thompson, Tonkes [11] or Cheng [2] where we can find some maximum principles and so called Wirtinger's inequalities for partial difference equations and further, we can see Stehlík [17] where maximum principles for second order dynamic operators of the elliptic type on time scales are studied.

We study nonlinear semidiscrete equations with discrete space and continuous time. First, we concentrate on the motivation for the study. Therefore, we derive the conservation law for this structure of variables. Then we deal with initial-boundary value problems. We present existence and uniqueness results and prove maximum and minimum principles and their applications, e.g. the boundedness of solution, the approximation, uniqueness and uniform stability results. Finally, we consider the initial value problem and prove the local existence and uniqueness of bounded solution with the help of the contraction principle.

5.1 Semidiscrete Conservation Law

Many equations have the form of conservation laws in natural sciences. Conservation laws often occur in physics, chemistry and technology. They reflect the balance of the quantity of some magnitude during the process. We can mention the well-known energy conservation law or the dynamic conservation law in physics.

We derive the semidiscrete conservation law for the discrete space variable. It leads to partial semidiscrete equations that we want to study. It is the motivation why the research of these problems is important and meaningful.

Continuous conservation laws are contained for example in Logan [10].

Hence, we consider the one dimensional discrete space. We simulate it by integers $\mathbb{Z}$. Further, we suppose the semidiscrete quantity $u = u(x, t)$ which changes continuously in the time and which is distributed in the discrete space. The magnitude u expresses the density or the concentration of the mass or of the population, energy etc. We denote by $\varphi = \varphi(x, t)$ the flux of u and we define it as follows

$$\varphi(i, t), \quad i \in \mathbb{Z}, \quad t \in [0, +\infty).$$

This quantifies the amount of u that overpasses between the positions $x = i$ and $x = i + 1$ in the time t. We assume that the positive direction is for increasing x.

Now we are prepared to derive the conservation law. We consider an arbitrary space segment between $x = i$ and $x = j$ when $i < j$. The total amount of the quantity in this segment is

$$\sum_{x=i}^{j} u(x,t).$$

We express the time change of the total amount in the section between $x = i$ and $x = j$. The change is influenced by the amount that flows into the segment and by the amount produced by sources in this segment. The amount that flows into in the time t is equal to

$$\varphi(i-1,t) - \varphi(j,t). \tag{5.1}$$

The amount produced by sources distributed in the segment in the time t is

$$\sum_{x=i}^{j} f(x,t) \tag{5.2}$$

where $f(x,t)$ denotes the source function that expresses amount produced by a source placed in x in the time t. If we put (5.1) and (5.2) together we get the "sum" form of the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x=i}^{j} u(x,t) = \varphi(i-1,t) - \varphi(j,t) + \sum_{x=i}^{j} f(x,t).$$

Now we modify the "sum" form of the conservation law as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x=i}^{j} u(x,t) &= -\left(\varphi(j,t) - \varphi(i-1,t)\right) + \sum_{x=i}^{j} f(x,t), \\
\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x=i}^{j} u(x,t) &= -\Bigg(\varphi(j,t) \underbrace{-\varphi(j-1,t) + \varphi(j-1,t)}_{=0} \underbrace{-\varphi(j-2,t) + \varphi(j-2,t)}_{=0} - \dots \\
&\qquad \underbrace{-\varphi(i,t) + \varphi(i,t)}_{=0} - \varphi(i-1,t)\Bigg) + \sum_{x=i}^{j} f(x,t), \\
\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x=i}^{j} u(x,t) &= -\Bigg(\underbrace{\varphi(j,t) - \varphi(j-1,t)}_{=\nabla_x \varphi(j,t)} + \underbrace{\varphi(j-1,t) - \varphi(j-2,t)}_{=\nabla_x \varphi(j-1,t)} + \varphi(j-2,t) - \dots \\
&\qquad + \underbrace{\varphi(i,t) - \varphi(i-1,t)}_{=\nabla_x \varphi(i,t)}\Bigg) + \sum_{x=i}^{j} f(x,t), \\
\sum_{x=i}^{j} \frac{\partial u(x,t)}{\partial t} &= -\sum_{x=i}^{j} \nabla_x \varphi(x,t) + \sum_{x=i}^{j} f(x,t)
\end{aligned}$$

and finally we get

$$\sum_{x=i}^{j}\left[\frac{\partial u(x,t)}{\partial t}+\nabla_x\varphi(x,t)-f(x,t)\right]=0. \tag{5.3}$$

Because (5.3) has to be satisfied for all space segments then

$$\frac{\partial u(x,t)}{\partial t}+\nabla_x\varphi(x,t)=f(x,t) \tag{5.4}$$

holds. This is the conservation law in the local form.

The equality (5.4) is essentially the partial semidiscrete equation with two unknown functions $u(x,t)$ and $\varphi(x,t)$. We have to add the relation between these functions. Then we get different types of first-order semidiscrete equations.

We study the case of

$$\varphi(x,t)=F(x,t,u(x,t)) \quad \text{when} \quad F:\mathbb{R}^3\to\mathbb{R}$$

that leads to the nonlinear semidiscrete transport equation

$$\frac{\partial u(x,t)}{\partial t}+\nabla_x F\left(x,t,u(x,t)\right)=f(x,t). \tag{5.5}$$

Remark 5.1. *Let us notice that the orientation of the space difference depends only on the definition of the flux φ. If we define $\varphi(i,t)$ as the flux between positions $x=i-1$ and $x=i$ we get the equation*

$$\frac{\partial u(x,t)}{\partial t}+\Delta_x F\left(x,t,u(x,t)\right)=f(x,t).$$

Remark 5.2. *If the function $F(x,t,u(x,t))$ is given by*

$$F(x,t,u(x,t))=ku(x,t), \quad k>0,$$

then we get the linear equation which we study in Section 2.

5.2 Existence and Uniqueness for Initial-Boundary Value Problem

In this paragraph we search for some conditions that guarantee the existence or existence and uniqueness of solution for nonlinear initial-boundary value problems. First, we need some auxiliary claims from the theory of ordinary differential equations. We refer to Kelley, Peterson [9] for their further study.

Primarily we investigate the following ordinary initial value problem

$$\begin{cases} u'(t)=g(t,u(t)), \quad g:I\times\mathbb{R}^n\to\mathbb{R}^n, \\ u(t_0)=u_0, \quad u_0\in\mathbb{R}^n, \end{cases} \tag{5.6}$$

when $I\subset\mathbb{R}$ is an interval. We state three theorems about the existence and uniqueness of solution of (5.6).

THEOREM 5.3 ([9], THEOREM 8.27). *Assume $t_0 \in \mathbb{R}$, $u_0 \in \mathbb{R}^n$ and let the function $g(\tau, \omega)$ be continuous on the rectangle*

$$Q = \{(\tau, \omega) \in \mathbb{R} \times \mathbb{R}^n : \quad |\tau - t_0| \leq a, \quad \| \omega - u_0 \| \leq b\}. \tag{5.7}$$

Then the initial value problem (5.6) *possesses a solution $u(x, t)$ on $[t_0 - \alpha, t_0 + \alpha]$ where*

$$\alpha = \min\left\{a, \frac{b}{M}\right\} \quad \text{and} \quad M = \max_{(\tau,\omega)\in Q} \| g(\tau, \omega) \| . \tag{5.8}$$

REMARK 5.4. *Theorem 5.3 is often called Cauchy–Peano theorem about local existence.*

THEOREM 5.5 ([9], THEOREM 8.13). *Let the function $g(\tau, \omega)$ be continuous on the rectangle Q given by* (5.7) *and let $g(\tau, \omega)$ satisfy a uniform Lipschitz condition with respect to ω on Q, i.e. there exists some constant $L > 0$ such that*

$$\| g(\tau, \omega_1) - g(\tau, \omega_2) \| \leq L \| \omega_1 - \omega_2 \|$$

holds for all $(\tau, \omega_1) \in Q$, $(\tau, \omega_2) \in Q$. Then the initial value problem (5.6) *possesses a unique solution $u(x, t)$ on $[t_0 - \alpha, t_0 + \alpha]$ where α is given by* (5.8).

