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Diploma Thesis

Analysis of Bistable Equation and Its Generalizations

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Diplomová práce

Analýza bistabilní rovnice a jejích zobecnění

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Declaration

I do hereby declare that the entire thesis is my original work and that I have used only the cited sources.

Pilsen, 17^{th} May 2013

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Abstract (EN)
This diploma thesis focuses on bistable equation $u_t = \varepsilon^2 u_{xx} - F'(u)$ that models the dynamics of phase transition at some critical temperature. It is based on work of Drábek and Robinson (Drábek, P. and Robinson, S.B.: *Continua of local minimizers in a non-smooth model of phase transition*, 2011) that offers an explanation to the phenomenon of *slow dynamics*.
In Chapter 1 we present some known results, that we work with and generalize in the next chapters. In Chapter 2 we abandon the physics motivated case of double-well potential and unravel the behaviour of the model also for multi-well potentials. Solution diagram is used in order to describe the stationary solutions; its properties are examined in Chapter 3. In Chapter 4 we open the issue of non-smooth potentials that enable an existence of manifolds of solutions. This result, known for non-smooth double-well potential, is generalized for potentials of other type. Determining the number of manifolds reveals an interesting connection to basic graph theory. The thesis contains a number of results that we consider being original. Their short summary builds Chapter 5.

Key words: bistable equation, phase transition, slow dynamics, multi-well potential, n-well potential, manifolds of solution, continua of solution, k-walk in n-path;

Abstrakt (CZ)
Diplomová práce se zabývá bistabilní rovnicí $u_t = \varepsilon^2 u_{xx} - F'(u)$, pomocí níž lze modelovat dynamiku skupenské přeměny za určité kritické teploty. Vychází z poznatků publikovaných Drábkem a Robinsonem (Pavel Drábek a Stephen B. Robinson: *Continua of local minimizers in a non-smooth model of phase transition*, 2011), která vysvětluje fenomén *pomalé dynamiky*.
V úvodní kapitole jsou představeny některé známé výsledky, na něž se v dalších kapitolách navazuje a které se rozvíjí. V kapitole druhé se práce oprostí od fyzikálně motivovaného případu potenciálu se dvěma zdroji a rozkrývá chování modelu i pro potenciály vícezdrojové. Pro popis stacionárních řešení modelu je použito diagramu řešení (v závislosti na parametrech), jehož vlastnosti jsou zkoumány v kapitole třetí. Čtvrtá kapitola potom otevírá problematiku nehladkých potenciálů, které umožňují vznik variet řešení. Výsledek, známý pro nehladký dvouzdrojový potenciál, je zobecňován pro další typy potenciálů. Určování počtu variet stacionárních řešení pak nabízí zajímavé propojení do elementární teorie grafů. Práce obsahuje velké množství výsledků, které považujeme za původní. Jejich shrnutí je věnována pátá, závěrečná kapitola.

Klíčová slova: bistabilní rovnice, fázová přeměna, pomalá dynamika, vícezdrojový potenciál, n-well potenciál, variety řešení, kontinua řešení, k-sled v n-cestě;

Abstrakt (SI)
Diplomska naloga obravnava bistabilno enačbo $u_t = \varepsilon^2 u_{xx} - F'(u)$, ki jo je mogoče uporabiti za modeliranje faznega prehoda pri dani kritični temperaturi. Temelji na ugotovitvah, objavljenih Drábkom in Robinsonom (Drábek, P. in Robinson, S.B.: *Continua of local minimizers in a non-smooth model of phase transition*, 2011), ki ponujajo pojasnitev pojava *počasne dinamike.* V uvodnem poglavju so predstavljene nekatere znane rezultate, ki jih posplošujemo v sledujočih poglavjih. V drugem poglavju delo zapušča fizikalno motiviran primer dvovirnega potenciala in razkriva obnašanje tudi za večvirne potenciale. Za opis stacionarnih rešitev modela se uporablja diagram rešitev, njegove lastnosti se pregledujejo v tretjem poglavju. Četrte poglavje potem odpira vprašanje negladkih potencialov, ki omogočajo obstoj mnogoterosti rešitev. Ta rezultat, znan za negladki dvovirni potencial, je posplošen v druge vrste potencialov. Določevanje števila mnogoterosti razkriva zanimivo povezavo z osnovno teorijo grafov. Diplomska naloga vsebuje številne rezultate, za katere menimo, da so originalne. Njihov povzetek je naveden v zadnjem, petem poglavju.

Ključne besede: bistabilna enačba, fazni prehod, počasna dinamika, večvirni potencial, n-virni potencial, mnogoterosti rešitev, kontinua rešitev, k-sprehod v n-poti;

Contents

"It is a warm summer evening in Ancient Greece..."

Sheldon Lee Cooper

Chapter 1

Introduction

The Lyapunov (energy) functional

$$J_\varepsilon = \frac{\varepsilon^2}{2}\int_0^1 (u_x)^2 \,\mathrm{d}x + \int_0^1 F(u(x))\,\mathrm{d}x, \quad u \in W^{1,2}(0,1), \tag{1.1}$$

and its corresponding gradient system

$$\begin{gathered} u_t = \varepsilon^2 u_{xx} - F'(u), \quad x \in (0,1), t > 0, \\ u_x(0) = u_x(1) = 0, \end{gathered} \tag{1.2}$$

is perhaps the simplest model for phase transition at an appropriate temperature that enables the coexistence of two phases of a substance. Function F represents the free energy. This context leads to a natural choice of the function F being a coercive function with two equal global minima (usually at ± 1). Parameter ε plays the role of a scaling parameter and is supposed to attain only small values. This model can be found as *bistable equation* for example in [PT01] or [Fus90].

What makes the model interesting from the purely mathematical point of view is the lack of an asymptotic model for $\varepsilon \to 0$. There is simply no limit differential equation which would capture the limit behaviour of the system, as the set of critical points of J_ε grows without bound with vanishing parameter ε.

To characterize the model we investigate the critical points of J_ε for different choices of F. What happens if there are more than two minimizers of F? What if their values are unequal? Those are questions that are answered in Chapter 2 of this work. The critical points of (1.1) for either smooth or non-smooth potentials F can be effectively described using solution diagrams. Proving some of their properties is the main content of Chapter 3.

A special attention is dedicated to *non-smooth* potentials. Besides the nonlinear diffusion, the use of potentials, that lost their smoothness at their minimizers, might offer an explanation to the *slow dynamics*. This is a phenomenon that occurs at the phase transition process, however is not satisfactorily explained by the classical model. Chapter 4 covers a review of published results, followed by their generalization for non-smooth multi-well potentials. The generalization then opens a connection to the graph theory.

We do not usually comment the basic methods of mathematical analysis used in the thesis. The reader can find them in any analysis handbook, we recommend [ZC04a], [ZC04b] as well as [CL55] and [Eva98] for ordinary and partial differential equations.

1.1 Critical Points of the Lyapunov Functional

Drábek and Robinson offer in [DR11] a brief summary of the critical points of J_ε with $F \in C^2(\mathbb{R})$ a double-well potential. We follow this approach to enable the reader to get a clear introduction to the problem. Some issues, that are worth deeper thinking or require larger comment, are performed in detail in the next chapters, although for a different, triple-well potential.

The critical points of the functional J_ε is a common concept of describing the model. The Euler-Lagrange equation enables us to look for them as a weak solutions of the homogeneous Neumann boundary value problem

$$\begin{gathered} -\varepsilon^2 u''(x) + F'(u(x)) = 0, \\ u_x(0) = u_x(1) = 0, \\ x \in (0,1). \end{gathered} \tag{1.3}$$

The usual regularity argument yields that every weak solution of (1.3) is actually twice continuously differentiable and therefore is the classical solution (see e.g. [DM05] or [DM13]).

1.2 Smooth Double-Well Potential

In this section we sum up and comment the approach to the problem and results given in [DR11] for the smooth double-well potential F, which is

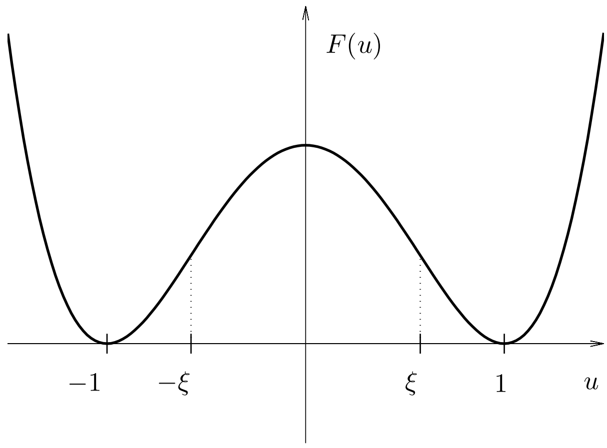

Figure 1.1: An example of standard double-well potential.

a $C^2(\mathbb{R})$ non-negative even function. It has a local maximum at 0, two global minima at ± 1 and two inflection points $\pm\xi$, so that F is strictly convex in $(-\infty, -\xi) \cup (\xi, +\infty)$ and strictly concave in $(-\xi, \xi)$. An example of such potential is shown in Figure 1.1. A frequently used simple example is the polynomial

$$F(u) = (1 - u^2)^2.$$

For any F having the properties above, the functional (1.1) has three constant critical points $-1, 0$ and 1 for all admissible values of ε. It is not difficult to see that ± 1 are global minimizers, as

$$J_\varepsilon(-1) = J_\varepsilon(1) = 0 \leq J_\varepsilon(u) \quad \forall u \in W^{1,2}(0, 1),$$

for arbitrary $\varepsilon > 0$. The constant function 0 is a saddle point. The authors of [DR11] suggest to prove that $J_\varepsilon(\delta) < J_\varepsilon(0) < J_\varepsilon(\delta \sin n\pi x)$ for large fixed n and all δ small enough. We will perform the proof in detail later for a triple-well potential.

For certain values of ε, the functional J_ε has also non-constant stationary points. The critical points of the functional are solutions of the corresponding Euler-Lagrange equation which is the homogeneous Neumann boundary problem (1.3). We can look for them using the shooting method. We analyse the initial value problem

$$\begin{aligned} -\varepsilon^2 u'' + F'(u) &= 0, \\ u(0) &= \theta, \\ u'(0) &= 0, \end{aligned} \tag{1.4}$$

for different values of parameters θ and ε and investigate for which values the solution satisfies the condition $u'(1) = 0$. As $F \in C^2(\mathbb{R})$, the assumptions of the Existence and Uniqueness Theorem for (1.4) are fulfilled. This justifies the shooting method and also ensures that for $\theta \in \{0, \pm 1\}$ and $\varepsilon > 0$ we get only the constant solution of (1.4).
For any $\theta > 1$ we have $F'(\theta) > 0$ and therefore it follows from (1.4) that the solution must be strictly convex, and thanks to $u'(0) = 0$ strictly increasing. Therefore $u'(x) \neq 0$, $\forall x > 0$, so it can not satisfy $u'(1) = 0$ for any scaling parameter ε. Similar argument leads to conclusion that no solution is obtained for $\theta < -1$.
To explore the solutions for the remaining values of $\theta \in (-1, 0) \cup (0, 1)$ we employ the first integral of (1.4):

$$\frac{\varepsilon^2}{2}(u'(x))^2 = F(u(x)) - F(\theta). \tag{1.5}$$

Potential F is even, so we can restrict our consideration to $\theta \in (-1, 0)$. The case $\theta \in (0, 1)$ will follow from the symmetry. As

$$F'(\theta) > 0, \quad \text{for } \theta \in (-1, 0),$$

the solution is strictly convex and together with $u'(0) = 0$ also strictly increasing until it reaches the value 0 at some point $x_0 = x_0(\theta)$. The strictly increasing solution $u = u(x)$ has strictly increasing inverse $x = x(u)$, for which we deduce from (1.5) that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 = \frac{\varepsilon^2}{2}\frac{1}{F(u) - F(\theta)}.$$

Using the separation-of-variables we can express x_0 as

$$x_0 = \frac{\varepsilon}{\sqrt{2}} \int_{\theta}^{0} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}, \tag{1.6}$$

which is positive due to $\theta < 0$ and the positivity of the integrand.

The Existence and Uniqueness Theorem for (1.4) together with the symmetry of F allow us to extend the solution as follows:

$$u(x) := \begin{cases} -u(2x_0 - x) & x \in (x_0, 2x_0), \\ u(4x_0 - x) & x \in (2x_0, 4x_0), \end{cases}$$

see Figure 1.2.
If we continue this process, we come to an n-nodal solution (for an arbitrary

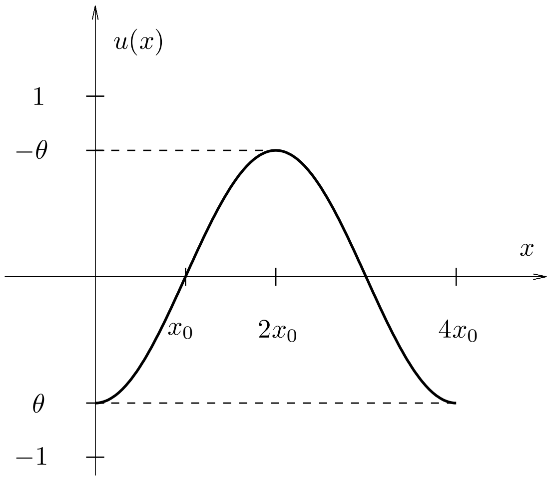

Figure 1.2: Solution of the initial value problem (2.7) with $\theta \in (-1, 0)$.

$n \in \mathbb{N}$), which can be scaled to $[0, 1]$. The scaling is accomplished using the parameter ε. Thus, for any $\theta \in (-1, 0) \cup (0, 1)$ we find a sequence

$$\varepsilon_1(\theta) > \varepsilon_2(\theta) > \cdots > \varepsilon_n(\theta) > \cdots > 0,$$

such that the boundary value problem (1.3) with $\varepsilon = \varepsilon_n(\theta)$ has an n-nodal solution with $u(0) = \theta$.
We focus more on the solutions for the double-well potential in Chapter 3. In the following Chapter 2, we try to generalize the problem for multi-well potentials.

Chapter 2

Stationary Solutions for Multi-Well Smooth Potentials

If we generalize the previous problem by admitting potentials of other type, a lot of previously proven properties will be preserved, however some new ones arise. The clue property we assume in this section is that

$$F \in C^2(\mathbb{R}).$$

This ensures, that the initial value problem (1.4) has a unique solution. Moreover, this enables us to determine all the solutions of the boundary value problem (1.3) for various initial values and parameters using the shooting method.
Two more restrictions for F are required. First we assume

$$F(x) \geq 0 \quad \forall x \in \mathbb{R},$$

so that the functional (1.1) remains non-negative. Second we assume that F is coercive, that is

$$\lim_{x \to \pm\infty} F(x) = +\infty.$$

We will need some further limitations connected with the inflection points, which will be discussed at each particular case separately.

The approach to the generalization is the following: We start with triple-well potential with equal (global) minima, then followed by triple-well potential with one elevated local minimum. The multi-well potential that has arbitrary finite number of local minima is treated afterwards.

2.1 Standard Triple-Well Potential

Definition 2.1. *Let non-negative coercive $F \in C^2(\mathbb{R})$ fulfil:*

1. *F is even,*
2. *F has local minima at 0 and ± 1 and $F(0) = F(\pm 1) = 0$,*
3. *The only inflection points of F are $\pm\xi_1, \pm\xi_2$ which are ordered*
$$0 < \xi_1 < \mu < \xi_2 < 1, \tag{2.1}$$
where $\pm\mu$ are the only local maxima points,
4. *$F''(x) = 0 \iff x \in \{\pm\xi_1, \pm\xi_2\}$.*

Then we call F standard triple-well potential.

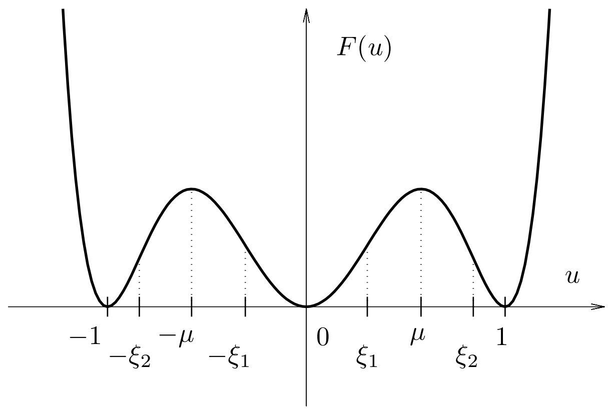

Figure 2.1: An example of standard triple-well potential.

Hence F is strictly convex on $(-\infty, -\xi_2) \cup (-\xi_1, \xi_1) \cup (\xi_2, +\infty)$ and strictly concave on $(-\xi_2, -\xi_1) \cup (\xi_1, \xi_2)$. A good simple example of such F is the polynomial

$$F(u) = u^2\left(1 - u^2\right)^2, \tag{2.2}$$

for which the inflection and maximum points have the following values

$$\xi_1 = \sqrt{\frac{1}{15}(6 - \sqrt{21})} \approx 0.307,\ \mu = \frac{\sqrt{3}}{3} \approx 0.577,\ \xi_2 = \sqrt{\frac{1}{15}(6 + \sqrt{21})} \approx 0.840.$$

An example of the standard triple-well potential is shown in Figure 2.1.

2.1.1 Constant Critical Solutions of Triple-Well Potential

Let F be a standard triple-well potential, then the functional

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_0^1 (u_x)^2\,\mathrm{d}x + \int_0^1 F(u)\,\mathrm{d}x,$$

has five constant critical points: $0, \pm\mu, \pm 1$. Similarly to the double-well potential, the minimizers of F (in this case the points $0, \pm 1$) are isolated global minimizers of J_ε, due to

$$J_\varepsilon(0) = J_\varepsilon(\pm 1) = 0 \leq J_\varepsilon(u), \quad \forall u \in W^{1,2}(0,1),$$

for any $\varepsilon > 0$, whereas the maximizers $\pm\mu$ are isolated saddles.

Proposition 2.2. *The constant critical points $\pm\mu$ of the functional J_ε, where F is a standard triple-well potential, are isolated saddle points.*

Proof. Thanks to the symmetry of F it suffices to show that the proposition holds for $+\mu$. As proposed in [DR11], we show that

$$J_\varepsilon(\mu+\delta) < J_\varepsilon(\mu) < J_\varepsilon(\mu + \delta\sin n\pi x),$$

for all δ small enough and n large but fixed. The first inequality is obvious; the first term in J_ε is equal to zero as $(\mu+\delta)_x \equiv 0$ and $F(\mu+\delta) < F(\mu)$. As for the second inequality, we have

$$\begin{aligned} J_\varepsilon(\mu + \delta\sin n\pi x) = \frac{\varepsilon^2}{2}\int_0^1 |\delta n\pi\cos n\pi x|\,\mathrm{d}x + \int_0^1 F(\mu+\delta\sin n\pi x)\,\mathrm{d}x \geq \\ \geq \frac{\varepsilon^2\delta^2 n^2\pi^2}{4} + \min\left\{F(\mu-\delta), F(\mu+\delta)\right\}. \end{aligned} \tag{2.3}$$

Without loss of generality, we may assume that the minimum is equal to $F(\mu+\delta)$. The Taylor theorem suggests that there exists $c \in U_\delta(\mu)$ such that

$$F(\mu+\delta) = F(\mu) + \frac{F''(c)}{2}\delta^2. \tag{2.4}$$

Take n so large that

$$0 > \frac{1}{2}F''(c) > -\frac{1}{4}\varepsilon^2 n^2\pi^2. \tag{2.5}$$

It follows from (2.3), (2.4) and (2.5) that

$$J_\varepsilon(\mu + \delta \sin n\pi x) > \frac{1}{4}\varepsilon^2\delta^2 n^2\pi^2 + F(\mu) - \frac{1}{4}\varepsilon^2\delta^2 n^2\pi^2 = J_\varepsilon(\mu).$$

The proof is finished. □

2.1.2 Construction of the Non-Constant Critical Points

Similarly as in the double-well potential case the non-constant solutions appear for some values of ε. The process of finding them remains the same; the Euler-Lagrange equation corresponds to the following boundary value problem

$$\begin{aligned} -\varepsilon^2 u'' + F'(u) &= 0, \\ u'(0) = u'(1) &= 0. \end{aligned} \tag{2.6}$$

We can find its solutions again using the shooting method for the initial value problem

$$\begin{aligned} -\varepsilon^2 u'' + F'(u) &= 0, \\ u(0) &= \theta, \\ u'(0) &= 0. \end{aligned} \tag{2.7}$$

As $F \in C^2(0,1)$, the assumptions for the uniqueness of the solution are fulfilled which justifies the use of the method. We show that for suitable values of parameters θ and ε, we can reach the desired goal $u'(1) = 0$.

Solution of the Initial Value Problem

We know, that for $\theta \in \{0, \pm\mu, \pm 1\}$ we get a constant solution, which satisfies $u'(1) = 0$ for any $\varepsilon > 0$. Simply we get no solution for $|\theta| > 1$, as $\theta > 1$ gives us strictly convex solution and $\theta < -1$ strictly concave solution, which cannot satisfy $u'(1) = 0$.

Hence only values of $\theta \in (-1, -\mu) \cup (-\mu, 0) \cup (0, \mu) \cup (\mu, 1)$ remain to be examined. Since F is even, we can focus on $\theta \in (-1, -\mu) \cup (-\mu, 0)$ only, as the rest will come from the symmetry. Notice that the following lemma holds for all even potentials.

Lemma 2.3. *Let v be a solution of the initial value problem (2.7) with F an even potential*[1] *F and $\theta \in [-1, 0]$, then $u(x) = -v(x)$ is a solution of (2.7) with initial value in $[0, 1]$.*

[1]For the exact definition of a general potential see Definition 2.23.

For the description of the solutions of (2.7) we use again the first integral,

$$\frac{\varepsilon^2}{2}|u'(x)|^2 = F(u(x)) - F(\theta). \tag{2.8}$$

For $\theta \in (-1, -\mu) : F'(\theta) > 0$, so the solution is convex and increasing, until it reaches $x_0 = x_0(\theta)$, for which $u(x_0) = -\mu$. This increasing function has an increasing inverse function so we can express x_0 from (2.8) as follows:

$$x_0 = \frac{\varepsilon}{\sqrt{2}} \int_{\theta}^{-\mu} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}. \tag{2.9}$$

The Existence Theorem allows us to extend the solution beyond x_0, as a solution of

$$\begin{aligned} -\varepsilon^2 u'' + F'(u) &= 0, \\ u(x_0) &= -\mu, \\ u'(x_0) &= \frac{\sqrt{2}}{\varepsilon}\sqrt{F(-\mu) - F(\theta)} > 0. \end{aligned}$$

However, as $F'(u) < 0, u \in (-\mu, 0)$, the solution is concave which suppresses the increase of the solution. We show that the suppress is sufficient so that the solution does not reach 0.

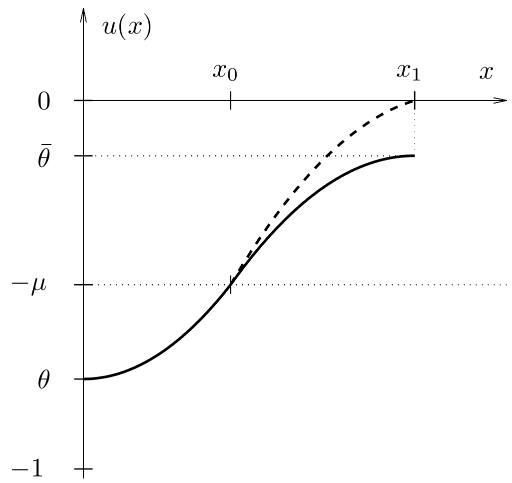

Figure 2.2: Illustration to the Proposition 2.4. The dashed variant does not occur in the case of F being standard triple-well potential.

Proposition 2.4. *The solution of the initial value problem (2.7) with F a standard triple-well potential, where $\theta \in (-1, -\mu)$, remains negative.*

Proof. We proceed via contradiction. Let us suppose that there is some $x_2 > 0$ for which $u(x_2) = 0, u'(x_2) \geq 0$ for u a solution of the initial value problem (2.7) with $u(0) = \theta \in (-1, -\mu)$. However, if $u'(x_2) = 0$ for some non-trivial solution, we get a straightforward contradiction with the Existence and Uniqueness Theorem. Therefore it remains that $u'(x_2) > 0$.
Substituting the points 0 and x_2 into the first integral (2.8) yields

$$F(\theta) = -\frac{\varepsilon^2}{2} \left| u'(x_2) \right|^2 ,$$

which is a contradiction with the non-negativity of F. □

Remark 2.5. *The equality $F(-1) = F(0)$ and its corollary $F(\theta) > F(0)$ for all admissible values of θ play the clue role here. Losing this equality will lead to new types of solutions in the next chapter.*

The next proposition excludes the option that the solution is bounded but strictly increasing in $(0, +\infty)$.

Proposition 2.6. *Let u be a solution of (2.7), $\theta \in (-1, -\mu)$. Then there exists $x_1 \in (x_0, +\infty)$, $\bar{\theta} \in (-\mu, 0)$ such that $u(x_1) = \bar{\theta}, u'(x_1) = 0$ and $u'(x) > 0, x \in (0, x_1)$.*

In other words, only the case sketched with solid line in Figure 2.2 occures for all θ.

Proof. As for the first integral (2.8), $u'(x_1) = 0$ implies $F(u(x_1)) = F(\theta)$. Let us define[2] function $\varphi : (-1, -\mu) \to (-\mu, 0)$ with the implicit formula

$$F(\varphi(\theta)) = F(\theta).$$

Thanks to the properties of F, φ is a one-to-one mapping. Hence for every $\theta \in (-1, -\mu)$ there exists unique $\varphi(\theta) \in (-\mu, 0)$ and $x_1 > x_0, x_1 \leq +\infty$ such that solution u of (2.7) satisfies $u(x_1) = \varphi(\theta)$, if $x_1 \in \mathbb{R}$ or at least $\lim_{x \to +\infty} u(x) = \varphi(\theta)$ if $x_1 = +\infty$ and $u'(x) > 0, x \in (0, x_1)$.
The solution u has an increasing inverse and we can express x_1 from the first integral (2.8) as follows

$$x_1 = \frac{\varepsilon}{\sqrt{2}} \int\limits_{\theta}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} > x_0. \tag{2.10}$$

This is a singular integral and it might happen that it does not converge. We exclude this case in the following lemma which we formulate rather general

[2] This definition is a special case of Definition 2.19.

for the sake of its further use. The assumptions of the lemma hold true in our case and are easy to be verified.

Lemma 2.7. *Let* $F(\theta) = F(\varphi(\theta))$ *and* $F(x) > F(\theta), \quad \forall x \in (\theta, \varphi(\theta))$. *Then*

$$x_1 = \frac{\varepsilon}{\sqrt{2}} \int_{\theta}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} < +\infty.$$

Proof. To simplify our notation we set $\bar{\theta} = \varphi(\theta)$.
We split the interval of integration into 3 subintervals using some small but fixed δ and examine each of them separately,

$$I(\theta) = \int_{\theta}^{\bar{\theta}} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} = \int_{\theta}^{\theta+\delta} \cdots + \int_{\theta+\delta}^{\bar{\theta}-\delta} \cdots + \int_{\bar{\theta}-\delta}^{\bar{\theta}} \cdots .$$

We denote the integrals over the subintervals by I_1, I_2, I_3, respectively. There is no need to examine I_2 which is finite due to a positive continuous integrand over a compact interval, as $F(s) - F(\theta) \geq c > 0$ for $s \in [\theta + \delta, \bar{\theta} - \delta]$, for any value of δ.
For $s \in (\theta, \theta + \delta)$ the Taylor's Theorem implies

$$F(s) = F(\theta) + F'(c_1)(s - \theta), \quad \text{where } \theta < c_1 < s < \theta + \delta,$$

which we can substitute into the integral

$$I_1(\theta) = \int_{\theta}^{\theta+\delta} \frac{\mathrm{d}s}{\sqrt{F'(c_1)(s - \theta)}} \geq 0.$$

The integrand can be estimated from above with

$$\frac{1}{\sqrt{K_1}} \int_{\theta}^{\theta+\delta} \frac{\mathrm{d}s}{\sqrt{(s - \theta)}} = \frac{2}{\sqrt{K_1}} \sqrt{\delta},$$

where

$$K_1 := \min_{u \in U_\delta(\theta)} |F'(u)|.$$

Such K_1 is strictly positive for all admissible θ for some appropriately chosen δ, thus $0 < I_1(\theta) < +\infty$.
Using the same procedure for I_3 and taking δ smaller if necessary, we get

$$0 \leq I_3(\theta) \leq \frac{2}{\sqrt{K_3}}\sqrt{\delta} < +\infty,$$

where $K_3 := \min_{u \in U_\delta(\bar{\theta})} |F'(u)| > 0$.

Hence $I(\theta) < +\infty$, thus $x_1 < +\infty$. □

Since $x_1 \in \mathbb{R}$, we have $u'(x_0) = 0$ and the proof of Proposition 2.6 is finished. □

We have constructed the solutions which are called *kinks*.

Definition 2.8. *Let $f \in C^1[a, b]$ for some $a < b$. We call f a kink, if it is strictly monotone and its derivative vanishes at the points a and b.*

Loosely speaking, kink is a function that ensures a smooth transition between two different levels.

Remark 2.9. *As there is no symmetry of F with respect to μ, the kink solution has in general no symmetry with respect to its inflection point, i.e. $x_0 \neq \frac{1}{2}x_1$ in general.*

We can now construct the solution in the same way for $\theta \in (-\mu, 0)$. We use a more straightforward method. We claim, that if we start at $\bar{\theta} = \varphi(\theta)$ with zero derivative, the solution of the initial value problem is equal to the reversed original kink, starting at θ.

Proposition 2.10. *The solution of*

$$\begin{gathered} -\varepsilon^2 v'' + F'(v) = 0, \\ v'(0) = 0, \\ v(0) = \bar{\theta} \in (-\mu, 0), \end{gathered}$$

with F a standard triple-well potential can be expressed as $v(x) = u(x_1 - x)$, where u is the solution of the same equation. The initial conditions are $u(0) = \theta \in (-1, -\mu)$, $u'(0) = 0$ and $x_1 > 0$ the first point where the derivative of u vanishes and $\bar{\theta} = \varphi(\theta)$.

Proof. The function $v(x) = u(x_1 - x)$ is well-defined as for any $\theta \in (-1, -\mu)$ the value $x_1 = x_1(\theta)$ is finite.
Thanks to φ being a one-to-one mapping, for any $\bar{\theta} \in (-\mu, 0)$ there exists a solution $u(x)$ of the initial value problem (2.7) with u starting at the point $\theta = \varphi^{-1}(\bar{\theta}) \in (-1, -\mu)$. Such u arrives to $\bar{\theta}$ with zero derivative at some positive finite x_1 .