REMARK 5.6. *Theorem 5.5 is often called Picard-Lindelöf theorem about local existence and uniqueness.*

THEOREM 5.7 ([9], COROLLARY 8.64). *Assume that $h : [0, +\infty) \to (0, +\infty)$ is continuous and there is a $v_0 \in [0, +\infty)$ such that*

$$\int_{v_0}^{+\infty} \frac{\mathrm{d}s}{h(s)} = +\infty.$$

Let the function $g : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let

$$\| g(\tau, \omega) \| \leq h (\| \omega \|)$$

hold for all $(\tau, \omega) \in [t_0, +\infty) \times \mathbb{R}^n$. Then for all $u_0 \in \mathbb{R}^n$ with $\| u_0 \| \leq v_0$ all solutions of (5.6) *exist on $[t_0, +\infty)$.*

Now we get back to semidiscrete equations. We study the global existence of the following initial-boundary value problem

$$\begin{cases} u_t(x, t) + \nabla_x F(x, t, u(x, t)) = f(x, t), \quad x \in \mathbb{Z}, \quad x > x_b, \quad t \in (0, +\infty), \\ \\ u(x, 0) = \phi(x), \quad x \in \mathbb{Z}, \quad x > x_b, \\ u(x_b, t) = \xi(t), \quad t \in [0, +\infty), \end{cases} \tag{5.9}$$

for given functions $F : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$, $\phi : \mathbb{Z} \to \mathbb{R}$ and $\xi : [0, +\infty) \to \mathbb{R}$ with $\xi \in \mathcal{C}([0, +\infty)) \cap \mathcal{C}^1(0, +\infty)$.

REMARK 5.8. *In the following text we always assume that* $\xi \in \mathcal{C}([0,+\infty)) \cap \mathcal{C}^1(0,+\infty)$.

THEOREM 5.9. *Let* $F(\chi,\tau,\omega)$ *be continuous for all* $\chi \in \mathbb{Z}$, $\chi \geq x_b$, *on* $[0,+\infty) \times \mathbb{R}$ *and let* $f(\chi,\tau)$ *be continuous for all* $\chi \in \mathbb{Z}$, $\chi \geq x_b$, *on* $[0,+\infty)$. *Assume that for all* $\chi \in \mathbb{Z}$, $\chi \geq x_b$, *there exist* $K = K(\chi) > 0$ *and* $L = L(\chi) > 0$ *such that for all* $\tau \in [0,+\infty)$ *and* $\omega \in \mathbb{R}$ *there is*

$$\begin{aligned} |F(\chi,\tau,\omega)| &\leq K(\chi), \\ |f(\chi,\tau)| &\leq L(\chi). \end{aligned}$$

Then the initial-boundary value problem (5.9) *possesses a solution* $u(x,t)$ *for all* $x \in \mathbb{Z}$, $x \geq x_b$, *and* $t \in [0,+\infty)$.

Proof. We prove the statement by induction on $x \in \mathbb{Z}$, $x \geq x_b$.

1. For $x = x_b$ we put $u(x_b,t) = \xi(t)$.
2. Let us have a solution $u(\overline{x},t)$ for all $\overline{x} \in \mathbb{Z}$, $x_b \leq \overline{x} < x$, on $[0,+\infty)$. Then from (5.9) we get for fixed x the following ordinary initial value problem

$$\text{(5.10)} \qquad \begin{cases} u_t(x,t) = f(x,t) + F(x-1,t,u(x-1,t)) - F(x,t,u(x,t)), \\ u(x,0) = \phi(x), \quad \phi(x) \in \mathbb{R}, \end{cases}$$

where $F(x-1,t,u(x-1,t))$ is a given function of t from the induction hypothesis. From continuity assumptions on F and f and from the fact that the composition of continuous functions is also continuous we get that the right-hand side of differential equation in (5.10) is continuous in t on $[0,+\infty)$.

- Theorem 5.3 gives us the existence of local solution $u(x,t)$ on some small interval $[0,\delta]$, $\delta > 0$.
- Next, we can make the following estimate

$$\begin{aligned} &|f(x,t) + F(x-1,t,u(x-1,t)) - F(x,t,u(x,t))| \\ &\quad \leq |f(x,t)| + |F(x-1,t,u(x-1,t))| + |F(x,t,u(x,t))| \\ &\quad \leq L(x) + K(x-1) + K(x) = H > 0. \end{aligned}$$

  Consequently, if we define $h(s) = H$ and $v_0 = |\phi(x)|$ then the assumptions of Theorem 5.7 are satisfied.

Finally, the local solution $u(x,t)$ can be extended on the interval $[0,+\infty)$. □

Theorem 5.9 guarantees only the existence of a global solution. If we want even uniqueness of this solution we can use Theorem 5.5 from which the following assertion follows.

THEOREM 5.10. *Assume that the assumptions of Theorem 5.9 are satisfied. Moreover, let $F(\chi, \tau, \omega)$ satisfy the uniform Lipschitz condition with respect to ω on $[0, +\infty) \times \mathbb{R}$. Then the initial-boundary value problem* (5.9) *possesses a unique solution $u(x,t)$ for all $x \in \mathbb{Z}$, $x \geq x_b$, and $t \in [0, +\infty)$.*

Proof. The existence of a solution $u(x,t)$ for all $x \in \mathbb{Z}$, $x \geq x_b$, and $t \in [0, +\infty)$ is the direct consequence of Theorem 5.9. Therefore, it remains to prove that $u(x,t)$ is unique.

Suppose by contradiction that there is a $t_0 \in [0, +\infty)$ such that there exist two distinct solutions $u_1(x,t)$ and $u_2(x,t)$ for $t \in [t_0, t_0 + \epsilon]$, $\epsilon > 0$, and $u_1(x,t) = u_2(x,t)$ on $[0, t_0]$. Let $\overline{x} \in \mathbb{Z}$, $\overline{x} > x_b$, be the smallest one for which $u_1(\overline{x}, t)$ and $u_2(\overline{x}, t)$ are distinct. Denote $u_{t_0} = u_1(\overline{x}, t_0)$.

For this $\overline{x}$ we have the following ordinary initial value problem

$$\begin{cases} u_t(\overline{x}, t) = f(\overline{x}, t) + F(\overline{x} - 1, t, u(\overline{x} - 1, t)) - F(\overline{x}, t, u(\overline{x}, t)), \\ u(\overline{x}, t_0) = u_{t_0}. \end{cases} \tag{5.11}$$

The right-hand side of equation in (5.11) is unique because $F(\overline{x} - 1, t, u(\overline{x} - 1, t))$ is unique by definition of $\overline{x}$. Further, the right-hand side satisfies the uniform Lipschitz condition with respect to u and therefore, the assumptions of Theorem 5.5 are satisfied on $[t_0, +\infty)$. From Theorem 5.5 we get that there exists unique solution $u(\overline{x}, t)$ on $[t_0, t_0 + \delta]$, $\delta > 0$, but we assume there are two distinct solutions $u_1(\overline{x}, t)$ and $u_2(\overline{x}, t)$ on $[t_0, t_0 + \epsilon]$. Hence, we get a contradiction on $[t_0, t_0 + \min\{\delta, \epsilon\}]$. □

We can study another type of semidiscrete equations. The following theorem gives us the existence of solution for the problem

$$\begin{cases} u_t(x,t) + F(x, t, \nabla_x u(x,t)) = f(x,t), \quad x \in \mathbb{Z}, \quad x > x_b, \quad t \in (0, +\infty), \\ \\ u(x, 0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \quad x \in \mathbb{Z}, \quad x > x_b, \\ u(x_b, t) = \xi(t), \quad \xi : [0, +\infty) \to \mathbb{R}. \end{cases} \tag{5.12}$$

when $F : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$.

THEOREM 5.11. *Let $F(\chi, \tau, \omega)$ be continuous for all $\chi \in \mathbb{Z}$, $\chi \geq x_b$, on $[0, +\infty) \times \mathbb{R}$ and let $f(\chi, \tau)$ be continuous for all $\chi \in \mathbb{Z}$, $\chi \geq x_b$, on $[0, +\infty)$. Assume that for all $\chi \in \mathbb{Z}$, $\chi \geq x_b$, there exist $K = K(\chi) > 0$ and $L = L(\chi) > 0$ such that for all $\tau \in [0, +\infty)$ and $\omega \in \mathbb{R}$ there is*

$$|F(\chi, \tau, \omega)| \leq K(\chi),$$
$$|f(\chi, \tau)| \leq L(\chi).$$

Then the initial-boundary value problem (5.12) *possesses a solution $u(x,t)$ for all $x \in \mathbb{Z}$, $x \geq x_b$, and $t \in [0, +\infty)$.*

Proof. The statement can be proved analogously as Theorem 5.9 with the help of Theorems 5.3 and 5.7. □

We prove the uniqueness of solution for (5.12) from the maximum and minimum principles in Subsection 5.4.4.

5.3 Maximum and Minimum Principles

In this subsection we consider the initial-boundary value problem (5.9). We derive maximum and minimum principles that are strong tools for differential and difference equations. We show that on several applications. We prove these principles for upper and lower solutions.

Definition 5.12. *A classical solution of the following initial-boundary value problem with inequalities*

$$\begin{cases} v_t(x,t) + \nabla_x F(x,t,v(x,t)) \le f(x,t), \\ \\ v(x,0) \le \phi(x), \\ v(x_b,t) \le \xi(t) \end{cases}$$

is called the lower solution of (5.9)*. A classical solution of*

$$\begin{cases} w_t(x,t) + \nabla_x F(x,t,w(x,t)) \ge f(x,t), \\ \\ w(x,0) \ge \phi(x), \\ w(x_b,t) \ge \xi(t) \end{cases}$$

is called the upper solution of (5.9)*.*

We state an auxiliary lemma that helps us later to prove the maximum principle.