So v is well-defined for all admissible θ as $x_1 \in \mathbb{R}$. It holds for v that $v(0) = u(x_1) = \bar{\theta}$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}v(0) = -\frac{\mathrm{d}}{\mathrm{d}x}u(x_1) = -u'(x_1) = 0.$$

Next, we show that the function v satisfies the equation,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}v(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(u(x_1 - x)\big) = (-1)^2\frac{\mathrm{d}}{\mathrm{d}x}u(x_1 - x) = u''(x_1 - x),$$

and

$$F'(v(x)) = F'(u(x_1 - x)),$$

therefore

$$-\varepsilon^2 v''(x) + F'(v(x)) = -\varepsilon^2 u''(x_1 - x) + F'(u(x_1 - x))) = 0,$$

which completes the proof. □

We have obtained all non-trivial negative solutions of (2.7) that satisfy $u'(x_d) = 0$ at some $x_d > 0$. Searching the positive solutions constructively would require the bijection φ in its extended form,

$$F(\varphi(\theta)) = F(\theta),$$

and

$$\varphi(\theta) \in \begin{cases} (-\mu, 0) & \text{for } \theta \in (-1, -\mu), \\ (\mu, 1) & \text{for } \theta \in (0, \mu). \end{cases}$$

The extension of φ is not necessary, since we obtain the positive solutions using Lemma 2.3. However, to ease our further work we use φ in its extended form, defined for both the intervals $(-1, -\mu)$ and $(0, \mu)$.

Remark 2.11. *For a general potential F we will define φ in a more general way in Definition 2.19.*

Remark 2.12. *Generalization of Proposition 2.10 can be proven. The proposition holds for a general potential F, suitably defined function φ and initial value $\bar{\theta} \in \operatorname{Im} \varphi$.*

Scaling of the Solution

As we know that the non-constant solutions of the initial value problem are kinks which ensure the transition between two levels, we can concatenate them repeatedly. In general, each such concatenation of kinks is a solution of the boundary value problem for some suitable ε.
The solution concatenated from n kinks is a solution to the boundary value problem if and only if $nx_1 = 1$. As the length x_1 of a kink depends on the initial value θ, we get a functional dependence of $\varepsilon_n = \varepsilon_n(\theta)$ with the following bind. The non-constant solution of the boundary value problem (2.6) with $\varepsilon = \varepsilon_n(\theta)$ consists of n kinks and $u(0) = \theta$. The other boundary value can be either $\theta, \varphi(\theta)$ or $\varphi^{-1}(\theta)$.[3] The formula for $\varepsilon_n(\theta)$ can be obtained easily from (2.10):

$$\varepsilon_n(\theta) = \frac{\sqrt{2}}{nI(\theta)}, \tag{2.11}$$

where

$$I(\theta) = \int\limits_{l_\theta} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} \geq 0.$$

Here l_θ denotes either the interval $(\theta, \varphi(\theta))$ or $(\varphi^{-1}(\theta), \theta)$.[4]

To finish this section we determine the type of those critical points we have constructed.

Proposition 2.13. *Each concatenation u of single kinks constructed above is a critical point of the saddle type of J_ε.*

Proof. We show the idea of the proof.
The same argument as Drábek and Robinson use for the standard double-well potential in [DR11] can be applied. Shifting the solution (as shown on Figure 2.3) yields some function $u_1 \in W^{1,2}(0,1)$ for which $J_\varepsilon(u_1) < J_\varepsilon(u)$ while u_2 created by reflecting a part of u near its maximum (see Figure 2.4) gives $J_\varepsilon(u) < J_\varepsilon(u_2)$.

□

[3] It is not difficult to determine that $u(1) = \theta$ if n is even. For n odd, $u(1) = \varphi(\theta)$ for $\theta \in \mathrm{Dom}\ \varphi$ and $u(1) = \varphi^{-1}(\theta)$ for $\theta \in \mathrm{Im}\ \varphi$. The enumeration is complete, as there are no non-constant solutions to the boundary value problem starting at $\theta \in \mathbb{R}\backslash(\mathrm{Dom}\ \varphi \cup \mathrm{Im}\ \varphi)$. We have used the invert φ^{-1} without saying a word about the invertibility of φ. But it is a corollary of its injectivity, which is straightforward, as F is monotone on both of the intervals that build $\mathrm{Dom}\ \varphi$ and its images are disjoint. An assertion of invertibility of φ for a general potential will follow.

[4] The interval l_θ is determined uniquely as $\mathrm{Dom}\ \varphi \cap \mathrm{Im}\ \varphi = \emptyset$.

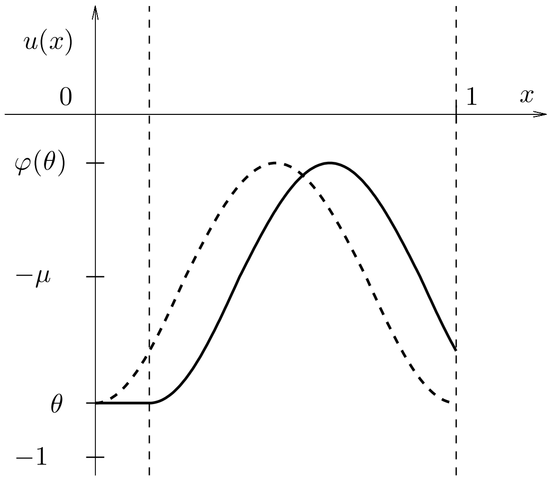

Figure 2.3: Shift of the stationary point to lower energy level.

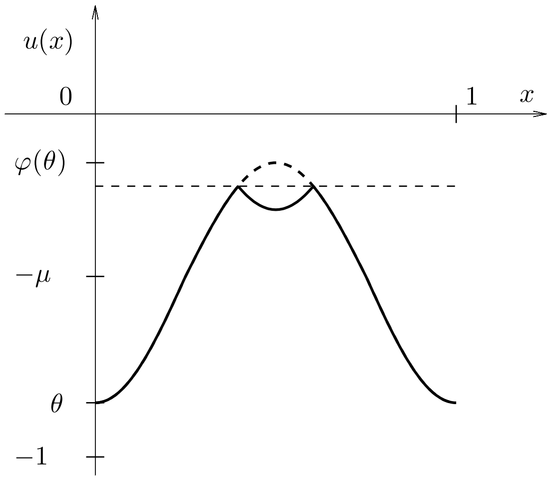

Figure 2.4: Reflection of the stationary point to higher energy level.

2.2 Triple-Well Potential with Elevated Middle Minimum

Let us make another stop towards a general potential. In the previous section we showed that the equality $F(0) = F(\pm 1)$ is crucial to ensure that the kink solutions cannot cross the local minima of F. In this section we show what happens if the value of $F(0)$ is elevated to some $m > 0$ so that it still remains a local minimum but it is no longer a global minimizer of F.

Definition 2.14. *Let $F \in C^2(\mathbb{R})$ satisfy*

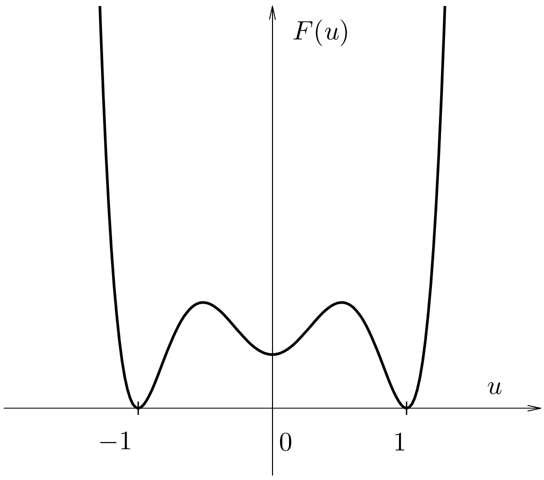

Figure 2.5: An example of triple-well m-potential.

1. *F is even,*

2. *F has local minima at 0 and ± 1 and $F(0) = m > 0 = F(\pm 1)$,*

3. *The only inflection points of F are $\pm\xi_1, \pm\xi_2$ for which*

$$0 < \xi_1 < \mu < \xi_2 < 1,$$

   where $\pm\mu$ are the only local maxima points,

4. *$F''(x) = 0 \iff x \in \{\pm\xi_1, \pm\xi_2\}$.*

Then we call F a triple-well m-potential.

Again, F is strictly convex in $(-\infty, -\xi_2) \cup (-\xi_1, \xi_1) \cup (\xi_2, +\infty)$ and strictly concave in $(-\xi_2, -\xi_1) \cup (\xi_1, \xi_2)$. A polynomial example of such F is of a form

$$F(u) = (u^2 + m)\big(1 - u^2\big)^2, \tag{2.12}$$

where $m \in \big(0, \frac{1}{2}\big)$, see Figure 2.5. The inflection and maximum points are

$$\xi_{1,2} = \sqrt{\frac{6 - 3m \mp \sqrt{9m^2 - 6m + 21}}{15}},\ \mu = \sqrt{\frac{1 - 2m}{3}}.$$

2.2.1 Constant Critical Points of Triple-Well m-Potential

Because the global behaviour of F remains the same for the points $\pm\mu, \pm 1$, the constant solutions equal to these values preserve their character, i.e. ± 1 are isolated global minimizers and $\pm\mu$ are isolated saddle points. A slight qualitative change occurs at the critical point $u \equiv 0$.

Theorem 2.15. *The constant* 0 *is a local minimum of the functional* J_ε *with* F *a triple-well m-potential.*

We rewrite this claim to clarify what we are going to prove. There exists $\delta > 0$ such that for all functions $y(x)$ in δ-neighbourhood of the constant 0 it holds that

$$J_\varepsilon(y(x)) \geq J_\varepsilon(0) = m.$$

We remind the topology of the Sobolev space.

$$y(x) \in \overline{K_{W^{1,2}(0,1)}(0,\delta)},$$

means that

$$||y(x)||_{W^{1,2}(0,1)} = \left(\int_0^1 |y'(x)|^2 \,\mathrm{d}x + \int_0^1 |y(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} < \delta.$$

Let us introduce at first two following lemmas.

Lemma 2.16. *Let* J_ε *be a functional defined in (1.1) with* F *a triple-well m-potential and* $a \in (0,1)$ *be a point such that* $F(a) = F(0) = m$. *Let* $p(x) \in W^{1,2}(0,1)$ *be a function such that there exists* $x_0 \in [0,1] : |p(x_0)| < \frac{a}{2}$ *and*

$$J_\varepsilon(p(x)) < J_\varepsilon(0). \tag{2.13}$$

Then it follows that $p(x) \in A$, *where*

$$A := \Big\{ y(x) \in W^{1,2}(0,1), \exists x_1, x_2 \in [0,1] : \big(y(x_1) = \frac{a}{2}, y(x_2) = a\big) \\ \text{or } \big(y(x_1) = -\frac{a}{2}, y(x_2) = -a\big) \Big\}.$$

Proof. The proof is quite simple. Any function $w(x) \in W^{1,2}(0,1)$ with its range in $[-a, a]$ gives $F(w(x)) \geq m$ for all $x \in [0,1]$ and therefore

$$J_\varepsilon(w(x)) \geq \int_0^1 F(w(x))\,\mathrm{d}x \geq m = J_\varepsilon(0).$$

Hence if $p(x)$ is such that $J_\varepsilon(p) < m$, then there exists x_3 such that $p(x_3) > a$ (or $p(x_3) < -a$). The function p is continuous due to the embedding of $W^{1,2}(0,1)$ into $C[0,1]$. Hence our assumption $|p(x_0)| < \frac{a}{2}$ and Intermediate Value Theorem yield that there exist $x_1, x_2 \in [0,1]$ such that either $p(x_1) = \frac{a}{2}$ and $p(x_2) = a$ or $p(x_1) = -\frac{a}{2}$ and $p(x_2) = -a$. □

Lemma 2.17. *Let $y(x) \in L^2(0,1)$, then it holds that*

$$||y||_{L^1(0,1)} \leq ||y||_{L^2(0,1)}. \tag{2.14}$$

We find it useful to remind the norms,

$$||y||_{L^1(0,1)} = \int_0^1 |y(x)|\,\mathrm{d}x,$$

and

$$||y||_{L^2(0,1)} = \left(\int_0^1 |y(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

For short we will denote the L^p-norm only with p ($||y||_p = ||y||_{L^p(0,1)}$).

Proof of Lemma (2.17). We remind the Holder inequality

$$||uv||_1 \leq ||u||_p ||v||_{p'}, \tag{2.15}$$

where p and p' are conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Let $p = 2, u = y$ and $v = 1$, then $p' = 2$ and we get from (2.15) that

$$||y||_{L^1(0,1)} = \int_0^1 |y(x)|\,\mathrm{d}x \leq \left(\int_0^1 y^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}} \cdot \mu((0,1))^{\frac{1}{2}} = ||y||_{L^2(0,1)}. \tag{2.16}$$

Here $\mu((0,1))$ denotes the Lebesgue measure of the interval $(0,1)$ and is equal to 1. □

Proof of the Theorem (2.15). We proceed via contradiction. We assume that in every δ-neighbourhood of 0 there exists some function $p(x)$ with lower energy. We want to reach a contradiction. Actually, we show that all functions that have lower energy lie outside some δ_0 neighbourhood of 0.

Let $p(x) \in W^{1,2}(0,1)$ have a lower energy than m, i.e. let (2.13) hold. For the case $|p(x)| > \frac{a}{2}$ for all $x \in [0,1]$, then p lies outside a ball,

$$||p||_{W^{1,2}(0,1)} \leq \frac{a}{2} =: \delta_0.$$

In the latter case ($|p(x)| \leq \frac{a}{2}$ for some x), the assumptions of Lemma 2.16 are fulfilled and we get that $p(x) \in A$. As $p'(x) \in L^1(0,1)$, it holds that

$$\int_{x_1}^{x_2} p'(x)\,\mathrm{d}x = p(x_2) - p(x_1).$$

As $p(x) \in A$, we have $p(x_2) - p(x_1) = \pm\frac{a}{2}$. Without loss of generality we can assume[5] that $p(x_2) - p(x_1) = \frac{a}{2}$. We can write

$$\frac{a}{2} = \int_{x_1}^{x_2} p'(x)\,\mathrm{d}x \leq \int_{x_1}^{x_2} |p'(x)|\,\mathrm{d}x \leq \int_{0}^{1} |p'(x)|\,\mathrm{d}x.$$

But the last term is nothing else but the L_1 norm of $p'(x)$.
Lemma 2.17 gives us

$$\frac{a}{2} \leq ||p'||_1 \leq ||p'||_2 \leq ||p||_{W^{1,2}(0,1)},$$

as the last inequality follows from the definition of the norm in $W^{1,2}(0,1)$. Hence all functions with lower energy lie outside a ball with radius $\delta_0 = \frac{a}{2}$ and we reached a contradiction. □

2.2.2 Construction of the Non-Constant Critical Points

The method of constructing the non-constant stationary solution remains the same. We investigate the solutions of the initial value problem (2.7).
The same argumentation as before excludes existence of solution for $|\theta| > 1$, as well as the Existence and Uniqueness Theorem denies existence of non-constant solution for $\theta \in \{0, \pm\mu, \pm 1\}$. The symmetry of function F again

[5] We can take $-p$ instead of p if necessary. Thanks to the symmetry of F, we get $J_\varepsilon(-p) = J_\varepsilon(p)$ and clearly $||p|| = ||-p||$ for any norm.

allows us to restrict our consideration to $\theta \in (-1, -\mu) \cup (-\mu, 0)$.
We start with exploring the case $\theta \in (-1, -\mu)$. As F is increasing in $(-1, -\mu)$, from (2.8) we get that the solution must be convex and hence increasing until it reaches the null point of F' which is $-\mu$ at some $x_0 = x_0(\theta)$. It can be computed as in the previous case (2.9). Beyond x_0 the solution must be concave which means that it reduces its growth. The crucial difference between this potential and the standard triple-well potential is that the solution now can cross the local minimum of the potential. In other words, the Proposition 2.4 does not hold for triple-well m-potentials.