Lemma 5.13. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ satisfy*

(F1) *$F(\chi, \tau, \omega)$ is increasing in χ, i.e.*

$$\text{for all} \quad \chi_1 < \chi_2 \quad \text{there is} \quad F(\chi_1, \tau, \omega) \le F(\chi_2, \tau, \omega), \tag{5.13}$$

(F2) *$F(\chi, \tau, \omega)$ is strictly increasing in ω, i.e.*

$$\text{for all} \quad \omega_1 < \omega_2 \quad \text{there is} \quad F(\chi, \tau, \omega_1) < F(\chi, \tau, \omega_2). \tag{5.14}$$

Then the following hold,

$$\text{if} \quad F(\chi_1, \tau, \omega_1) \le F(\chi_2, \tau, \omega_2) \quad \text{then} \quad \chi_1 \le \chi_2 \quad \text{or} \quad \omega_1 \le \omega_2, \tag{5.15}$$

$$\text{if} \quad F(\chi_1, \tau, \omega_1) < F(\chi_2, \tau, \omega_2) \quad \text{then} \quad \chi_1 < \chi_2 \quad \text{or} \quad \omega_1 < \omega_2. \tag{5.16}$$

Proof. We prove only (5.15). The proof of (5.16) is analogous. Let us suppose by contradiction that

$$\chi_1 > \chi_2 \quad \text{and} \quad \omega_1 > \omega_2.$$

Then we have

$$F(\chi_2, \tau, \omega_2) \overset{(5.13)}{\le} F(\chi_1, \tau, \omega_2) \overset{(5.14)}{<} F(\chi_1, \tau, \omega_1),$$

a contradiction with the assumption of $F(\chi_1, \tau, \omega_1) \le F(\chi_2, \tau, \omega_2)$. □

Now we prove fundamental assertions of this section, the maximum and minimum principles.

Theorem 5.14. *Let $u(x,t)$ be a lower solution of* (5.9) *where $F(\chi,\tau,\omega)$ satisfies assumptions* (F1) *and* (F2) *and $f(\chi,\tau)=0$ identically. Then*

$$u(x,t) \quad \leq \sup_{\substack{x\in\mathbb{Z}, x\geq x_b \\ t\in[0,+\infty)}} \{\phi(x),\xi(t)\}\,.$$

holds for all $x\in\mathbb{Z}$, $x\geq x_b$ and for all $t\in[0,+\infty)$.

Proof. First, we denote

$$M = \sup_{\substack{x\in\mathbb{Z}, x\geq x_b \\ t\in[0,+\infty)}} \{\phi(x),\xi(t)\} \tag{5.17}$$

for the sake of brevity. We prove the theorem by contradiction. Let us assume that there exist $x_0\in\mathbb{Z}$, $x_0\geq x_b$ and $t_0\in[0,+\infty)$ such that

$$u(x_0,t_0) > M. \tag{5.18}$$

If $x_0 = x_b$ or $t_0 = 0$ then we have the contradiction with the definition (5.17) of M. Therefore, we suppose that $x_0 > x_b$ and $t_0 > 0$.

Now we use the definition of lower solution and assumptions (5.18) and the fact that $F(\chi,\tau,\omega)$ is strictly increasing in ω

$$\begin{aligned} u_t(x_0,t_0) + \nabla_x F(x_0,t_0,u(x_0,t_0)) &\leq 0, \\ u_t(x_0,t_0) + \underbrace{F(x_0,t_0,u(x_0,t_0))}_{>F(x_0,t_0,M)} &\leq F(x_0-1,t_0,u(x_0-1,t_0)) \end{aligned} \tag{5.19}$$

and thus,

$$u_t(x_0,t_0) < F(x_0-1,t_0,u(x_0-1,t_0)) - F(x_0,t_0,M). \tag{5.20}$$

Now there are two possibilities.

1. If there is

$$F(x_0-1,t_0,u(x_0-1,t_0)) > F(x_0,t_0,M)$$

then from (5.16) in Lemma 5.13 and from the assumptions (F1) and (F2) we get

$$x_0 - 1 > x_0 \quad \text{or} \quad u(x_0-1,t_0) > M.$$

The first case does not occur. Then the second one,

$$u(x_0 - 1, t_0) > M,$$

holds and we define

$$x_1 = x_0 - 1 \quad \text{and} \quad t_1 = t_0.$$

2. The second possibility is the case of

$$F(x_0 - 1, t_0, u(x_0 - 1, t_0)) \leq F(x_0, t_0, M).$$

From (5.20) there is

$$u_t(x_0, t_0) < 0.$$

Therefore, the function $u(x_0, t)$ is strictly decreasing in $t = t_0$ and we can define

$$\overline{t_0} = \inf \{\tau = [0, t_0] : \quad u(x_0, t) \text{ is strictly decreasing on the interval } (\tau, t_0)\}.$$

We know that

$$u(x_0, \overline{t_0}) > u(x_0, t_0) > M. \tag{5.21}$$

We have two possibilities again. If $\overline{t_0} = 0$ then (5.21) leads to the contradiction with the definition (5.17) of M via the initial condition $\phi(x)$. If $\overline{t_0} > 0$ then there is necessarily

$$u_t(x_0, \overline{t_0}) = 0 \tag{5.22}$$

and from (5.19) we get

$$F(x_0, \overline{t_0}, u(x_0, \overline{t_0})) \leq F(x_0 - 1, \overline{t_0}, u(x_0 - 1, \overline{t_0})).$$

Then the assertion (5.15) of Lemma 5.13 implies

$$x_0 \leq x_0 - 1 \quad \text{or} \quad u(x_0, \overline{t_0}) \leq u(x_0 - 1, \overline{t_0}).$$

The first inequality does not occur again and hence, we have

$$u(x_0, \overline{t_0}) \leq u(x_0 - 1, \overline{t_0}). \tag{5.23}$$

Inequalities (5.21) and (5.23) give us

$$M < u(x_0, t_0) < u(x_0, \overline{t_0}) \leq u(x_0 - 1, \overline{t_0}).$$

Consequently, in this case we define

$$x_1 = x_0 - 1 \quad \text{and} \quad t_1 = \overline{t_0}.$$

We have $u(x_1, t_1) > M$. We continue iteratively in the same way. We observe that after at most $x_0 - x_b$ steps we get the contradiction with the definition (5.17) of M. □

REMARK 5.15. *The maximum principle holds even for a more general problem*

$$\begin{cases} u_t(x,t) + \nabla_x F(x,t,u(x,t)) + G(x,t,u(x,t)) \leq f(x,t), \\ \\ u(x,0) \leq \phi(x), \\ u(x_b,t) \leq \xi(t) \end{cases}$$

when we consider the same assumptions as in Theorem 5.14 and moreover, $G : \mathbb{R}^3 \to \mathbb{R}$ *and* $f : \mathbb{R}^2 \to \mathbb{R}$ *satisfy*

$$\begin{aligned} G(\chi,\tau,\omega) &\geq 0, \\ f(\chi,\tau) &\leq 0 \end{aligned}$$

for all $\chi \in \mathbb{Z}$, $\chi \geq x_b$, $\tau \in [0,+\infty)$, $\omega \in \mathbb{R}$. *This can be proved by a simple application of Theorem 5.14.*

The minimum principle follows. We can prove it in the same way as the maximum principle in Theorem 5.14.

THEOREM 5.16. *Let* $u(x,t)$ *be an upper solution of* (5.9) *where* $F(\chi,\tau,\omega)$ *satisfies*

(F3) $F(\chi,\tau,\omega)$ *is decreasing in* χ,

(F4) $F(\chi,\tau,\omega)$ *is strictly increasing in* ω,

and $f(\chi,\tau) = 0$ *identically. Then*

$$\inf_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{\phi(x), \xi(t)\} \quad \leq \quad u(x,t)$$

holds for all $x \in \mathbb{Z}$, $x \geq x_b$ *and for all* $t \in [0,+\infty)$.

Proof. The statement can be proved analogously as Theorem 5.14. □

REMARK 5.17. *Also the minimum principle holds even for a more general problem*

$$\begin{cases} u_t(x,t) + \nabla_x F(x,t,u(x,t)) + G(x,t,u(x,t)) \geq f(x,t), \\ \\ u(x,0) \geq \phi(x), \\ u(x_b,t) \geq \xi(t) \end{cases}$$

when we consider the same assumptions as in Theorem 5.16 and moreover, $G : \mathbb{R}^3 \to \mathbb{R}$ *and* $f : \mathbb{R}^2 \to \mathbb{R}$ *satisfy*

$$G(\chi,\tau,\omega) \leq 0,$$
$$f(\chi,\tau) \geq 0$$

for all $\chi \in \mathbb{Z}$, $\chi \geq x_b$, $\tau \in [0,+\infty)$, $\omega \in \mathbb{R}$. We can prove it by a simple application of Theorem 5.16.