Proposition 2.18. *There exists $\theta^* \in (-1, -\mu)$ such that for $\theta \in (-1, \theta^*)$ the solution of the initial value problem (2.7) reaches 0 with positive derivative, whereas for $\theta \in [\theta^*, 0)$ the solution remains negative. Moreover, $\theta^* = -a$.*

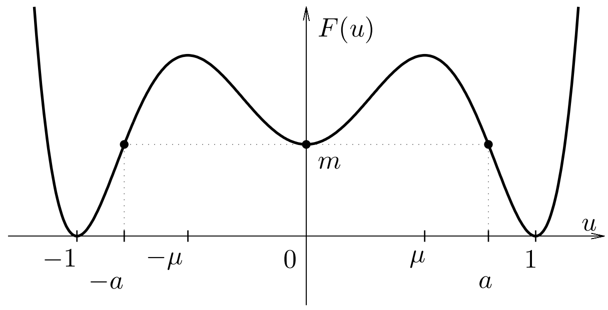

Figure 2.6: Significant points of the triple-well m-potential.

Proof. From the first integral (2.8) we get that the solution that reaches 0 at some x_1 has the derivative

$$u'(x_1) = \frac{\sqrt{2}}{\varepsilon}\sqrt{m - F(\theta)}. \tag{2.17}$$

This term makes sense only if $F(\theta) < m = F(0)$, that is for $\theta \in (-1, -a)$ only. Hence the solutions starting in $[-a, 0)$ can never reach 0. $\square$

Similarly as in the previous section, we would like to employ the function φ. It determines the maximum of the solution starting at θ. Let us introduce the following definition, independent of the number of local minima of F.

Definition 2.19. *Let $F \in C^2(\mathbb{R})$ be a potential, i.e. non-negative, coercive function with finite number of local extremes. Then we define a mapping $\tilde{\varphi}$ in the domain*

$$\text{Dom } \tilde{\varphi} := \{x \in \mathbb{R}, F'(x) > 0, \exists y > x : F(y) = F(x)\},$$

with

$$\tilde{\varphi}(x) := \min\{y > x : F(y) = F(x)\},$$

and its restriction

$$\varphi(x) = \tilde{\varphi}|_B,$$

where

$$B := \{x \in \text{Dom } \tilde{\varphi} : F'(\tilde{\varphi}(x)) \neq 0\}.$$

Loosely speaking, φ assigns the nearest value at the same level of F, whilst it avoids assigning the local minima of F. See also the graph of $\varphi(\theta)$ in Figure 2.7.

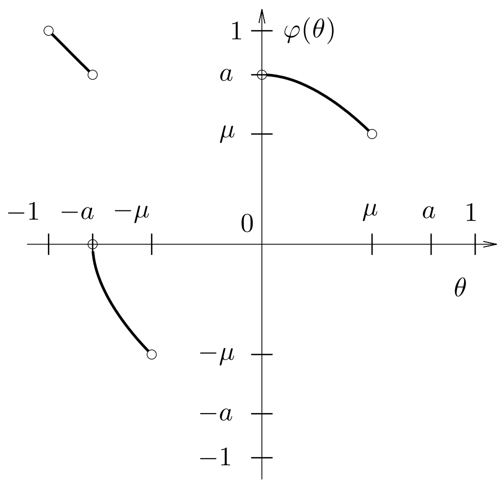

Figure 2.7: Graph of the function φ for the triple-well m-potential.

Proposition 2.20. *The mapping φ is single-valued for every $x \in$ Dom φ and it holds that*

1. $\varphi(x) > x$,
2. $F(\varphi(x)) = F(x)$,
3. $F(y) > F(x), \quad \forall y \in (x, \varphi(x))$,
4. $F'(\varphi(x)) < 0$.

Proof. We show that even $\tilde{\varphi}$ is single valued. The question is whether for all $x \in \text{Dom } \tilde{\varphi}$ the set $L_x := \{y > x : F(y) = F(x)\}$ has its minimum.
The set L_x is non-empty as $x \in \text{Dom } \tilde{\varphi}$. The fact that the potential F has only finite number of local extremes implies that L_x is finite. A non-empty finite set of numbers has its unique minimum thus $\tilde{\varphi}$ is single-valued and so is its restriction φ.
The first and second property of φ are trivial.
The third can be proven via contradiction. Let for $y \in (x, \varphi(x))$ it holds $F(y) \leq F(x)$. If $F(y) = F(x)$ we get a contradiction with the definition of φ. Let $F(y) < F(x)$. As $F'(x) > 0$, there exists $x_h \in (x, x + \delta) : F(x_h) > F(x)$ and thanks to the continuity of F there exists $x_s \in (x_h, y) : F(x_s) = F(x)$. But as $x < x_h < x_s < y < \varphi(x)$, we get a contradiction with the definition of φ.
We prove the fourth property again via contradiction. As $F'(\varphi(x)) = 0$ is excluded by the definition of Dom φ, let us expect that $F'(\varphi(x)) > 0$. But then there must exist $x_h \in (x, x + \delta_1)$ and $x_l \in (\varphi(x) - \delta_2, \varphi(x))$ such that

$$F(x_h) > F(x) = F(\varphi(x)) > F(x_l).$$

But then there is some $x_s \in (x, \varphi(x))$ at the same level of F $(F(x_s) = F(x))$ and we come into contradiction with the same argument as above. □

For the triple-well m-potential we get

$$\begin{aligned} B &= \{-a\}, \\ \text{Dom } \tilde{\varphi} &= (-1, -\mu) \cup (0, \mu), \\ \text{Dom } \varphi &= (-1, -a) \cup (-a, -\mu) \cup (0, \mu). \end{aligned}$$

Now we should justify that φ really gives us the maximum value of the kink. However, almost all the construction work has been already done. For solution crossing 0, i.e. $\theta \in (-1, -a)$, we use the symmetry of F and the same argumentation as at the standard double-well potential applies. For $\theta \in (-a, 0)$ the situation is the same as at the standard triple-well potential. Therefore we get a kink solutions for both cases. Their finiteness is proven using the Lemma 2.7.

Proposition 2.21. *A solution of the initial value problem (2.7) satisfying $\theta \in \text{Dom } \varphi$ is bounded and reaches its minimum θ and its maximum $\varphi(\theta)$.*

Proof. We have discussed already, that the solution u of the (2.7) starting at $\theta \in \text{Dom } \varphi$ arrives at $\varphi(\theta)$ with vanishing derivative. It remains to show that $\text{Im } u \subseteq [\theta, \varphi(\theta)]$.
At first we exclude $y \in (-\infty, \theta)$ from the image of u. If there exists some

$x_t \in (0, +\infty)$ such that $F(x_t) = y \in (-\infty, \theta)$, then thanks to the continuity of the solution it must also cross some $y_l \in (\theta - \delta, \theta) : F(y_l) < F(\theta)$. Substituting $u(x) = y$ we get a contradiction with the first integral, as the right-hand side of (2.8) becomes negative. The same argument excludes also $(\varphi(\theta), +\infty)$. □

It remains to justify why we have excluded the point $-a$ from the domain of φ. Let us introduce the following proposition.

Proposition 2.22. *The solution of the initial value problem (2.7) with triple-well m-potential F and $\theta = -a \in (-1, -\mu)$ such that $F(-a) = m = F(0)$, does not achieve its maximum.*

Proof. We know that the solution is increasing. We use the formula for the length of the interval, in which the solution is strictly increasing (2.10),

$$x_1 = \frac{\varepsilon}{\sqrt{2}} I(-a) = \frac{\varepsilon}{\sqrt{2}} \int_{-a}^{0} \frac{\mathrm{d}s}{\sqrt{F(s) - F(-a)}}.$$

We want to show that the solution is strictly increasing in $[0, +\infty)$, i.e. we need to prove that $I(-a) = +\infty$. For this purpose we split the integration into three subintervals $(-a, 0) = (-a, -a + \delta) \cup (-a + \delta, -\delta) \cup (-\delta, 0)$ where δ is fixed, sufficiently small. Let us denote the integrals over subintervals with I_1, I_2, I_3, respectively. Then it is enough to explore I_3, as $I_1 \geq 0$ and $I_2 \geq 0$ since we integrate positive integrand over a positive interval. And for $s \in (-\delta, 0)$ from the Taylor's Theorem we get

$$F(s) = F(0) + F'(0)s + F''(c)\frac{s^2}{2},$$

where $-\delta < s < c < 0$. As the first derivative of F vanishes at 0 and $F(-a) = F(0)$ we can rewrite and integrate

$$I_3 = \int_{-\delta}^{0} \frac{\mathrm{d}s}{\sqrt{F(s) - F(-a)}} = \int_{-\delta}^{0} \frac{\sqrt{2}\,\mathrm{d}s}{\sqrt{|F''(c)||s|}} = \sqrt{\frac{2}{|F''(c)|}} \big[\ln s\big]_0^{\delta} = +\infty,$$

as $|F''(c)| \geq K > 0$ in some small δ neighbourhood of 0. Hence the integral diverges due to function, thus $x_1 = +\infty$. □

A simple corollary of this proposition is that there is no solution of the boundary value problem (2.6) with $u(0) = -a$. The proof is identical for the general case; the solution of the initial value problem (2.7) starting at

$\theta \in (\text{Dom } \tilde{\varphi} \setminus \text{Dom } \varphi)$ does not achieve its maximum.

Similarly as in the previous case we can concatenate the kinks and scale them using the parameter ε to fit into the interval $[0, 1]$. The same argumentation as in the previous case will lead us to the conclusion that these concatenations are critical points of the saddle type for appropriate values of $\varepsilon_n(\theta)$.

2.3 Standard Multi-Well Potential

Studying some typical behaviour of the stationary solutions of J_ε with two types of triple-well potentials should suggest, what phenomena we meet in a general case. With the general case we mean an n-well potential, a $C^2(\mathbb{R})$ function F, which has the suitable geometry.
We introduce the following general definition so we can use it in further chapters.

Definition 2.23. *Let $n \geq 2$ and a non-negative and coercive $F \in C^1(\mathbb{R})$ have*

1. *n local minimizers ν_i, $i = 1, 2, \ldots, n$,*
2. *$n-1$ local maximizers μ_i, $i = 1, 2, \ldots, n-1$,*
3. *$2n-2$ inflection points ξ_i, $i = 1, 2, \ldots, 2n-2$, which are ordered in the following way:*

$$\begin{aligned} \nu_1 < \xi_1 < \mu_1 < \xi_2 < \nu_2 < \cdots < \nu_i < \xi_{2i-1} < \mu_i < \xi_{2i} < \nu_{i+1} < \ldots \\ \cdots < \nu_{n-1} < \xi_{2n-3} < \mu_{n-1} < \xi_{2n-2} < \nu_n, \end{aligned} \tag{2.18}$$

4. *$F''(x) = 0$ only at the inflection points of F,*

then we call F a (multi-well) potential.
Moreover, if $F \in C^2(\mathbb{R})$, then we call F standard (also smooth) multi-well potential.

Remark 2.24. *The demand on the number of inflection points is too strong. It suffices to expect, that there is a finite number of inflection points between two neighbouring maxima. Then it is guaranteed that $|F''(y)| \geq C > 0$ for some small neighbourhood of a local extreme.*

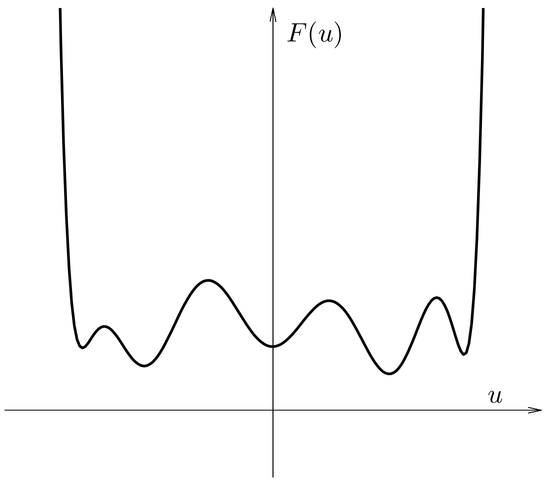

Figure 2.8: An example of a multi-well potential.

Like in the previous sections we use the shooting method for the appropriate initial value problem (1.4), i.e. $-\varepsilon^2 u'' + F'(u) = 0$, with zero initial derivative and $u(0) = \theta$.
We remind the function φ defined in Definition 2.19. To the properties proven in Proposition 2.20 we add the invertibility of φ.

Proposition 2.25. *The function φ is invertible. Moreover, for all $\bar{\theta} \in \mathrm{Im}\ \varphi$, the solution of the initial value problem 2.7 with $\bar{\theta}$ has its minimum $\varphi^{-1}(\bar{\theta})$ and maximum $\bar{\theta}$.*

Proof. We show that φ is injective. Let $x_a, x_b \in \mathrm{Dom}\ \varphi$ and $\varphi(x_a) = \varphi(x_b)$. From the properties of φ it follows that both x_a and x_b is the nearest lower values at the same level, i.e. $F(\varphi(x_a)) = F(x_a) = F(x_b)$. But the maximum of the set $L := \{y < \varphi(x_a) : F(y) = F(\varphi(x_a))\}$ exists and is unique, therefore $x_a = x_b$. The function φ is injective and invertible.
The claim that $\varphi^{-1}(\bar{\theta})$ gives the minimum of the kink starting at $\bar{\theta} \in \mathrm{Im}\ \varphi$ is a corollary of the Proposition 2.21 and an analogy of Proposition 2.10. for a general potential.
We implicitly expect the existence of the solution in some sufficiently long right neighbourhood of 0. It it justifiable thanks to assumptions of Global Existence Theorem are fulfilled (see [CL55]). □

We would show the same way as for the triple-well m-potential that the boundary value problem (2.6) can have a non-constant solution if and only $\theta \in \mathrm{Dom}\ \varphi \cup \mathrm{Im}\ \varphi$. These solutions are finite kinks. Again, we can come to an implicit formula for a strictly monotone solution to the initial value problem (2.7).

Proposition 2.26. *Let $\theta \in \mathrm{Dom}\ \varphi$. Then the solution $u(x)$ to (2.7) is given by*

$$x = \frac{\varepsilon}{\sqrt{2}} \int\limits_{\theta}^{u(x)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}, \quad x \in (0, x_1),$$

where

$$x_1 = \frac{\varepsilon}{\sqrt{2}} \int\limits_{\theta}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}.$$

Similarly, $\theta \in \mathrm{Im}\ \varphi$ gives a strictly decreasing solution $v(x)$,

$$x = \frac{\varepsilon}{\sqrt{2}} \int\limits_{v(x)}^{\theta} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}, \quad x \in (0, x_2),$$

where x_2 is given by

$$x_2 = \frac{\varepsilon}{\sqrt{2}} \int\limits_{\varphi^{-1}(\theta)}^{\theta} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}.$$

The finiteness of x_1, x_2 follows from Lemma 2.7, hence both $u(x)$ and $v(x)$ are well-defined.
Again, we can reverse the kinks to get new solutions. See Proposition 2.10 and Remark 2.12.

We find it useful to proceed all possible combinations to make our discussion about the stationary points of J_ε complete. According to the derivative of F, we can distinguish three options:

1. $F'(\theta) = 0$.
   Then $u''(0) = 0$ and hence $u \equiv \theta$ in $(0, 1)$ and again extremal points of F generate constant stationary solutions. The constant solutions equal to any of local maximizers are saddle points, the proof is identical as in Proposition 2.2. Global minimizers of F are global minimizers of J_ε, which can be shown directly from the definition of J_ε. The solutions equal to local (but not global) minimizers are local minimizers. The proof here would follow the proof of Theorem 2.15.

2. $F'(\theta) > 0$.
   Similarly as in the previous sections, the function φ helps us to distinguish three qualitatively different cases.