Assumptions on $G(\chi,\tau,\omega)$ and $f(\chi,\tau)$ from Remark 5.15 and Remark 5.17 can be satisfied together, i.e. maximum and minimum principles can hold together, only if $G(\chi,\tau,\omega) = 0$ and $f(\chi,\tau) = 0$ identically.

5.4 Applications of Maximum and Minimum Principles

In next paragraphs we study basic applications of maximum and minimum principles presented in Theorems 5.14 and 5.16.

5.4.1 Sign Preservation

PROPOSITION 5.18. *Let $u(x,t)$ be an upper solution of* (5.9) *where $F(\chi,\tau,\omega)$ satisfies assumptions* (F3) *and* (F4)*, $f(\chi,\tau) = 0$ identically and*

$$\phi(x) \geq 0,$$
$$\xi(t) \geq 0.$$

Then there is

$$u(x,t) \geq 0$$

for all $x > x_b$, $x \in \mathbb{Z}$, and for all $t \in [0,+\infty)$.

Proof. The statement is the direct consequence of Theorem 5.16. □

5.4.2 Boundedness of Solution

We want to use the maximum and minimum principles together. Hence, we have to satisfy the assumptions of Theorem 5.14 and Theorem 5.16. The difference between these assumptions is in the monotonicity of the function $F(\chi,\tau,\omega)$ in the variable χ and in inequalities in the definition of lower and upper solution. Therefore, we consider the following problem

$$\text{(5.24)} \qquad \begin{cases} u_t(x,t) + \nabla_x F(t,u(x,t)) = 0, \quad x \in \mathbb{Z}, \quad x > x_b, \quad t \in (0,+\infty), \\ \\ u(x,0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \quad x \in \mathbb{Z}, \quad x > x_b, \\ u(x_b,t) = \xi(t), \quad \xi : [0,+\infty) \to \mathbb{R}, \end{cases}$$

for $F : \mathbb{R}^2 \to \mathbb{R}$.

Now we can state the result about the boundedness of solution of (5.24).

THEOREM 5.19. *Let $u(x,t)$ be a solution of* (5.24) *when the function $F(\tau,\omega)$ is strictly increasing in the second variable and initial-boundary conditions $\phi(x)$ and $\xi(t)$ are bounded. Then $u(x,t)$ is also bounded and for all $x \in \mathbb{Z}$, $x \geq x_b$ and for all $t \in [0,+\infty)$ there is*

$$\inf_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{\phi(x), \xi(t)\} \quad \leq \quad u(x,t) \quad \leq \quad \sup_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{\phi(x), \xi(t)\} .$$

Proof. The statement is the direct consequence of Theorem 5.14 and Theorem 5.16. □

EXAMPLE 5.20. Let us consider the function

$$F(\chi,\tau,\omega) = \arctan\omega$$

and let

$$\phi(x) \geq 0 \text{ be bounded,}$$
$$\xi(t) \geq 0 \text{ be bounded.}$$

Hence, $F(\chi,\tau,\omega)$ is continuous, bounded, strictly increasing in ω and satisfies the uniform Lipschitz condition with respect to ω.

Then the assumptions of Theorem 5.10, Proposition 5.18 and Theorem 5.19 are satisfied. Thus, there exists a unique global solution $u(x,t)$ of (5.9). Furthermore, $u(x,t)$ is nonnegative and bounded. ■

EXAMPLE 5.21. Consider the function

$$F(\chi,\tau,\omega) = \overline{F}(\omega) = \begin{cases} -1, & \omega < -1, \\ -\sqrt[3]{-\omega}, & -1 \leq \omega < 0, \\ \sqrt[3]{\omega}, & 0 \leq \omega < 1, \\ 1, & 1 < \omega, \end{cases}$$

and let

$$\phi(x) \geq 0 \text{ be bounded,}$$
$$\xi(t) \geq 0 \text{ be bounded.}$$

In this case $F(\chi,\tau,\omega)$ is continuous, bounded, strictly increasing in ω but it does not satisfy the Lipschitz condition. Thus, from Theorem 5.9 there exists a global solution $u(x,t)$ of (5.9) which is nonnegative and bounded but we lose the guarantee of uniqueness. ■

5.4.3 Approximation of Solution

In this paragraph we consider the initial-boundary value problem (5.12). We investigate the possibility of approximation of solution. For this case we have to derive special maximum and minimum principles. Let us suppose the following nonlinear initial-boundary value problem

$$
(5.25) \qquad \begin{cases} u_t(x,t) + k(x,t)\nabla_x F(u(x,t)) = f(x,t), \quad x \in \mathbb{Z}, \quad x > x_b, \quad t \in (0,+\infty)\,, \\ \\ u(x,0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \quad x \in \mathbb{Z}, \quad x > x_b, \\ u(x_b,t) = \xi(t), \quad \xi : [0,+\infty) \to \mathbb{R}, \end{cases}
$$

when $k, f : \mathbb{R}^2 \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$. Primarily we state the maximum and minimum principles for (5.25) analogously as above.

THEOREM 5.22. *Consider the problem* (5.25) *where $F(\omega)$ is increasing on $\mathbb{R}$, $f(\chi,\tau) = 0$ identically and*

$$
k(\chi,\tau) \geq 0.
$$

1. *If $u(x,t)$ is a lower solution then*

$$
u(x,t) \quad \leq \sup_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{\phi(x), \xi(t)\}\,.
$$

for all $x \in \mathbb{Z}$, $x \geq x_b$ and for all $t \in [0,+\infty)$.

2. *If $u(x,t)$ is an upper solution then*

$$
\inf_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{\phi(x), \xi(t)\} \quad \leq \quad u(x,t)
$$

for all $x \in \mathbb{Z}$, $x \geq x_b$ and for all $t \in [0,+\infty)$.

REMARK 5.23. *The lower and upper solutions are defined analogously as in Definition 5.12.*

Proof. The statement can be proved analogously as Theorems 5.14 and 5.16. □

And now the approximation theorem follows.

THEOREM 5.24. *Let $u(x,t)$ be a classical solution of* (5.12) *where the partial derivative $F_\omega(\chi,\tau,\omega)$ is continuous and*

$$
F_\omega(\chi,\tau,\omega) \geq 0.
$$

Let $v(x,t)$ be a lower solution and $w(x,t)$ be an upper solution. Then

$$
v(x,t) \leq u(x,t) \leq w(x,t)
$$

is satisfied for all $x \in \mathbb{Z}$, $x \geq x_b$ and for all $t \in [0,+\infty)$.

Proof. We prove the statement with help of the minimum principle from Theorem 5.22. We define two auxiliary functions

$$\overline{v}(x,t) = u(x,t) - v(x,t) \quad \text{and} \quad \overline{w}(x,t) = w(x,t) - u(x,t)$$

and we investigate their sign.

1. First, we study the function $\overline{v}(x,t)$. Because $v(x,t)$ is the lower solution we get for all $x \in \mathbb{Z}$, $x \geq x_b$ and for all $t \in [0,+\infty)$

$$0 \leq u_t(x,t) + F(x,t,\nabla_x u(x,t)) - v_t(x,t) - F(x,t,\nabla_x v(x,t)).$$

Thanks to the assumptions on F we can use the mean value theorem and we can continue with our estimate,

$$\begin{aligned} 0 &\leq u_t(x,t) + F(x,t,\nabla_x u(x,t)) - v_t(x,t) - F(x,t,\nabla_x v(x,t)) \\ &= (u(x,t) - v(x,t))_t + F_\omega(x,t,\theta)\left[\nabla_x u(x,t) - \nabla_x v(x,t)\right] \\ &= \overline{v}_t(x,t) + F_\omega(x,t,\theta)\nabla_x \overline{v}(x,t). \end{aligned}$$

For initial and boundary conditions we have

$$\overline{v}(x,0) = \underbrace{u(x,0)}_{=\phi(x)} - \underbrace{v(x,0)}_{\leq\phi(x)} \geq 0,$$

$$\overline{v}(x_b,t) = \underbrace{u(x_b,t)}_{=\xi(t)} - \underbrace{v(x_b,t)}_{\leq\xi(t)} \geq 0.$$

Thus, $\overline{v}(x,t)$ satisfies assumptions of Theorem 5.22 which implies

$$\overline{v}(x,t) \geq 0, \quad \text{i.e.} \quad v(x,t) \leq u(x,t).$$

2. For the function $\overline{w}(x,t)$ it is similar. The function $w(x,t)$ is the upper solution and therefore, for all $x \in \mathbb{Z}$, $x \geq x_b$ and for all $t \in [0,+\infty)$ we have

$$\begin{aligned} 0 &\leq w_t(x,t) + F(x,t,\nabla_x w(x,t)) - u_t(x,t) - F(x,t,\nabla_x u(x,t)) \\ &= (w(x,t) - u(x,t))_t + F_\omega(x,t,\theta)\left[\nabla_x w(x,t) - \nabla_x u(x,t)\right] \\ &= \overline{w}_t(x,t) + F_\omega(x,t,\theta)\nabla_x \overline{w}(x,t). \end{aligned}$$

Initial and boundary conditions are

$$\overline{w}(x,0) = \underbrace{w(x,0)}_{\geq\phi(x)} - \underbrace{u(x,0)}_{=\phi(x)} \geq 0,$$

$$\overline{w}(x_b,t) = \underbrace{w(x_b,t)}_{\geq\xi(t)} - \underbrace{u(x_b,t)}_{=\xi(t)} \geq 0.$$

Theorem 5.22 gives

$$\overline{w}(x,t) \geq 0, \quad \text{i.e.} \quad u(x,t) \leq w(x,t).$$

Hence, we get

$$v(x,t) \leq u(x,t) \leq w(x,t)$$

for all $x \in \mathbb{Z}$, $x \geq x_b$ and for all $t \in [0,+\infty)$. □

REMARK 5.25. *If we assume that for all $\chi \in \mathbb{Z}$, $\chi \geq x_b$, and for all $\tau \in [0,+\infty)$ there exits $L = L(\chi,\tau) > 0$ such that for all ω_1, $\omega_2 \in \mathbb{R}$ there is*

$$F(\chi,\tau,\omega_1) - F(\chi,\tau,\omega_2) \leq L(\omega_1 - \omega_2)$$

instead of assumptions on F in Theorem 5.24 then the statement also holds.

5.4.4 Uniqueness of Solution

THEOREM 5.26. *Consider the problem* (5.12) *where $F(\chi,\tau,\omega)$ is continuous, $F_\omega(\chi,\tau,\omega)$ is continuous and*

$$F_\omega(\chi,\tau,\omega) \geq 0.$$

Assume that for all $\chi \in \mathbb{Z}$, $\chi \geq x_b$, there exist $K = K(\chi) > 0$ and $L = L(\chi) > 0$ such that for all $\tau \in [0,+\infty)$ and $\omega \in \mathbb{R}$ there is

$$\begin{aligned} |F(\chi,\tau,\omega)| &\leq K(\chi), \\ |f(\chi,\tau)| &\leq L(\chi). \end{aligned}$$

Then (5.12) *possesses a unique classical solution.*

Proof. The existence of a solution is the direct consequence of Theorem 5.11. It remains to prove the uniqueness.

Let us assume there exist two distinct solutions $u_1(x,t)$ and $u_2(x,t)$. Then $u_1(x,t)$ and $u_2(x,t)$ are both also lower and upper solutions. Then from Theorem 5.24 we get

$$u_1(x,t) \leq u_2(x,t) \quad \text{and} \quad u_2(x,t) \leq u_1(x,t)$$

and consequently, there is $u_1(x,t) = u_2(x,t)$ which is the contradiction. □

We illustrate Theorems 5.24 and 5.26 with a simple example.

Example 5.27. Let us consider the continuous function

$$F(\chi, \tau, \omega) = \arctan \omega.$$

again. Then there is

$$F_\omega(\chi, \tau, \omega) = \frac{1}{1+\omega^2} \geq 0$$

and continuous. Therefore, $F(\chi, \tau, \omega)$ satisfies assumptions of Theorems 5.24 and 5.26 and then there exists a unique solution of (5.12). ■

5.4.5 Approximation and Uniqueness of Solution for Linear Equation

If we use the maximum principle from Theorem 5.14 we can also prove approximation and uniqueness results for the following linear initial-boundary value problem

(5.26)

$$\begin{cases} u_t(x,t) + \nabla_x \left[k(x,t)u(x,t)\right] = f(x,t), \quad x \in \mathbb{Z}, \quad x > x_b, \quad t \in (0,+\infty), \quad k, f : \mathbb{R}^2 \to \mathbb{R}, \\ \\ u(x,0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \quad x \in \mathbb{Z}, \quad x > x_b, \\ u(x_b,t) = \xi(t), \quad \xi : [0,+\infty) \to \mathbb{R}. \end{cases}$$

Theorem 5.28. *Let $k(\chi, \tau)$ be monotone in the variable χ. Let $u(x,t)$ be a classical solution of* (5.26) *and let $v(x,t)$ be a lower solution and $w(x,t)$ an upper solution. Then*

$$v(x,t) \leq u(x,t) \leq w(x,t)$$

holds for all $x \in \mathbb{Z}$, $x \geq x_b$ and for all $t \in [0,+\infty)$.

Proof. We can prove the statement analogously as Theorem 5.24. □

Theorem 5.29. *Let $k(\chi, \tau)$ be monotone in χ. Then the problem* (5.26) *possesses at most one classical solution.*

Proof. We need to prove only the uniqueness. We can prove it analogously as Theorem 5.26. □

5.4.6 Uniform Stability for Linear Equation

We suppose in this paragraph the following linear initial-boundary value problem

$$\begin{cases} u_t(x,t) + \nabla_x \left[c(t)u(x,t)\right] = 0, \quad x \in \mathbb{Z}, \quad x > x_b, \quad t \in (0,+\infty), \quad c : \mathbb{R} \to \mathbb{R}, \\ \\ u(x,0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \quad x \in \mathbb{Z}, \quad x > x_b, \\ u(x_b,t) = \xi(t), \quad \xi : [0,+\infty) \to \mathbb{R}. \end{cases} \tag{5.27}$$

We ask what happens with the solution $u(x,t)$ if we change the initial-boundary conditions $\phi(x)$ and $\xi(t)$.

THEOREM 5.30. *Let $u_1(x,t)$ be a classical solution of* (5.27) *with the initial condition $\phi_1(x)$ and with the boundary condition $\xi_1(t)$. Let $u_2(x,t)$ be a classical solution of* (5.27) *with the initial condition $\phi_2(x)$ and with the boundary condition $\xi_2(t)$. Then*

$$\sup_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} |u_1(x,t) - u_2(x,t)| \quad \leq \quad \sup_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\} \tag{5.28}$$

holds.

Proof. Let us suppose the function $v(x,t)$ defined by

$$v(x,t) = u_1(x,t) - u_2(x,t).$$

Then $v(x,t)$ solves the problem (5.27) with the initial condition $\phi_1(x) - \phi_2(x)$ and with the boundary condition $\xi_1(t) - \xi_2(t)$.

Assumptions of the maximum principle in Theorem 5.14 are satisfied and hence, we get

$$\begin{aligned} u_1(x,t) - u_2(x,t) &= v(x,t) \leq \sup_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{\phi_1(x) - \phi_2(x), \xi_1(t) - \xi_2(t)\} \\ &\leq \sup_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\} . \end{aligned} \tag{5.29}$$

Similarly, assumptions of the minimum principle in Theorem 5.16 are satisfied. Therefore, there is

$$\begin{aligned} u_1(x,t) - u_2(x,t) &= v(x,t) \geq \inf_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{\phi_1(x) - \phi_2(x), \xi_1(t) - \xi_2(t)\} \\ &\geq - \sup_{\substack{x \in \mathbb{Z}, x \geq x_b \\ t \in [0,+\infty)}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\} . \end{aligned} \tag{5.30}$$

Inequalities in (5.29) and (5.30) give us (5.28). □

COROLLARY 5.31. *Let $\{u_n(x,t)\}_{n=1}^{+\infty}$ be a sequence of solutions $u_n(x,t)$ of* (5.27) *with the initial condition $\phi_n(x)$ and with the boundary condition $\xi_n(t)$. Assume that*

$$\phi_n(x) \to \phi(x) \text{ for } x \in \mathbb{Z}, x \geq x_b \quad \text{and} \quad \xi_n(t) \rightrightarrows \xi(t) \text{ for } t \in [0,+\infty)$$

and assume that $u(x,t)$ is a solution of (5.27) *with the initial condition $\phi(x)$ and with the boundary condition $\xi(t)$. Then*

$$u_n(x,t) \rightrightarrows u(x,t) \quad \text{for} \quad x \in \mathbb{Z}, x \geq x_b \quad \text{and} \quad t \in [0,+\infty) .$$

Proof. The statement follows directly from Theorem 5.30. □

5.5 Existence and Uniqueness for Initial Value Problem

In previous subsections we solved initial-boundary value problems with the left-boundary condition. In this paragraph we deal with the initial value problem

$$\begin{cases} u_t(x,t) + \nabla_x F(t, u(x,t)) = 0, \quad x \in \mathbb{Z}, \quad t \in (0, +\infty), \\ u(x,0) = \phi(x), \quad x \in \mathbb{Z}, \end{cases} \tag{5.31}$$

when $F : \mathbb{R}^2 \to \mathbb{R}$ and $\phi : \mathbb{Z} \to \mathbb{R}$.