   (a) $\theta \in \text{Dom } \varphi$,
   (b) $\theta \notin \text{Dom } \varphi \wedge \forall x \in (\theta, +\infty) : F(x) > F(\theta)$,
   (c) $\theta \notin \text{Dom } \varphi \wedge \exists x > \theta : F(x) = F(\theta)$.

   Let us discuss the three possibilities:

   (a) If $\theta \in \text{Dom } \varphi$ it immediately follows from the definition of φ that $F'(\varphi(\theta)) < 0$ and the solution is a finite kink.
   (b) When $\theta \notin \text{Dom } \varphi \wedge F(x) > F(\theta) \quad \forall x > \theta$,
   the first integral (2.8) suggests that there is no point $x_1 > 0$ where the derivative vanishes. The boundary value problem (2.6) has no solution.
   (c) The last case $\theta \notin \text{Dom } \varphi \wedge \exists x > \theta : F(x) = F(\theta)$ together with the definition of φ gives that for such x it must hold that $F'(x) = 0$.[6] Hence we get an infinite kink[7]. The proof is identical as the proof of Proposition 2.22. No solution for the boundary value problem.

   Just briefly to the last case.

3. $F'(\theta) < 0$.

   We distinguish among three cases.

   (a) If $\theta \in \text{Im } \varphi$ then there exists $\bar{\theta} = \varphi^{-1}(\theta)$ and we get a finite kink.
   (b) $\theta \notin \text{Im } \varphi \wedge F(x) > F(\theta) \quad \forall x \in (-\infty, \theta)$.
   The solution remains concave hence decreasing as there is no point at which the derivative vanishes. We obtain no solution of the boundary value problem (2.6).
   (c) $\theta \notin \text{Im } \varphi \wedge \exists x \in (-\infty, \theta) : F(x) = F(\theta)$. It follows similarly as before that this case leads to an infinite kink, again we get no solution for the boundary value problem.

[6] In other words, using the definition of $\tilde{\varphi}$: $\theta \in B$, i.e. $F'(\tilde{\varphi}(\theta)) = 0$.

[7] By an infinite kink we mean a strictly monotone solution of the initial value problem (2.7) for which it holds that $u(x) \to u_0 \in \mathbb{R}, u'(x) \to 0$ as $x \to +\infty$.

The whole diversification of the initial values is shown on Figure 2.9 with an example of standard multi-well potential. Each of colours stands for one of cases discussed above. Only cases 1, 2a and 3a lead to a critical point of J_ε.

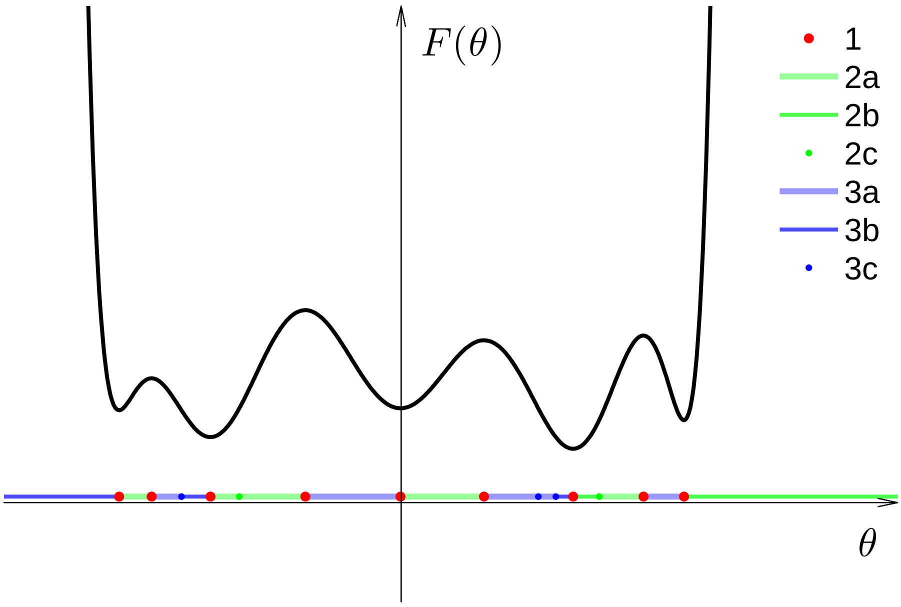

Figure 2.9: The initial values distinguished by colors to show the different behaviour of the solution of the initial value problem (2.7).

Again, these kinks can be concatenated repeatedly and then scaled to the basic interval. The following assertion brings a basic property of the solution.

Proposition 2.27. *Let u be a solution of the initial value problem (2.7) with a multi-well potential and $\theta \in \mathrm{Dom}\ \varphi \cup \mathrm{Im}\ \varphi$. Then the number of inflection points of a kink u between its neighbouring extremes is determined by the number of extremes of F on $(\min u, \max u)$.*

The proof is simple, as the intervals of monotonicity of F determine the convexity (and concavity) of the solution.
The same procedure as at Proposition 2.13 proves that every concatenation of kinks is a saddle point. We can now state the final theorem.

Theorem 2.28. *The critical points of the Lyapunov functional J_ε defined in (1.1) with F a standard multi-well potential are concatenations of kinks and constant solutions. All the non-trivial solutions are saddles, the constant solutions corresponding to the local (global) minimizers of F are local (global) minimizers of J_ε, the local maximizers of F are saddle points of J_ε.*

Chapter 3

Solution Diagram for Smooth Potentials

Our attention now focuses on an effective way of describing the set of critical points of J_ε with a standard (smooth) potential, introduced in the previous chapter. They are characterized by their initial value θ. To depict the critical points we use solution diagrams—a set of couples (θ, ε) that determine a critical point of J_ε, or equivalently a solution of the homogeneous Neumann boundary value problem

$$\begin{aligned} -\varepsilon^2 u'' + F'(u) &= 0, \\ u'(0) = u'(1) &= 0. \end{aligned} \tag{3.1}$$

First, we introduce the definition of the solution diagram.

Definition 3.1. *The set*

$$SD := \big\{(\theta, \varepsilon) \in \mathbb{R} \times \mathbb{R}^+ : \textit{(3.1) with parameter } \varepsilon \\ \textit{has a solution } u \in C^2[0,1] \textit{ with } u(0) = \theta\big\}$$

is called a solution diagram of J_ε with potential F.

For short we often say a *solution diagram for the potential F.*

The authors of [DR11] sketch the solution diagram for the standard double-well potential without any detailed clarification. In the following chapter we justify the properties of their sketch (see Figure 3.2) and generalize its properties. Our goal is to be able to sketch the diagram for an arbitrary smooth potential.

There are some constant solutions for every potential. Corresponding rays $r_\theta = \{(\theta, \varepsilon), \varepsilon > 0\}$ build the trivial part of the solution diagram.
There are two ways how to look at the diagram. Either as a (multi-valued) mapping $\theta = \theta(\varepsilon)$ or $\varepsilon = \varepsilon(\theta)$. We prefer the latter option. We have shown that the non-trivial part of SD can be split into disjoint subsets $\varepsilon_n = \varepsilon_n(\theta)$, where $n \in \mathbb{N}$ denotes the number of kinks the solution consists of.

3.1 Solution Diagram for Smooth Double-Well Potential

We start with the simplest, double-well potential. First we handle the constant solutions. We have shown that there are constant solution equal to 0 and ± 1 for all $\varepsilon > 0$. Hence the rays

$$\begin{aligned} r_{-1} &= \{(-1, \varepsilon), \varepsilon > 0\}, \\ r_0 &= \{(0, \varepsilon), \varepsilon > 0\}, \\ r_1 &= \{(1, \varepsilon), \varepsilon > 0\}, \end{aligned}$$

build the trivial part of the solution diagram. We remind again that there are no non-trivial solutions for $\theta \in \{0, \pm 1\}$. We have already shown that the boundary value problem does not have any solution for $|\theta| > 1$ either. Moreover, we show that we can restrict our investigation to $\theta \in (-1, 0)$, as the other part of the solution diagram comes from the symmetry.

Proposition 3.2. *Let SD be a solution diagram for a double-well potential F. Then it is symmetric with respect to the ε axis, i.e.*

$$(-\theta, \varepsilon) \in SD \qquad \Longleftrightarrow \qquad (\theta, \varepsilon) \in SD.$$

Proof. It suffices to show, that for u a solution of the boundary value problem with $u(0) = \theta$, $-u$ is also a solution. This comes immediately from generalized Lemma 2.3. □

Remark 3.3. *Proposition 3.2 holds for any even potential F.*

3.1.1 Basic Properties of the Non-Trivial Part of the Solution Diagram

For $\theta \in (-1, 0)$ the solution u of the initial value problem (1.4) reaches 0 at some point x_0. This point was determined exactly in (1.6) as

$$x_0 = \frac{\varepsilon}{\sqrt{2}} \int_{\theta}^{0} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}. \tag{3.2}$$

The symmetry of potential F enables simple extension of u to a kink with $u(x) = u(2x_0 - x)$ for $x \in (x_0, 2x_0)$. Initial value problem solution created by concatenating n kinks is a solution of the boundary value problem (3.1) if and only if

$$2nx_0 = 1.$$

Examples of such solutions are given in Figure 3.1.

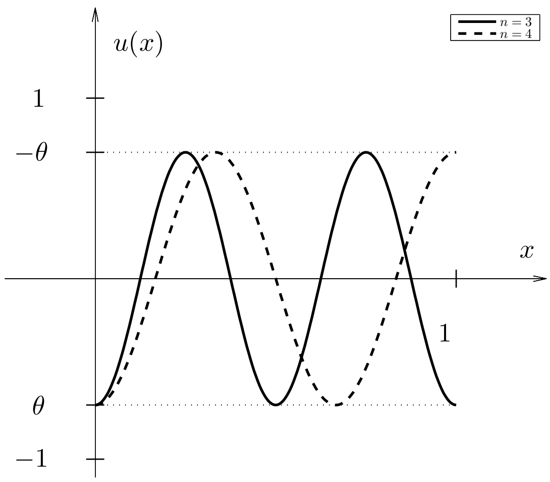

Figure 3.1: Concatenated solution to boundary value problem (3.1) with $\varepsilon = \varepsilon_n(\theta)$ for $n = 3$ (dashed) and $n = 4$ (solid).

For each $n \in \mathbb{N}$ we get easily from (3.2) the formula for ε_n,

$$\varepsilon_n(\theta) = \varepsilon = \frac{\sqrt{2}}{2n \displaystyle\int_{\theta}^{0} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}}. \tag{3.3}$$

For more comfortable work we denote[1]

$$I(\theta) := 2\int_{\theta}^{0} \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}}, \tag{3.5}$$

so we get

$$\varepsilon_n(\theta) = \frac{\sqrt{2}}{nI(\theta)}. \tag{3.6}$$

The next proposition gives us some useful properties that are necessary for sketching the diagram.

Proposition 3.4. *Let ε_n be given by (3.3). Then for each n it holds that*

1. $\varepsilon_n(\theta)$ *is continuous on* $(-1, 0)$,
2. $\lim\limits_{\theta\to -1^+} \varepsilon_n(\theta) = 0$,
3. $\lim\limits_{\theta\to 0^-} \varepsilon_n(\theta) \in \mathbb{R}$.

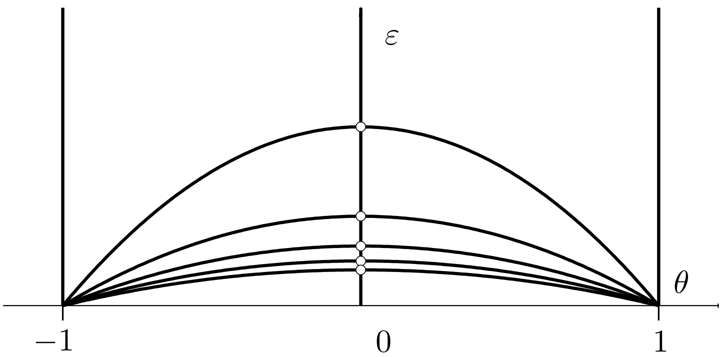

Figure 3.2: The sketch of solution diagram satisfying all properties proven in this section. For the sake of lucidity we sketch only first five branches $\varepsilon_n(\theta)$.

[1]For a general potential, we set

$$I(\theta) := \int_{\theta}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}}. \tag{3.4}$$

Proof. 1. Let us discuss the continuity of $I(\theta)$, defined in (3.5) at some given point $\theta_0 \in (-1, 0)$, i.e.

$$\lim_{\theta \to \theta_0} 2 \int_{\theta}^{0} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}. \tag{3.7}$$

As the integrand is singular, proving the continuity of (3.7) is not trivial. Using the substitution $s = \theta v$ we get

$$2 \int_{\theta}^{0} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} = 2 \int_{0}^{1} \frac{-\theta \, \mathrm{d}v}{\sqrt{F(\theta v) - F(\theta)}}. \tag{3.8}$$

We have a pointwise convergence

$$\lim_{\theta \to \theta_0} \frac{-2\theta}{\sqrt{F(\theta v) - F(\theta)}} = \frac{-2\theta_0}{\sqrt{F(\theta_0 v) - F(\theta_0)}},$$

for any θ_0. By Lebesgue Theorem, to show that $\lim_{\theta \to \theta_0} I(\theta) = I(\theta_0)$, it suffices to find an integrable majorant to the integrand in (3.8). Let us take $\delta > 0$ such that $U(\theta_0, \delta) \subset (-1, 0)$. We assume that

$$\theta \in U(\theta_0, \delta/2).$$

Take δ_0 small enough that $(1 - \delta_0)\theta \in U(\theta_0, \delta)$. Then we define the integrable majorant to the integrand in (3.8) as follows.

$$M(v) \stackrel{def}{=} \begin{cases} \dfrac{2\sqrt{-\theta_0 + \delta/2}}{\sqrt{C_m \delta_0}}, & \text{for } v \in (0, 1 - \delta_0], \\[2ex] \dfrac{2\sqrt{-\theta_0 + \delta/2}}{\sqrt{C_m (1 - v)}}, & \text{for } v \in (1 - \delta_0, 1), \end{cases} \tag{3.9}$$

where $C_m := \min_{x \in U(\theta_0, \delta)} F'(x)$ is positive.

We show that $M(v)$ really majorizes (3.8). Using the Taylor expansion we get

$$F(\theta v) = F(\theta) + F'(c)(\theta - \theta v), \tag{3.10}$$

for $\theta v - \theta < \theta \delta_0$ and $c \in (\theta, \theta v)$, hence

$$1-\delta_0 < v < \frac{c}{\theta} < 1.$$

Thanks to (3.10) we can estimate

$$\frac{-2\theta}{\sqrt{F(\theta v)-F(\theta)}} \le \frac{2\sqrt{-\theta}}{\sqrt{C_m(1-v)}}, \quad v \in (1-\delta_0, 1),$$

and as $\theta \in U(\theta_0, \delta/2)$, we get an estimate independent of θ,

$$\frac{-2\theta}{\sqrt{F(\theta v)-F(\theta)}} \le \frac{2\sqrt{-\theta_0+\delta/2}}{\sqrt{C_m(1-v)}}, \quad v \in (1-\delta_0, 1).$$

The monotonicity of F gives

$$F(\theta v) \ge F((1-\delta_0)\theta) > F(\theta), v \in (0, 1-\delta_0],$$

for every $\theta \in (-1, 0)$, so we can estimate

$$F(\theta v) - F(\theta) \ge F((1-\delta_0)\theta) - F(\theta) \ge C_m(-\theta)\delta_0.$$

The last inequality uses (3.10) again. Considering that we take θ from $\delta/2$-neighbourhood of θ_0, we get

$$\frac{-2\theta}{\sqrt{F(\theta v)-F(\theta)}} \le \frac{2\sqrt{-\theta_0+\delta/2}}{\sqrt{C_m\delta_0}}, \quad v \in (0, 1-\delta_0].$$

The integrability of the majorant M is straightforward as

$$\int_0^1 M(v)\,\mathrm{d}v = \int_0^{1-\delta_0} \frac{2\sqrt{-\theta_0+\delta/2}\,\mathrm{d}v}{\sqrt{C_m\delta_0}} + \int_{1-\delta_0}^1 \frac{2\sqrt{-\theta_0+\delta/2}\,\mathrm{d}v}{\sqrt{C_m(1-v)}} =$$

$$= 2\sqrt{\frac{-\theta_0+\delta/2}{C_m\delta_0}}(1-\delta_0) + 4\sqrt{\frac{-\theta_0+\delta/2}{C_m}}\sqrt{\delta_0} < +\infty.$$

Hence $I(\theta) \to I(\theta_0)$ for $\theta \to \theta_0$. And as $I(\theta_0) > 0$ for all $\theta_0 \in (-1, 0)$, we get no discontinuities of $\varepsilon_n(\theta) = \frac{\sqrt{2}}{nI(\theta)}$ either.

2. Thanks to (3.6) it suffices to show that

$$\lim_{\theta \to -1^+} I(\theta) = +\infty.$$

Using (3.8) and the Fatou lemma[2] we get

$$\lim_{\theta \to -1^+} 2 \int_0^1 \frac{-\theta \,\mathrm{d}v}{\sqrt{F(\theta v) - F(\theta)}} \geq 2 \int_0^1 \frac{\mathrm{d}v}{\sqrt{F(-v)}} = 2 \int_0^1 \frac{\mathrm{d}v}{\sqrt{F(v)}}.$$

We show that the last integral diverges. As the integrand is not bounded at the left neighbourhood of 1 we focus on its behaviour in there. Let $\delta \in (0,1)$, then

$$2 \int_0^1 \frac{\mathrm{d}v}{\sqrt{F(v)}} = 2 \int_0^{1-\delta} \frac{\mathrm{d}v}{\sqrt{F(v)}} + 2 \int_{1-\delta}^1 \frac{\mathrm{d}v}{\sqrt{F(v)}}, \tag{3.11}$$

and examine the first one, as the latter is finite.
Using the Taylor expansion, we get

$$F(v) = \frac{F''(c)}{2}(v-1)^2, \tag{3.12}$$

where $1 - \delta < v < c < 1$, for sufficiently small δ.
Substituting from (3.12) into the first of the integrals in (3.11) one gets

$$2 \int_{1-\delta}^1 \sqrt{\frac{2}{F''(c)}} \frac{1}{|v-1|} \mathrm{d}v \geq 2 \sqrt{\frac{2}{K_m}} \left[\ln|v-1|\right]_{1-\delta}^1 = +\infty,$$

where $K_m := \min_{x \in U_\delta(1)} F''(x)$ is positive for δ chosen small enough.
Hence $\lim_{\theta \to -1^+} I(\theta) = +\infty$ and from (3.6) $\lim_{\theta \to -1^+} \varepsilon_n(\theta) = 0$.

3. Again, using (3.6) we show an equivalent claim

$$\lim_{\theta \to 0^-} I(\theta) \geq C > 0.$$

[2]Fatou lemma:

$$\int_D \liminf_{n \to +\infty} f_n \leq \liminf_{n \to +\infty} \int_D f_n,$$

for integrable functions f_n.

The existence of the limit is given by the continuity of $I(\theta)$ and it remains to show that only that it does not vanish.

Using the Mean Value Theorem twice gives

$$|F(s)-F(\theta)| = |F'(c_1)(s-\theta)| = |(F'(c_1)-F'(0))(s-\theta)| = |F''(c_2)c_1(s-\theta)|, \tag{3.13}$$

for some c_1, c_2 satisfying $\theta < c_1 < s < 0$ and $\theta < c_1 < c_2 < 0$.
The equality (3.13) with the related inequalities allows us to estimate $|F(s) - F(\theta)|$ from above with

$$|F(s) - F(\theta)| \leq K_M(-\theta)(s - \theta), \tag{3.14}$$

where $K_M := \max_{x \in U_\delta(0)} |F''(x)|$ is a positive constant. The estimate (3.14) together with (3.5) gives

$$I(\theta) \geq \frac{2}{\sqrt{K_M(-\theta)}} \int_\theta^0 \frac{\mathrm{d}s}{\sqrt{s-\theta}} = \frac{2}{\sqrt{K_M(-\theta)}} \frac{\sqrt{-\theta}}{\frac{1}{2}} = \frac{4}{\sqrt{K_M}} > 0.$$

Hence $\lim_{\theta \to 0^-} I(\theta) \in \mathbb{R}^+$, then also $\lim_{\theta \to 0^-} \varepsilon_n(\theta) \in \mathbb{R}^+$.

□

Corollary 3.5. *The limit*

$$\lim_{\theta \to 0} \varepsilon_n(\theta) \tag{3.15}$$

exists and is finite.

Proof. The claim is a simple corollary of the two previous propositions. Thanks to Proposition 3.2 it holds that

$$\lim_{\theta \to 0^-} \varepsilon_n(\theta) = \lim_{\theta \to 0^+} \varepsilon_n(\theta),$$

therefore the limit (3.15) exists and is equal to the one-sided limits, which are finite as for Proposition 3.4. □

We know that the functions $\varepsilon_n(\theta)$ are continuous on $(-1, 0)$ with a finite limit at 0. We show even more about the limit.

Proposition 3.6. *There exist constants $C_1, C_2 > 0$ such that*

$$0 < \frac{C_1}{n} < \lim_{\theta \to 0} \varepsilon_n(\theta) < \frac{C_2}{n}.$$

Proof. Proposition 3.4 gives immediately that $C_2 = \dfrac{\sqrt{2K_M}}{4}$.
For the latter inequality we start with (3.13) and its lower estimate,

$$|F(s) - F(\theta)| > K_m(-s)(s - \theta),$$

where $K_m = \min_{x \in U_\delta(0)} |F''(x)|$. For sufficiently small δ the constant K_m is positive.
We substitute the estimate into (3.5) and we get

$$I(\theta) = 2\int_\theta^0 \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} \leq \frac{2}{\sqrt{K_m}} \int_\theta^0 \frac{\mathrm{d}s}{\sqrt{(0-s)(s-\theta)}} = \frac{2\pi}{\sqrt{K_m}}, \tag{3.16}$$

where the integral has been computed using the following lemma.

Lemma 3.7. *Let $a < b$, then*

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{(b-x)(x-a)}} = \pi.$$

Proof. We use the integration formula from [Bar74],

$$\int \frac{\mathrm{d}x}{\sqrt{kx^2 + lx + m}} = -\frac{1}{\sqrt{-k}} \arcsin \frac{2kx + l}{\sqrt{l^2 - 4km}}, \tag{3.17}$$

for $k < 0, l^2 - 4km > 0$. In our case,

$$-x^2 + (a+b)x - ab = kx^2 + lx + m,$$

i.e. $k = -1 < 0$ and $l^2 - 4km = (a+b)^2 - 4ab = (a-b)^2 > 0$. Therefore,

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{(b-x)(x-a)}} = \left[-\arcsin \frac{-2x + a + b}{b - a} \right]_a^b = -\arcsin(-1) + \arcsin(1) = \pi.$$

□

Hence $I(\theta) \leq \dfrac{2\pi}{\sqrt{K_m}}$ and thus

$$\varepsilon_n(\theta) \geq \frac{\sqrt{2K_m}}{2n\pi},$$

for all sufficiently small negative values of θ. So the same estimate holds also for the limit and $C_1 = \dfrac{\sqrt{K_m}}{\sqrt{2}\pi}$. □

The set $(\theta, \varepsilon_n(\theta))$ is a graph of piecewise continuous single-valued mapping for every n. Hence it is justifiable to call them *branches*. Speaking about branches we mean the function $\varepsilon_n(\theta)$ or its graph. As they are closely related, we emphasize only where it is important to distinguish.

3.1.2 Branches of the Solution Diagram as a Functional Sequence

Now we look at the branches $\varepsilon_n(\theta)$ as a sequence of functions.

Proposition 3.8. *Let $\varepsilon_n(\theta)$ be branches of the solution diagram for a standard double-well potential. Then*

$$\lim_{n\to+\infty} \varepsilon_n(\theta) = 0,$$

for any $\theta \in (-1, 0)$, i.e. pointwise convergence.

Proof. Using the relation

$$\varepsilon_n(\theta) = \frac{\sqrt{2}}{nI(\theta)}, \tag{3.18}$$

the finiteness of the integral $I(\theta)$ gives together with the Squeeze Theorem the desired result. □

Lemma 3.9. *The sequence $\{M_n\}_{n\in\mathbb{N}}$, where M_i is the supremum of $\varepsilon_i(\theta)$ in $(-1, 0)$, converges towards 0.*

Proof. Each of ε_n is continuous in $(-1, 0)$ and the limits at the boundary points are finite. So its supremum over $(-1, 0)$ is finite. It follows from (3.18) that

$$M_i = \frac{M_1}{i}, \tag{3.19}$$

hence $M_n \to 0$. □

Corollary 3.10. *The functional sequence* $\varepsilon_n(\theta)$ *converges uniformly to the constant* 0*.*

Proof. It comes straightforward from the Sufficient Condition of Uniform Convergence as the sequence M_n majorises the functional sequence $\varepsilon_n(\theta)$ and $\lim_{n\to+\infty} M_n = 0$. □

This approach should help to characterize the behaviour of branches ε_n. It is straightforward to show that results of this section are valid for all smooth potentials.

3.1.3 Monotonicity of the Branches

The branches $\varepsilon_n(\theta)$ of the solution diagram are continuous in $(-1, 0)$ and we know that for the limits at the boundary points it holds

$$0 = \lim_{\theta\to-1^+} \varepsilon_n(\theta) < \lim_{\theta\to0^-} \varepsilon_n(\theta) \in \mathbb{R}.$$

We would like to show the monotonicity of the branches. It occurs that in general it does not hold true, however there is a simple criterion for the monotonicity.

Theorem 3.11 (**Sufficient Condition for Monotonicity of the Branches).** *Let* F *be a double-well potential. If there is no inflection point of* F' *in* $(-\xi, 0)$*, then the function* $\varepsilon_n(\theta)$ *is strictly increasing in* $(-1, 0)$*.*

We remind that $-\xi$ is the only inflection point of F on the negative half-axis. We find it important to emphasize that the usual choice of the double-well potential, $F(u) = (1 - u^2)^2$, fulfils the condition. This may suggest that the sufficient condition in Theorem 3.11 is rather natural.

Proof. We proceed the proof by showing that the derivative of $\varepsilon_n(\theta)$ exists and is positive in $(-1, 0)$.
Using (3.6) we can differentiate

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\varepsilon_n(\theta) = \frac{\sqrt{2}}{n}\frac{-1}{(I(\theta))^2}\frac{\mathrm{d}}{\mathrm{d}\theta}I(\theta).$$

Equivalently we show that $\frac{\mathrm{d}}{\mathrm{d}\theta}I(\theta)$ exists and is negative in $(-1, 0)$. The idea of the proof follows [SW81, Lemma 3.2.]. We start with a substitution that simplifies the notation, as the variable θ no longer occurs in the integration limit.

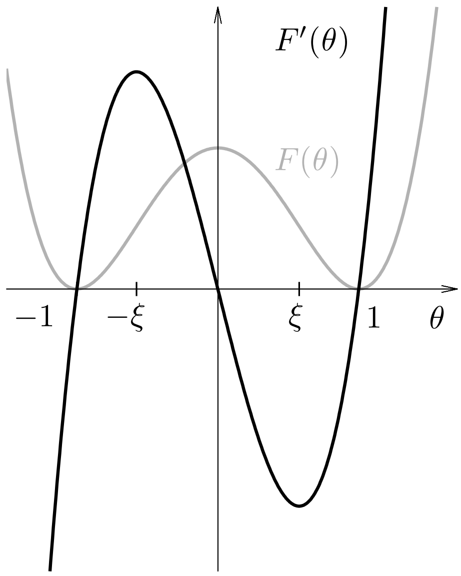

Figure 3.3: Standard double-well potential F and its derivative $F' =: f$.

$$I(\theta) = 2\int_{\theta}^{0} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} = \left| \begin{array}{c} s = \theta a \\ \mathrm{d}s = \theta\,\mathrm{d}a \end{array} \right| = 2\int_{0}^{1} \frac{-\theta\,\mathrm{d}a}{\sqrt{F(\theta a) - F(\theta)}}. \tag{3.20}$$

Now we can differentiate easily,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} I(\theta) = 2\int_{0}^{1} \frac{-\,\mathrm{d}a}{\sqrt{F(\theta a) - F(\theta)}} + 2\int_{0}^{1} \frac{-\theta(-\frac{1}{2})(aF'(\theta a) - F'(\theta))}{(F(\theta a) - F(\theta))^{\frac{3}{2}}} \mathrm{d}a.$$

Substituting back $s = \theta a$ we obtain

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}\theta}(\theta) &= 2\int_{\theta}^{0} \frac{F(s) - F(\theta) - \frac{1}{2}(sF'(s) - \theta F'(\theta))}{\theta(F(s) - F(\theta))^{\frac{3}{2}}}\,\mathrm{d}s = \\ &= 2\int_{\theta}^{0} \frac{(F(s) - \frac{1}{2}sF'(s)) - (F(\theta) - \frac{1}{2}\theta F'(\theta))}{\theta(F(s) - F(\theta))^{\frac{3}{2}}}\,\mathrm{d}s. \end{aligned} \tag{3.21}$$

As the denominator is negative due to $\theta < 0$, we need to show that the numerator is positive. This is an equivalent to the claim that

$$G(x) := F(x) - \frac{1}{2}xF'(x),$$

is strictly increasing. We remind that $s \in (\theta, 0)$ in (3.21). Since $F \in C^2(\mathbb{R})$, G is continuously differentiable. Therefore G is strictly increasing if and only if $G'(x) > 0$ on the interval $(-1, 0)$. We show that for $x \in (-1, 0)$ it holds

$$G'(x) = \frac{1}{2}(F'(x) - xF''(x)) > 0,$$

or equivalently,

$$F'(x) > xF''(x), \quad \forall x \in (-1, 0). \tag{3.22}$$

We use a geometrical method. If we denote $f(x) := F'(x)$, we can rewrite (3.22) as

$$f(x) > xf'(x), \quad \forall x \in (-1, 0). \tag{3.23}$$

We split the interval $(-1, 0)$ into two subintervals $(-1, -\xi)$ and $(-\xi, 0)$, where $-\xi$ is an inflection point of F, i.e. a local maximizer of f. Uniqueness of such point is ensured by the definition of double-well potential.
For $x \in (-1, -\xi)$ the inequality in (3.23) is trivial, as $f(x) > 0 > xf'(x)$. For $x \in (-\xi, 0)$ we use the geometrical interpretation of the derivative being the slope of a tangent line to the graph at x. Then it is not difficult to see the interpretation of $xf'(x)$ as a vertical segment determined by the horizontal axis and a shifted tangent line. See Figure 3.4.

We know that $f'(-\xi) = 0$ and $f'(0) < 0$. As for the assumption of the theorem there is no inflection point of f in $(-\xi, 0)$. Hence f' is strictly decreasing and f is strictly concave in $(-\xi, 0)$. For a strictly concave function it holds that all tangent lines to its graph lie over the graph (see e.g. [Roc97]). Let $t_{x^*}(x)$ be a tangent line at an arbitrary point $x^* \in (-\xi, 0)$. Then it holds that $t_{x^*}(0) > F(0) = 0$ and a line, parallel with the tangent and passing through the origin lies under the tangent.