We solve the existence and uniqueness of the classical solution of (5.31). We use the contraction principle to prove it.

THEOREM 5.32 ([4], VĚTA 3.1). *Let $\mathcal{M}$ be a complete metric space with a metric d and $T : \mathcal{M} \to \mathcal{M}$ be an operator. If T is a contraction, i.e. there exists $\alpha \in (0,1)$ such that for all $v, w \in \mathcal{M}$*

$$d(T(v), T(w)) \leq \alpha d(v, w)$$

then there exists exactly one $v_0 \in \mathcal{M}$ such that

$$v_0 = T(v_0).$$

REMARK 5.33. *Moreover, one can show that*

$$v_0 = \lim_{n \to +\infty} v_n$$

when the sequence $\{v_n\}_{n=1}^{+\infty}$ is given recursively

$$v_{n+1} = T(v_n),$$

$v_1 \in \mathcal{M}$ is arbitrary. The point $v_0 \in \mathcal{M}$ is called fixed point of T.

First, we have to define our operator for the application of the contraction principle. Hence, let us consider the equation in (5.31)

$$u_t(x,t) + \nabla_x F(t, u(x,t)) = 0.$$

We integrate this equation with respect to the second variable

$$u(x,t) - \underbrace{u(x,0)}_{=\phi(x)} + \int_0^t \nabla_x F(\tau, u(x,\tau)) \mathrm{d}\tau = 0.$$

After the application of the initial condition we have the following fixed point problem

$$u(x,t) = \phi(x) - \int_0^t \nabla_x F(\tau, u(x,\tau)) \mathrm{d}\tau.$$

It motivates us to define our operator T as follows

$$T(u)(x,t) = \phi(x) - \int_0^t \nabla_x F(\tau, u(x,\tau))\mathrm{d}\tau. \tag{5.32}$$

Second, we have to specify our complete metric space $\mathcal{M}$. Let us denote $D = \mathbb{Z} \times [0,\delta]$ for some $\delta > 0$ and define $\mathcal{M}$ as

$$\begin{aligned} \mathcal{M} = \{v, \quad v : D \to \mathbb{R}, \quad & v_t(x,t) \text{ is continuous in } t \in [0,\delta] \text{ for all } x \in \mathbb{Z}, \\ & v(x,t) \text{ is bounded on } D, \\ & v_t(x,t) \text{ is bounded on } D\}. \end{aligned} \tag{5.33}$$

Now, we have to check that $\mathcal{M}$ is the complete metric space.

Remark 5.34. *We prove even more. We prove that $\mathcal{M}$ is Banach space, i.e. normed linear space which is complete with respect to its norm $\| \cdot \|_{\mathcal{M}}$.*

Lemma 5.35. *Define for all $v \in \mathcal{M}$*

$$\| v \|_{\mathcal{M}} = \sup_{(x,t)\in D} |v(x,t)| + \sup_{(x,t)\in D} |v_t(x,t)|. \tag{5.34}$$

Then $(\mathcal{M}, \| \cdot \|_{\mathcal{M}})$ is a normed linear space.

Remark 5.36. *For the sake of brevity we denote $\| \cdot \|_{\mathcal{M}}$ only by $\| \cdot \|$.*

Proof. The linearity is clear because $\mathcal{M}$ is the set of functions. It remains to prove that $\| \cdot \|$ is the norm. From (5.34) and from the boundedness of $v(x,t)$ and $v_t(x,t)$ on D we directly get that for all $v \in \mathcal{M}$ there is $\| v \| \in [0, +\infty)$. Now we verify properties of norm.

1. If we assume $\| v \| = 0$, i.e.

   $$\sup_{(x,t)\in D} |v(x,t)| + \sup_{(x,t)\in D} |v_t(x,t)| = 0,$$

   then there is necessarily

   $$\sup_{(x,t)\in D} |v(x,t)| = 0 \quad \text{and} \quad \sup_{(x,t)\in D} |v_t(x,t)| = 0$$

   which implies $v(x,t) = 0$ identically. On the other hand, if we assume $v(x,t) = 0$ identically then from (5.34) there is $\| v \| = 0$.

2. Let $\lambda \in \mathbb{R}$ and $v \in \mathcal{M}$ be arbitrary. Then we get

   $$\begin{aligned} \| \lambda v \| &= \sup_{(x,t)\in D} |\lambda v(x,t)| + \sup_{(x,t)\in D} |\lambda v_t(x,t)| \\ &= |\lambda| \sup_{(x,t)\in D} |v(x,t)| + |\lambda| \sup_{(x,t)\in D} |v_t(x,t)| = |\lambda| \, \| v \| . \end{aligned}$$

3. Let $v, w \in \mathcal{M}$ be arbitrary. Then with the help of triangle inequality for $|\cdot|$ and properties of the supremum there is

$$\begin{aligned} \| v + w \| &= \sup_{(x,t)\in D} |v(x,t) + w(x,t)| + \sup_{(x,t)\in D} |v_t(x,t) + w_t(x,t)| \\ &\leq \sup_{(x,t)\in D} (|v(x,t)| + |w(x,t)|) + \sup_{(x,t)\in D} (|v_t(x,t)| + |w_t(x,t)|) \\ &\leq \sup_{(x,t)\in D} |v(x,t)| + \sup_{(x,t)\in D} |w(x,t)| + \sup_{(x,t)\in D} |v_t(x,t)| + \sup_{(x,t)\in D} |w_t(x,t)| \\ &= \| v \| + \| w \| . \end{aligned}$$

Therefore, $\|\cdot\|$ is a norm and $(\mathcal{M}, \|\cdot\|)$ is a normed linear space. □

Now we deal with the completeness of $\mathcal{M}$. For this study we need some auxiliary assertions.

PROPOSITION 5.37 ([12], VĚTA 1.4). *The function sequence $\{f_n(t)\}_{n=1}^{+\infty}$ converges uniformly on M to the limit function $f(t)$ if and only if for all $\epsilon > 0$ there exits $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ and for all $t \in M$ there is*

$$|f_m(t) - f_n(t)| < \epsilon.$$

REMARK 5.38. *Proposition 5.37 is often called Bolzano–Cauchy criterion of the uniform convergence.*

PROPOSITION 5.39 ([6], VĚTA 17.1). *Let $\{f_n(t)\}_{n=1}^{+\infty}$ be a sequence of continuous functions on arbitrary interval $I \subseteq \mathbb{R}$ that converges uniformly on I to the limit function $f(t)$. Then $f(t)$ is continuous on I.*

PROPOSITION 5.40 ([6], VĚTA 17.1). *Let $\{f_n(t)\}_{n=1}^{+\infty}$ be a sequence of differentiable functions on an arbitrary closed interval $I = [a, b] \subseteq \mathbb{R}$ that converges on I to the limit function $f(t)$. If the function sequence $\{f_n'(t)\}_{n=1}^{+\infty}$ converges uniformly on I then the limit function $f(t)$ is differentiable and the following holds*

$$f'(t) = \left[\lim_{n\to+\infty} f_n(t)\right]' = \lim_{n\to+\infty} f_n'(t).$$

Now we are prepared to prove the completeness of $\mathcal{M}$.

LEMMA 5.41. *The normed linear space $(\mathcal{M}, \|\cdot\|)$ is complete with respect to its norm.*

Proof. We have to prove that every Cauchy sequence $\{v_n\}_{n=1}^{+\infty}$ (we denote it only by $\{v_n\}$ in the following text) in $(\mathcal{M}, \|\cdot\|)$ is convergent in $(\mathcal{M}, \|\cdot\|)$.