Therefore

$$f(x^*) > x^* f'(x^*), \quad \forall x^* \in (-\xi, 0),$$

and the chain of equivalences presented above brings us to the desired conclusion, that $\varepsilon_n(\theta)$ is increasing in $(-1, 0)$. $\square$

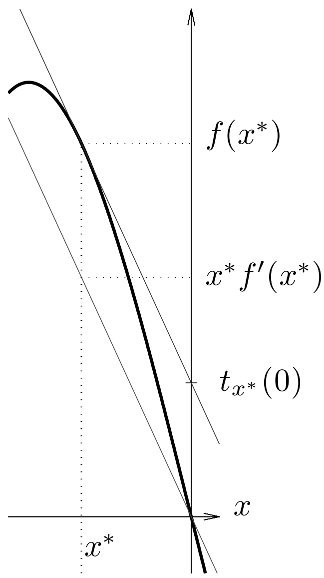

Figure 3.4: Geometrical interpretation of the inequality $f(x) > xf'(x)$.

Remark 3.12. *The condition in Theorem 3.11 is not necessary. It is possible to sketch f that breaks it, but still all tangent line intersects the vertical axis in positive values and thus (3.23) is fulfilled. An example of such f is*

$$f(x) = \begin{cases} -x(x+1) + \dfrac{1}{50}(\cos(4\pi x) - 1) & \textit{for } x \leq 0 \\ x(x-1) + \dfrac{1}{50}(\cos(4\pi x) - 1) & \textit{for } x > 0. \end{cases} \tag{3.24}$$

It is shown in Appendix A that $f \in C^1(\mathbb{R})$ and its prime function $F \in C^2(\mathbb{R})$ has the properties set in the definition of a double-well potential and hence it is a desired counterexample.

Now we are able to sketch the solution diagram at least roughly. See Figure 3.2.
The approach introduced in proof of Theorem 3.11 does not give us information about the limits of the derivative of $\varepsilon_n(\theta)$ at the boundary points -1 and 0. Further investigation of the limit behaviour of (3.21) for $\theta \to 0$ (or $\theta \to -1$) is far from being trivial. Therefore, this question, as well as for example the question of inflection points of ε_n, remains opened.

3.2 Solution Diagram for the Standard Multi-Well Potential

We skip showing the properties of solution diagram for triple-well potential, as the ordering of Chapter 2 may suggest and approach straightly the general problem. What can we say about the solution diagram of a standard multi-well potential? We use what we know about the critical points of J_ε from Section 2.3. Again, the solution diagram can be split to its trivial part which represents constant solutions and non-trivial part, representing concatenations of kinks. This implies that the non-trivial part is again built by separate branches $\varepsilon_n(\theta)$. We can express ε_n explicitly, as in (3.6):

$$\varepsilon_n(\theta) = \frac{\sqrt{2}}{nI(\theta)},$$

where $I(\theta)$ stands for

$$I(\theta) = \int\limits_{\theta}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}},$$

for $\theta \in \mathrm{Dom}\ \varphi$ and

$$I(\theta) = \int\limits_{\varphi^{-1}(\theta)}^{\theta} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}},$$

for $\theta \in \mathrm{Im}\ \varphi$.

For proving the properties of the branches for a general potential $F \in C^2(\mathbb{R})$, the function φ is crucial. We introduced some basic properties of φ in Proposition 2.20 and showed the invertibility of φ in Proposition 2.25. Some more properties of φ are summarized in the following lemma. The demand for F having a finite number of extremes is necessary to ensure, that both Dom φ and Im φ are piecewise connected sets.

Lemma 3.13. *Let $F \in C^2(\mathbb{R})$ be a standard potential. Let D_i, E_j for indices $i = \{1, 2, \ldots, \bar{\imath}\}, j = \{1, 2, \ldots, \bar{\jmath}\}$ be pairwise disjoint intervals such that $\bigcup_i D_i = \mathrm{Dom}\ \varphi$ and $\bigcup_j E_j = \mathrm{Im}\ \varphi$. We assume that D_i and E_j are maximal having this property. Then D_i, E_j are open sets (for all indices), for all sets D_i the function $\varphi|_{D_i}$ is continuous and strictly decreasing. There exists bijection*

$$b : \{1, 2, \ldots, \bar{i}\} \to \{1, 2, \ldots, \bar{j}\},$$

satisfying $\varphi(D_i) = E_{b(i)}$.

The proof does not bring anyhow deep ideas and therefore we postponed it to Appendix B.

We would like to ease our work using some sort of symmetry. The following assertion is a generalization of Proposition 3.2 for a standard multi-well potential.

Proposition 3.14. *Let SD be a solution diagram for a standard multi-well potential. Then it holds that*

$$(\theta, \varepsilon) \in SD \qquad \Longleftrightarrow \qquad (\varphi(\theta), \varepsilon) \in SD.$$

The assertion is a straightforward corollary of Proposition 2.10 and Remark 2.12. Notice that $\varphi = -id_{(-1,0)}$ for the double-well potential.
We can also prove the continuity of the branches and some of its limit behaviour.

Proposition 3.15. *Let the assumptions of Lemma 3.13 be fulfilled. Then for arbitrary* $n \in \mathbb{N}$ *the following assertions hold.*

1. *Branches* ε_n *are continuous in each of intervals* D_i, E_j.
2. *Let* $(\nu_k, \nu_k + \delta) \subseteq D_i$ *for some indices* i, k *and some* $\delta > 0$, *then*
$$\lim_{\theta \to \nu_k^+} \varepsilon_n(\theta) = 0.$$
3. *Let* $(\nu_k - \delta, \nu_k) \subseteq E_j$ *for some indices* j, k *and some* $\delta > 0$, *then*
$$\lim_{\theta \to \nu_k^-} \varepsilon_n(\theta) = 0.$$
4. *Let* $(\mu_k - \delta, \mu_k) \subseteq D_i$ *and* $(\mu_k, \mu_k + \delta) \subseteq E_j$ *for some triplet* i, j, k *and some* $\delta > 0$. *Then there exists the limit*
$$\lim_{\theta \to \mu_k} \varepsilon_n(\theta) =: L_{k,n},$$
and there exist constants $C_{1,k}, C_{2,k}$ *such that*
$$0 < \frac{C_{1,k}}{n} < L_{k,n} < \frac{C_{2,k}}{n}. \tag{3.25}$$

We remind, that $\nu_k, k \in \{1, 2, \ldots n\}$ and $\mu_l, l \in \{1, 2, \ldots, n-1\}$ are local minimizers and maximizers, respectively, of the potential F.

Proof. 1. The proof is proceed the same way as for the double-well potential (Proposition 3.4, 1.). However, we find it useful to show the proof of this general assertion.

Using (3.6) we transform the problem of continuity of $\varepsilon_n(\theta)$ to proving

$$\lim_{\theta\to\theta_0} I(\theta) = \lim_{\theta\to\theta_0} \int_{\theta}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}} = \int_{\theta_0}^{\varphi(\theta_0)} \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta_0)}} = I(\theta_0). \tag{3.26}$$

Let us take some fixed $\theta_0 \in D_i$ and let E_j be the set that contains $\varphi(\theta_0)$. Take $\delta, \overline{\delta} > 0$ such that $U(\theta_0, \delta) \subset D_i$ and $U(\varphi(\theta_0), \overline{\delta}) \subset E_j$. Such choice is possible, as both D_i, E_j are open sets and φ is continuous. We take $\delta', \delta'' > 0$, $\delta' < \delta$ and $\delta'' < \overline{\delta}$ such that $\theta \in U(\theta_0, \delta')$ and $\varphi(\theta) \in U(\varphi(\theta_0), \delta'')$.

Rewriting the limit in (3.26) using the substitution $s = (\varphi(\theta) - \theta)v + \theta$ (which gives $\mathrm{d}s = (\varphi(\theta) - \theta)\mathrm{d}v$), we get

$$\lim_{\theta\to\theta_0} \int_0^1 \frac{(\varphi(\theta)-\theta)\,\mathrm{d}v}{\sqrt{F((\varphi(\theta)-\theta)v+\theta)-F(\theta)}}. \tag{3.27}$$

The pointwise convergence of the integrand is ensured thanks to the continuity of F and φ. It remains to find an integrable majorant. The situation is more complicated with comparison to the double-well case as the integrand is unbounded at both integration limits. We suggest the majorant

$$M(v) = \begin{cases} \sqrt{\dfrac{R}{C_m}}\dfrac{1}{\sqrt{v}} & \text{for } v \in (0, \delta_0) \\[2ex] \max\left\{\sqrt{\dfrac{R}{C_m\delta_0}}, \sqrt{\dfrac{\overline{R}}{\overline{C_m}\delta_1}}\right\} & \text{for } v \in (\delta_0, 1-\delta_1) \\[2ex] \sqrt{\dfrac{\overline{R}}{\overline{C_m}}}\dfrac{1}{\sqrt{1-v}} & \text{for } v \in (1-\delta_1, 1), \end{cases}$$

where

$$\begin{aligned} C_m :=& \min_{x\in U(\theta_0,\delta)} F'(x), \\ \overline{C_m} :=& \min_{x\in U(\varphi(\theta_0),\delta)} -F'(x), \\ R :=& \; \varphi(\theta_0) - \theta_0 + \delta' + \delta'', \end{aligned} \tag{3.28}$$

are positive constants. The demands on $\delta_0, \delta_1 > 0$ will follow. The integrability of $M(v)$ is easy to be verified, we focus on proving that $M(v)$ really majorizes the integrand in (3.27).

We start with the first part. Take some $\delta_0 > 0$ small enough that $((\varphi(\theta)-\theta)v+\theta) \in U(\theta_0, \delta)$ for all $v \in (0, \delta_0)$. The Taylor formula gives

$$F((\varphi(\theta) - \theta)v + \theta) = F(\theta) + F'(c)(\varphi(\theta) - \theta)v,$$

where $c \in (\theta, (\varphi(\theta) - \theta)v + \theta)$. Then the following estimates in $(0, \delta_0)$ hold:

$$\frac{\varphi(\theta) - \theta}{\sqrt{F((\varphi(\theta) - \theta)v + \theta) - F(\theta)}} \le \frac{\varphi(\theta) - \theta}{\sqrt{C_m v(\varphi(\theta) - \theta)}} = \sqrt{\frac{\varphi(\theta) - \theta}{C_m v}},$$

where C_m is defined in (3.28). As we require a θ-independent estimate, we use the inequality

$$\varphi(\theta) - \theta \le \varphi(\theta_0) + \delta'' - (\theta_0 - \delta').$$

See our assumptions for δ' and δ'' from the beginning of this proof. Hence

$$\frac{\varphi(\theta) - \theta}{\sqrt{F((\varphi(\theta) - \theta)v + \theta) - F(\theta)}} \le \sqrt{\frac{\varphi(\theta_0) - \theta_0 + \delta' + \delta''}{C_m v}}.$$

We apply the same procedure also for the left neighbourhood of 1. Let δ_1 be chosen such that $((\varphi(\theta) - \theta)v + \theta) \in U(\varphi(\theta_0), \overline{\delta})$. Then for $v \in (1 - \delta_1, 1)$ one can write

$$\begin{aligned} F((\varphi(\theta) - \theta)v + \theta) &= F(\varphi(\theta)) + F'(d)((\varphi(\theta) - \theta)v + \theta - \varphi(\theta)) = \\ &= F(\theta) + F'(d)(\varphi(\theta) - \theta)(v - 1), \end{aligned}$$

where $d \in ((\varphi(\theta)-\theta)v+\theta, \varphi(\theta))$. Using $\overline{C_m}$ from (3.28) we get the estimate

$$\frac{\varphi(\theta)-\theta}{\sqrt{F((\varphi(\theta)-\theta)v+\theta)-F(\theta)}} \leq \frac{\varphi(\theta)-\theta}{\sqrt{\overline{C_m}(1-v)(\varphi(\theta)-\theta)}} = \sqrt{\frac{\varphi(\theta)-\theta}{\overline{C_m}(1-v)}}.$$

Again, the θ-independent estimate takes the form

$$\frac{\varphi(\theta)-\theta}{\sqrt{F((\varphi(\theta)-\theta)v+\theta)-F(\theta)}} \leq \sqrt{\frac{\varphi(\theta_0)-\theta_0+\delta'+\delta''}{\overline{C_m}}}\frac{1}{\sqrt{(1-v)}}.$$

The integrand is bounded in $(\delta_0, 1-\delta_1)$, as the denominator can be estimated from below with a positive constant,

$$F(\varphi(\theta)-\theta)v+\theta)-F(\theta) \geq c := \min\{C_m\delta_0, \overline{C_m}\delta_1\} > 0. \tag{3.29}$$

The inequality (3.29) would not hold true, if there were a local minimum of F in $(\theta_0, \varphi(\theta_0))$ getting *too close* to $F(\theta)$. But we chose δ_0, δ_1 such that for values $v \in (0, \delta_0)$ we remain in δ-neighbourhood of θ_0 and therefore in D_i (and for $v \in (1-\delta_1, 1)$ in δ-neighbourhood of $\varphi(\theta_0)$ and hence in E_j). This excludes the possibility that some local minimum of F gets too close to violate the estimate (3.29).

Hence $I(\theta) \to I(\theta_0) > 0$, for any $\theta_0 \in D_i$. Therefore $\varepsilon_n(\theta)$ is continuous in any D_i. The relation $\varphi(D_i) = E_j$ together with the strict monotonicity of φ and Proposition 3.14 give immediately also the continuity of $\varepsilon_n(\theta)$ in E_j.

2. Similarly as in Proposition 3.4 we show that $\lim_{\theta\to\nu_k^+} I(\theta) = +\infty$. Using $s = (\varphi(\theta)-\theta)v+\theta$ again, we get

$$\lim_{\theta\to\nu_k^+}\int_0^1 \frac{(\varphi(\theta)-\theta)\,\mathrm{d}v}{\sqrt{F((\varphi(\theta)-\theta)v+\theta)-F(\theta)}}.$$

We integrate only over $(0,\delta)$, as for every non-negative function f the inequality,

$$\int_0^\delta f(x)\,\mathrm{d}x \le \int_0^1 f(x)\,\mathrm{d}x, \tag{3.30}$$

holds. Again, using the Fatou lemma we get

$$\lim_{\theta\to\nu_k^+} \int_0^\delta \frac{(\varphi(\theta)-\theta)\,\mathrm{d}v}{\sqrt{F((\varphi(\theta)-\theta)v+\theta)-F(\theta)}} \ge \int_0^\delta \frac{(L-\nu_k)\,\mathrm{d}v}{\sqrt{F((L-\nu_k)v+\nu_k)-F(\nu_k)}}, \tag{3.31}$$

where L denotes $L := \lim_{\theta\to\nu_k^+} \varphi(\theta)$. We can express F in the neighbourhood of ν_k (for $v \in (0,\delta)$) with the Taylor polynomial:

$$F((L-\nu_k)v+\nu_k) = F(\nu_k) + \frac{F''(c)}{2}(L-\nu_k)^2 v^2. \tag{3.32}$$

Employing the positive constant

$$K_m := \min_{x\in U(\nu_k,\delta(L-\nu_k)+\nu_k)} F''(x)$$

(we may take δ smaller if necessary) and combining (3.30), (3.31) and (3.32) yields

$$\lim_{\theta\to\nu_k^+} I(\theta) \ge \sqrt{\frac{2}{K_m}} \int_0^\delta \frac{\mathrm{d}v}{v} = +\infty.$$

Hence $\lim I(\theta) = +\infty$ and $\lim \varepsilon_n(\theta) = 0$ for $\theta \to \nu_k^+$.