Let $\{v_n\} \subseteq \mathcal{M}$ be a Cauchy sequence. Then for all $\epsilon > 0$ there exits $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ there is

$$\| v_m - v_n \| < \epsilon,$$

i.e.

$$\sup_{(x,t)\in D} |v_m(x,t) - v_n(x,t)| + \sup_{(x,t)\in D} |(v_m)_t(x,t) - (v_n)_t(x,t)| < \epsilon. \tag{5.35}$$

The relation (5.35) implies that for all $\epsilon > 0$ there exits $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ the following holds

$$\sup_{(x,t)\in D} |v_m(x,t) - v_n(x,t)| < \epsilon, \tag{5.36}$$

$$\sup_{(x,t)\in D} |(v_m)_t(x,t) - (v_n)_t(x,t)| < \epsilon. \tag{5.37}$$

Finally, supremums in (5.36) and (5.37) give us that for all $\epsilon > 0$ there exits $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ and for all $(x,t) \in D$ there is

$$|v_m(x,t) - v_n(x,t)| < \epsilon, \tag{5.38}$$

$$|(v_m)_t(x,t) - (v_n)_t(x,t)| < \epsilon. \tag{5.39}$$

Now from Bolzano–Cauchy criterion of the uniform convergence (Proposition 5.37) we get that for all fixed $x \in \mathbb{Z}$ both function sequences $\{v_n(x,t)\}$ and $\{(v_n)_t(x,t)\}$ converge uniformly to their limit functions, $t \in [0, \delta]$, i.e.

$$v_n(x,t) \rightrightarrows v(x,t), \quad v(x,t) = \lim_{n\to+\infty} v_n(x,t),$$

$$(v_n)_t(x,t) \rightrightarrows w(x,t), \quad w(x,t) = \lim_{n\to+\infty} (v_n)_t(x,t).$$

From Proposition 5.40 we directly get that there is $w(x,t) = v_t(x,t)$, i.e.

$$(v_n)_t(x,t) \rightrightarrows v_t(x,t), \quad t \in [0, \delta]. \tag{5.40}$$

Now we make the limit transition for $m \to +\infty$ in relations (5.38) and (5.39) and then for all $\epsilon > 0$ there exits $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and for all $(x,t) \in D$ there is

$$|v_n(x,t) - v(x,t)| \le \epsilon, \tag{5.41}$$

$$|(v_n)_t(x,t) - v_t(x,t)| \le \epsilon. \tag{5.42}$$

At this moment it suffices to verify that $v : D \to \mathbb{R}$ satisfies $v \in \mathcal{M}$ and that $\{v_n\} \subseteq \mathcal{M}$ converges to v in the norm $\| \cdot \|$.

1. Functions $(v_n)_t$ are continuous in the variable t on $[0,\delta]$ because $v_n \in \mathcal{M}$. From (5.40) and from Proposition 5.39 we immediately have that the limit function v_t is also continuous on $[0,\delta]$.

   Let $\epsilon > 0$ be arbitrary and fixed. Then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and for all $(x,t) \in D$ there is

$$\begin{aligned} |v(x,t)| &= |v(x,t) - v_{n_0+1}(x,t) + v_{n_0+1}(x,t)| \\ &\leq \underbrace{|v(x,t) - v_{n_0+1}(x,t)|}_{\overset{(5.41)}{\leq \epsilon}} + \underbrace{|v_{n_0+1}(x,t)|}_{\leq K_{n_0+1}} \leq \epsilon + K_{n_0+1} \end{aligned}$$

   because $v_{n_0+1} \in \mathcal{M}$, i.e. $v_{n_0+1}(x,t)$ is bounded on D. Therefore, $v(x,t)$ is also bounded on D.

   Analogously we get

$$\begin{aligned} |v_t(x,t)| &= |v_t(x,t) - (v_{n_0+1})_t(x,t) + (v_{n_0+1})_t(x,t)| \\ &\leq \underbrace{|v_t(x,t) - (v_{n_0+1})_t(x,t)|}_{\overset{(5.42)}{\leq \epsilon}} + \underbrace{|(v_{n_0+1})_t(x,t)|}_{\leq \tilde{K}_{n_0+1}} \leq \epsilon + \tilde{K}_{n_0+1}. \end{aligned}$$

   Thus, $v_t(x,t)$ is also bounded on D and hence, $v \in \mathcal{M}$.

2. It remains to prove that $\{v_n\}$ converges to $v \in \mathcal{M}$ in the norm $\| \cdot \|$. But it follows directly from (5.41) and (5.42) that for all $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n > n_0$ there is

$$\begin{aligned} \| v_n - v \| &= \underbrace{\sup_{(x,t)\in D} |v_n(x,t) - v(x,t)|}_{\overset{(5.41)}{\leq \epsilon}} + \underbrace{\sup_{(x,t)\in D} |(v_n)_t(x,t) - v_t(x,t)|}_{\overset{(5.42)}{\leq \epsilon}} \\ &\leq 2\epsilon \end{aligned}$$

   which implies that $\{v_n\}$ converges to $v \in \mathcal{M}$ in the space $(\mathcal{M}, \| \cdot \|)$.

We have shown that every Cauchy sequence in $(\mathcal{M}, \| \cdot \|)$ is convergent in $(\mathcal{M}, \| \cdot \|)$ and consequently, the space $(\mathcal{M}, \| \cdot \|)$ is complete. □

THEOREM 5.42. *Space $(\mathcal{M}, \| \cdot \|)$ is a Banach space.*

Proof. The statement is the direct consequence of Lemma 5.35 and Lemma 5.41. □

Now we have our complete space and we can go back to our operator T defined by (5.32). We have to prove that T maps into $\mathcal{M}$ and that T is the contraction. We have to place some additional assumptions on functions ϕ and F from (5.31).

Lemma 5.43. *Assume that*

(A1) *the function* $\phi : \mathbb{Z} \to \mathbb{R}$ *is bounded, i.e. there exits* $K_1 \geq 0$ *such that for all* $x \in \mathbb{Z}$ *there is* $|\phi(x)| \leq K_1$,

(A2) *the function* $F : \mathbb{R}^2 \to \mathbb{R}$ *is continuous on* $[0, \delta] \times \mathbb{R}$,

(A3) *the function* $F : \mathbb{R}^2 \to \mathbb{R}$ *satisfies the Lipschitz condition in the second variable, i.e. there exists* $L > 0$ *such that for all* $\tau \in [0, \delta]$ *and for all* $\omega_1, \omega_2 \in \mathbb{R}$ *there is* $|F(\tau, \omega_1) - F(\tau, \omega_2)| \leq L|\omega_1 - \omega_2|$.

Then for all $v \in \mathcal{M}$ *there is* $T(v) \in \mathcal{M}$.

Proof. Let $v \in \mathcal{M}$ be arbitrary. First, we observe that $v(x, t)$ is bounded on D, i.e. there exists $K_2 \geq 0$ such that for all $(x, t) \in D$ there is

$$|v(x,t)| \leq K_2. \tag{5.43}$$

Now we check that $T(v) \in \mathcal{M}$ holds.

1. Let us suppose the partial derivative in t of the function $T(v)(x, t)$. Then we have

$$\begin{aligned} T(v)_t(x,t) &= \frac{\partial}{\partial t}\left(\phi(x) - \int_0^t \nabla_x F(\tau, v(x,\tau))\mathrm{d}\tau\right) \\ &= -F(t, v(x,t)) + F(t, v(x-1,t)) \end{aligned} \tag{5.44}$$

   for all $x \in \mathbb{Z}$. The assumption (A2) implies that $T(v)_t(x, \cdot)$ is continuous on $[0, \delta]$ because it is composition of continuous functions v and F (see e.g. Drábek, Míka [5]).

2. For all $(x, t) \in D$ the following holds

$$
\begin{aligned}
|T(v)(x,t)| &= \left| \phi(x) - \int_0^t \nabla_x F(\tau, v(x,\tau))\mathrm{d}\tau \right| \\
&\leq |\phi(x)| + \left| \int_0^t \nabla_x F(\tau, v(x,\tau))\mathrm{d}\tau \right| \\
&\leq |\phi(x)| + \int_0^t |F(\tau, v(x,\tau)) - F(\tau, v(x-1,\tau))| \,\mathrm{d}\tau \\
&\overset{(A3)}{\leq} |\phi(x)| + \int_0^t L\,|v(x,\tau) - v(x-1,\tau)| \,\mathrm{d}\tau \\
&\leq |\phi(x)| + L\int_0^t |v(x,\tau)| \,\mathrm{d}\tau + L\int_0^t |v(x-1,\tau)| \,\mathrm{d}\tau \\
&\overset{(A1),(5.43)}{\leq} K_1 + 2LK_2 \int_0^t \mathrm{d}\tau = K_1 + 2tLK_2 \leq K_1 + 2\delta LK_2.
\end{aligned}
$$

Therefore, the function $T(v)(x,t)$ is bounded on D.

3. For all $(x,t) \in D$ the following holds

$$
\begin{aligned}
|T(v)_t(x,t)| &\overset{(5.44)}{=} |F(t, v(x-1,t)) - F(t, v(x,t))| \\
&\overset{(A3)}{\leq} L\,|v(x-1,t) - v(x,t)| \\
&\leq L\,|v(x-1,t)| + L\,|v(x,t)| \\
&\overset{(5.43)}{\leq} 2LK_2,
\end{aligned}
$$

i.e. the function $T(v)_t(x,t)$ is also bounded on D.

Consequently, for all $v \in \mathcal{M}$ there is $T(v) \in \mathcal{M}$. □

THEOREM 5.44. *Let the assumptions* (A1), (A2), (A3) *be satisfied. Then for all* $v, w \in \mathcal{M}$ *the following holds*

$$
\| T(v) - T(w) \| \leq 2L(1+\delta) \| v - w \| .
$$

REMARK 5.45. *The statement implies when we can choose* $\delta > 0$ *so small that the constant* $2L(1+\delta) < 1$, *i.e.* $0 < \delta < \frac{1}{2L} - 1$, *that* T *is the contraction. It is easy to see that we can put* $0 < \delta < \frac{1}{2L} - 1$ *only if* $L < \frac{1}{2}$.

Proof. Let $v, w \in \mathcal{M}$ be arbitrary. Then the following estimate holds

$$\begin{aligned}
\| T(v) - T(w) \| &= \sup_{(x,t)\in D} |T(v)(x,t) - T(w)(x,t)| + \sup_{(x,t)\in D} |T(v)_t(x,t) - T(w)_t(x,t)| \\
&\overset{(5.32),(5.44)}{=} \sup_{(x,t)\in D} \left| \phi(x) - \int_0^t \nabla_x F(\tau, v(x,\tau)) \mathrm{d}\tau - \phi(x) + \int_0^t \nabla_x F(\tau, w(x,\tau)) \mathrm{d}\tau \right| \\
&\quad + \sup_{(x,t)\in D} |-\nabla_x F(t, v(x,t)) + \nabla_x F(t, w(x,t))| \\
&\leq \sup_{(x,t)\in D} \int_0^t |F(\tau, v(x,\tau)) - F(\tau, w(x,\tau))| \, \mathrm{d}\tau \\
&\quad + \sup_{(x,t)\in D} \int_0^t |F(\tau, v(x-1,\tau)) - F(\tau, w(x-1,\tau))| \, \mathrm{d}\tau \\
&\quad + \sup_{(x,t)\in D} |F(t, v(x,t)) - F(t, w(x,t)| \\
&\quad + \sup_{(x,t)\in D} |F(t, v(x-1,t)) - F(t, w(x-1,t)| \\
&\overset{(A3)}{\leq} \delta L \sup_{(x,t)\in D} |v(x,t) - w(x,t)| + \delta L \sup_{(x,t)\in D} |v(x-1,t) - w(x-1,t)| \\
&\quad + L \sup_{(x,t)\in D} |v(x,t) - w(x,t)| + L \sup_{(x,t)\in D} |v(x-1,t) - w(x-1,t)| \\
&\leq 2(1+\delta)L \sup_{(x,t)\in D} |v(x,t) - w(x,t)| \leq 2(1+\delta)L \| v - w \| .
\end{aligned}$$

□

THEOREM 5.46. *Let the assumptions* (A1), (A2), (A3) *be satisfied with* $L < \frac{1}{2}$ *and let* $0 < \delta < \frac{1}{2L} - 1$. *Then in the space* $(\mathcal{M}, \| \cdot \|)$ *there exists exactly one classical solution of the initial value problem* (5.31) *on* D.

REMARK 5.47. *Theorem 5.46 gives us exactly one bounded classical solution of* (5.31) *with bounded partial derivative in* t *on* D.

Proof. From Theorem 5.42 we know that $(\mathcal{M}, \| \cdot \|)$ is a Banach space. From Theorem 5.44 we know that the operator $T : \mathcal{M} \to \mathcal{M}$ is the contraction. Then by the contraction principle in Theorem 5.32 there exists exactly one fixed point of T in $\mathcal{M}$ that is the classical solution of (5.31).

□

EXAMPLE 5.48. Let the function $F : \mathbb{R} \to \mathbb{R}$ be given by

$$F(\omega) = \frac{1}{2+\theta} \arctan \omega$$

for arbitrarily small $\theta > 0$. Then F is continuous on $\mathbb{R}$ and for the derivative of F

$$F'(\omega) = \frac{1}{(2+\theta)(1+\omega^2)} \leq \frac{1}{2+\theta} = L < \frac{1}{2}$$

holds. From the mean value theorem we get that for all $\omega_1, \omega_2 \in \mathbb{R}$ there exists $\xi, \tilde{\xi} \in \mathbb{R}$ such that

$$F(\omega_1) - F(\omega_2) = F'(\xi)(\omega_1 - \omega_2) \leq L(\omega_1 - \omega_2) \leq L|\omega_1 - \omega_2|,$$

$$F(\omega_2) - F(\omega_1) = F'(\tilde{\xi})(\omega_2 - \omega_1) \leq L(\omega_2 - \omega_1) \leq L|\omega_1 - \omega_2|$$

and consequently, there is

$$|F(\omega_1) - F(\omega_2)| \leq L|\omega_1 - \omega_2|$$

for all $\omega_1, \omega_2 \in \mathbb{R}$. Therefore, if we put $\delta < \frac{\theta}{2}$ and $\phi : \mathbb{Z} \to \mathbb{R}$ bounded then we satisfy all assuptions (A1), (A2), (A3) and T is the contraction. Then, from Theorem 5.46 the initial value problem

$$\begin{cases} u_t(x,t) + \frac{1}{2+\theta} \nabla_x \arctan\left(u(x,t)\right) = 0, \\ u(x,0) = \phi(x) \end{cases}$$

has exactly one bounded classical solution with bounded partial derivative in t on $[0,\delta] \times \mathbb{R}$. ■

EXAMPLE 5.49. If we assume the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(\omega) = \frac{1}{2+\theta} \sin \omega$$

for arbitrarily small $\theta > 0$ then we can use the same procedure as in Example 5.48. We get that for $\delta < \frac{\theta}{2}$ and $\phi : \mathbb{Z} \to \mathbb{R}$ bounded the initial value problem

$$\begin{cases} u_t(x,t) + \frac{1}{2+\theta} \nabla_x \sin\left(u(x,t)\right) = 0, \\ u(x,0) = \phi(x) \end{cases}$$

has exactly one bounded classical solution with bounded partial derivative in t on $[0,\delta] \times \mathbb{R}$. ■

Conclusion

We have answered several questions from the theory of semidiscrete equations. The summary of results from Chapters 2-4 about linear equations (solution, uniqueness, preservation laws, periodicity) has been presented in Subsection 4.5. We repeat the summary for the review.

	(PDE)	(DS)	(DT)	(DE)
Explicit solution	$\checkmark$(Th. 1.1)	$\checkmark^{\dagger}$ (Th. 2.1)	$\checkmark$(Th. 3.1)	$\checkmark^{*}$ (Th. 4.5)
Uniqueness	$\checkmark$(Th. 1.3)	?	$\checkmark$(Th. 3.2)	$\checkmark^{*}$ (Th. 4.9)
Sign preservation	$\checkmark$(Th. 1.1)	$\checkmark^{\dagger}$ (Prop. 2.6)	$\times$ (Ex. 3.13)	$\checkmark^{*\dagger}$ (Prop. 4.11)
Integral preservation in x	$\checkmark$(Th. 1.6)	$\checkmark^{*\dagger}$ (Th. 2.7)	$\checkmark^{\dagger}$ (Th. 3.9)	$\checkmark^{*}$ (Th. 4.12)
Integral preservation in t	$\checkmark^{\dagger}$ (Th. 1.7)	$\checkmark^{*\dagger}$ (Th. 2.8)	?	$\checkmark^{*\dagger}$ (Th. 4.13)
Stochastic process	$\checkmark^{*}$	$\checkmark^{*\dagger}$	$\times$	$\checkmark^{*\dagger}$
Periodicity in x	$\checkmark$(Th. 1.1)	?	$\checkmark$(Th. 3.8)	?

Table 2: Properties summary of linear transport equations.

For nonlinear equations with discrete space and continuous time we have proved the existence or existence and uniqueness of solution of initial-boundary value problem and we have shown some maximum and minimum principles with their applications. Finally, we have studied the existence of solution for initial value problem.

But there are still many open questions. Let us mention some of them.

1. linear semidiscrete transport equations with nonconstant coefficients,
2. transport equations on general domains with time scale structure (see e.g. Bohner, Peterson [1]),
3. open questions for linear transport equation with discrete time and continuous space:
    - For this structure we do not have the sign preservation in general. Is there some initial condition for which the sign is preserved?
    - Can we solve this problem with weaker assumptions on the initial condition ϕ? We have needed $\phi \in \mathcal{C}^{\infty}(\mathbb{R})$.
4. open questions for nonlinear transport equation with discrete space and continuous time:
    - Can we generalize applications of maximum and minimum principle (approximation, uniqueness, uniform stability) to nonlinear problems?
    - Is there another maximum principle with distinct or weaker assumptions?
    - Can we prove the uniqueness of solution for initial-boundary value problem (5.9) from the maximum principle, i.e. without fulfillment of Lipschitz condition?
    - We can try to find better estimates in the proof of existence and uniqueness result for the nonlinear initial value problem.
5. nonlinear semidiscrete transport equations with discrete time and continuous space.

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