3. The proof contains of the same steps as the previous one and therefore we omit it.

4. The proof follows the ideas of proofs of Propositions 3.4 and 3.6. From the assumptions it straightly follows that $\varphi(D_i) = E_j$. Let us take $\delta', \delta'' > 0$ such that $\theta \in (\mu_k - \delta', \mu_k)$ and $\varphi(\theta) \in (\mu_k, \mu_k + \delta'')$. We split the integration

$$I(\theta) = \int_\theta^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}} = \int_\theta^{\mu_k} \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}} + \int_{\mu_k}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s)-F(\theta)}}, \tag{3.33}$$

denote the integrals with $I_1(\theta), I_2(\theta)$, respectively and examine each of them separately. Using the Mean Value Theorem twice for the denominator in I_1 gives

$$\begin{aligned} F(s) - F(\theta) = F'(c_1)(s-\theta) = (F'(c_1) - F'(\mu_k))(s-\theta) = \\ = F''(c_2)(c_1 - \mu_k)(s-\theta) \leq K_{M,k}(\mu_k - \theta)(s-\theta). \end{aligned} \tag{3.34}$$

It holds that $\theta < c_1 < s < \mu_k$ and $c_1 < c_2 < \mu_k$ and we define

$$K_{M,k} := \max_{x \in U(\mu_k, \delta)} |F''(x)| > 0.$$

Analogously, for I_2 one gets

$$\begin{aligned} F(s) - F(\theta) = F(s) - F(\varphi(\theta)) = F'(c_3)(s - \varphi(\theta)) = \\ = (F'(c_3) - F'(\mu_k))(s - \varphi(\theta)) = F''(c_4)(c_3 - \mu_k)(s - \varphi(\theta)) \leq \\ \leq K_{M,k}(\varphi(\theta) - \mu_k)(\varphi(\theta) - s), \end{aligned} \tag{3.35}$$

where $\mu_k < s < c_3 < \varphi(\theta)$ and $\mu_k < c_4 < c_3$. Substituting the estimates (3.34) and (3.35) into the integration (3.33), we get

$$I_1(\theta) \leq \frac{1}{\sqrt{K_{M,k}(\mu_k - \theta)}} \int_{\theta}^{\mu_k} \frac{\mathrm{d}s}{\sqrt{s-\theta}} = \frac{2}{\sqrt{K_{M,k}}},$$

and

$$I_2(\theta) \leq \frac{1}{\sqrt{K_{M,k}(\varphi(\theta) - \mu_k)}} \int_{\mu_k}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{\varphi(\theta) - s}} = \frac{2}{\sqrt{K_{M,k}}}.$$

Altogether $I(\theta) \leq \frac{4}{\sqrt{K_{M,k}}}$. Combining with (3.6) we get

$$C_{2,k} = \frac{\sqrt{2K_{M,k}}}{4}.$$

The second part of the proof copies the proof[3] of Proposition 3.6 and gives $C_{1,k} = \frac{\sqrt{2K_{m,k}}}{2\pi}$, where

[3] Again, it is necessary to split the integration into two parts using the local maximum μ_k.

$$K_{m,k} := \min_{x \in U(\mu_k, \delta)} |F''(x)|.$$

□

Thanks to the bijection between indices of D_i, E_j, we can use the notation D_i, E_i. (We just re-indexed the subsets of Im φ is such way, that the bijection b is an identity.)

Proposition 3.16. *Let F be a smooth potential, φ from Definition 2.19 and D_i, E_i maximal intervals such that their union build* Dom φ, Im φ, *respectively. Let $D_i = (x_{a_i}, x_{b_i}), E_i = (y_{b_i}, y_{a_i})$ and*

$$P = \bigcup_{i=1}^{\bar{i}} \{x_{a_i}, x_{b_i}, y_{b_i}, y_{a_i}\} \setminus \bigcup_{k=1}^{n-1} \mu_k,$$

a set of all boundary points except for the maximizers of F. Then

$$\lim_{\theta \to z^{\pm}} \varepsilon_n(\theta) = 0, \tag{3.36}$$

for any $z \in P$ and any $n \in \mathbb{N}$. By (3.36) we mean a one-sided limit, from above for lower boundary points x_{a_i}, y_{b_i} and from below for upper boundary points x_{b_i}, y_{a_i}.

In other words, if a boundary point is not a local maximizer of F, then branches ε_n emanate from 0 at this point.

Proof. The proof is a simple discussion of the cases that may occur. The lower boundary point x_{a_i} is either a local minimizer of F or a point for which

$$y_{a_i} = \lim_{x \to x_{a_i}^+} \varphi(x),$$

is a local minimizer of F. Proposition 3.15 (2. and 3.), together with Proposition 3.14 give that

$$\lim_{x \to x_{a_i}^+} \varepsilon_n(x) = \lim_{x \to y_{a_i}^-} \varepsilon_n(x) = 0.$$

There are two options for the points x_{b_i}, y_{b_i}. We get either $x_{b_i} = y_{b_i} = \mu_k$, for some k, which the proposition says nothing about. The second option is that there is a local minimizer of F at the same level as x_{b_i} and y_{b_i}. We show that also for this case, $\lim_{\theta \to x_{b_i}^-} I(\theta) = +\infty$.

Let F have a local minimizer $\nu_k \in (x_{b_i}, y_{b_i})$ satisfying $F(\nu_k) = F(x_{b_i})$. Then for any $\theta \in (x_{b_i} - \delta, x_{b_i})$ and some $\delta > 0$ small

$$I(\theta) = \int_{\theta}^{\varphi(\theta)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}} > \int_{\theta}^{\nu_k} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\theta)}}.$$

We show, that the last integral diverges. Substituting $s = (\nu_k - \theta)v + \theta$ we rewrite it as

$$\int_0^1 \frac{(\nu_k - \theta)\,\mathrm{d}v}{\sqrt{F((\nu_k - \theta)v + \theta) - F(\theta)}}.$$

Using the Fatou lemma we get

$$\lim_{\theta \to x_{b_i}^-} I(\theta) \geq \int_0^1 \frac{(\nu_k - x_{b_i})\,\mathrm{d}v}{\sqrt{F((\nu_k - x_{b_i})v + x_{b_i}) - F(x_{b_i})}}. \tag{3.37}$$

Taking some $\delta' > 0$ small enough, for $v \in (1 - \delta', 1)$ the Taylor expansion gives

$$F((\nu_k - x_{b_i})v + x_{b_i}) = F(\nu_k) + \frac{F''(c)}{2}(\nu_k - x_{b_i})^2(1-v)^2 \geq F(\nu_k) + \frac{K_m}{2}(\nu_k - x_{b_i})^2(1-v)^2, \tag{3.38}$$

for $1 - \delta' < v < \frac{c - x_{b_i}}{\nu_k - x_{b_i}} < 1$, and K_m is defined as

$$K_m := \min_{x \in (\nu_k - \delta'(\nu_k - x_{b_i}), \nu_k)} F''(x).$$

The relation (3.37) remains valid even if we reduce the integration interval to $(1 - \delta', 1)$. Then, substituting from (3.38), we get

$$\lim_{\theta \to x_{b_i}^-} I(\theta) \geq \sqrt{\frac{2}{K_m}} \int_{1-\delta'}^{1} \frac{\mathrm{d}v}{1 - v} = +\infty,$$

hence $\lim_{\theta \to x_{b_i}^-} \varepsilon_n(\theta) = 0$. □

What about the monotonicity of the branches? At the general potential also $\varphi(\theta)$ occurs in the formula for ε_n. The lack of information about φ' does not allow us to prove some variation of Theorem 3.11. Hence we are not able to determine the conditions for monotonicity of branches, neither their maximum. The monotonicity of $\varepsilon_n(\theta)$ remains an interesting open problem.

Example 3.17. *Let us consider F being a standard triple-well m-potential. Then* $\mathrm{Dom}\ \varphi = D_1 \cup D_2 \cup D_3$ *and* $\mathrm{Im}\ \varphi = E_1 \cup E_2 \cup E_3$*, where*

$$\begin{aligned} D_1 = &\ (-1,-a), \\ D_2 = &\ (-a,-\mu), \\ D_3 = &\ (0,\mu), \\ E_1 = &\ (a,1), \\ E_2 = &\ (-\mu,0), \\ E_3 = &\ (\mu,a). \end{aligned}$$

Then Proposition 3.16 gives us the following properties of branches of its solution diagram:

$$\lim_{\theta \to \sharp} \varepsilon_n(\theta) = 0,$$

where $\sharp$ stands for arbitrary of the following expressions:

$$-1^+, -a^-, -a^+, 0^-, 0^+, a^-, a^+, 1^-.$$

Proposition 3.15, 3. gives also the positivity of limits

$$\lim_{\theta \to \pm\mu} \varepsilon_n(\theta).$$

From the symmetry of F and Proposition 3.14 it follows that

$$(\theta, \varepsilon) \in SD \iff (\varphi(\theta), \varepsilon) \in SD \iff (-\theta, \varepsilon) \in SD.$$

So we can sketch the solution diagram, see Figure 3.5.

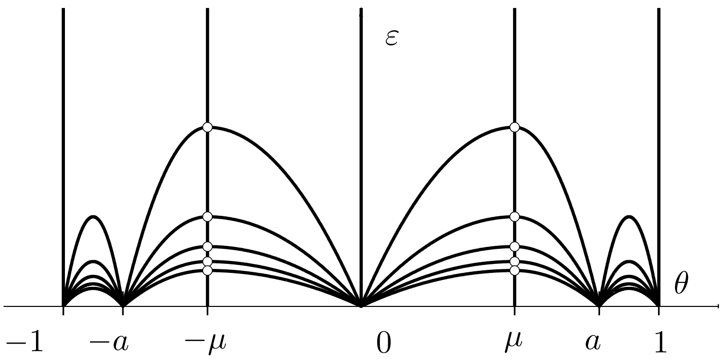

Figure 3.5: Sketch of the solution diagram for F a non-smooth triple-well m-potential. For the sake of lucidity we show only first five branches $\varepsilon_1(\theta)$ to $\varepsilon_5(\theta)$.

Chapter 4

Non-Smooth Potentials

For a comfortable work with the model of a phase transition, introduced in Chapter 1, a smoothly differentiable function F is usually chosen. However, the model with smooth potential F does not explain a phenomenon called *slow dynamics*. This discrepancy lead to searching for a more accurate model. It occurs that even a small change in the potential F ensures the existence of new stationary points of the functional J_ε and offer an explanation of the experimentally observed behaviour of the system. The *small change of the potential* is a loss of the C^2-smoothness of F at its minimizers[1].
Let us introduce the model

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial^2 u}{\partial x} - F'(u), \tag{4.2}$$

where the choice of a double-well potential $F(u) = |1 - u^2|^\alpha$, $\alpha \in (1, 2)$ is made. Drábek and Robinson ([DR11] and [Drá11]) have shown that the set of critical points of the Lyapunov functional gets much richer; manifolds of local minima occur in the non-smooth case. The existence of such manifolds of local minimizers of the functional J_ε corresponds to the *slow dynamics*, which was confirmed by various numerical experiments made by Otta [Ott07]. In the following chapter we show at first the sufficient condition for the new type solution to occur. Then, motivated by these results, we explain its ef-

[1] The same effect can be reached also considering the non-linear diffusion that leads to p-Laplace operator. The more general model

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial}{\partial x}\Big(\Big|\frac{\partial u}{\partial x}\Big|^{p-2}\frac{\partial u}{\partial x}\Big) + F'(u), \tag{4.1}$$

with $F = |1-u^2|^\alpha$ was studied, see e.g. [DMT11]. The mutual relation between parameters p and α plays the clue role in behaviour of the model. Note that (4.2) is a special case of the model (4.1) for the choice $p = 2$.

fects for potentials of various types.

We find it important to emphasize, that for a non-smooth potential, the most of previously derived properties hold. The functional J_ε with F a non-smooth potential[2] still has constant stationary points equal to local extrema of F and non-constant points taking the shape of kinks and their concatenations. Definition 2.19 can be considered also for non-smooth potentials; function φ can describe the non-constant solutions we have met already in the smooth case.

4.1 Conditions for the Existence of Continua of Solutions

In this section we set the assumptions for the potential to ensure the existence of the new type solution. As mentioned above, we examine potentials F that are C^2 everywhere except of its local minimizers, where it is C^1.

Let us focus on the discontinuity of F'' at the local minimizers of F. The following proposition suggests that a discontinuity of F'' of the first type (a jump) does not allow an existence of a non-constant solution. We remark that it is not a corollary of the Existence and Uniqueness Theorem, as its assumptions are not fulfilled.

Proposition 4.1. *Let be $F \in C^1(\mathbb{R})$ a coercive function. Let ν be a local minimizer of F, let F have a second derivative $F'' : \mathbb{R} \setminus \{\nu\} \to \mathbb{R}$, which is continuous everywhere but at ν, where F'' has a discontinuity of the first type, a jump. Then there is no non-constant solution of the initial value problem*

$$\begin{aligned} -\varepsilon^2 u'' + F'(u) &= 0, \\ u(0) &= \nu, \\ u'(0) &= 0. \end{aligned} \tag{4.3}$$

We remind that there is a constant solution to (4.3) due to $F'(\nu) = 0$.

Proof. We proceed via contradiction. Let us expect that there exists a non-constant solution $u(x)$ in some right neighbourhood of 0. Without loss of generality expect that $u(x) > \nu$ in $(0, \delta)$, for some $\delta > 0$ small enough[3]. Then using the separation-of-variables method, we come to a formula

[2]The exact definition of a non-smooth potential will be given by Definition 4.5.

[3]For $u(x) < \nu$ in $(0, \delta)$ only the integration limits in (4.4) would switch. The rest of the proof would be proceeded the same way.

$$x = \frac{\varepsilon}{\sqrt{2}} \int_{\nu}^{u(x)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}}, \quad x \in (0, \delta). \tag{4.4}$$

The differentiability properties of F ensures that we can use the Taylor expansion with the Lagrange form of the remainder in the right neighbourhood of ν:

$$F(s) = F(\nu) + \frac{F''(c)}{2}(s - \nu)^2, \tag{4.5}$$

for $\nu < c < s < \nu + \delta_0$. Let $u(x_0) = \nu_0$ for some $x_0 \in (0, \delta)$ and some $\nu_0 \in (\nu, \nu + \delta_0)$. Hence combining (4.4) and (4.5) it holds that

$$x_0 = \varepsilon\sqrt{2} \int_{\nu}^{\nu_0} \frac{\mathrm{d}s}{\sqrt{F''(c)}(s - \nu)}. \tag{4.6}$$

The jump discontinuity of F'' at ν ensures that

$$\lim_{y \to \nu^+} F''(y) < +\infty,$$

so the constant $K_M := \sup_{y \in (\nu, \nu + \delta_0)} |F''(y)| > 0$ is finite. Hence we can estimate

$$x_0 \geq \frac{\varepsilon\sqrt{2}}{\sqrt{K_M}} \int_{\nu}^{\nu_0} \frac{\mathrm{d}s}{s - \nu} = \frac{\varepsilon\sqrt{2}}{\sqrt{K_M}} \left[\ln(s - \nu)\right]_{\nu}^{\nu_0} = +\infty,$$

thus $x_0 = +\infty$, a contradiction with the assumption $x_0 \in (0, \delta)$. □

An example of the potential F that fulfils the assumptions given in Proposition 4.1 is the following:

$$F(u) = \begin{cases} (x+1)^2 & \text{for } x \in (-\infty, -1), \\ (1 - x^2)^2 & \text{for } x \in [-1, 1], \\ (x-1)^2 & \text{for } x \in (1, +\infty). \end{cases}$$

For this particular case $\nu = -1$, $F''(-1^-) = 2$ and $F''(-1^+) = 8$.

Removable discontinuity can be treated as a zero-jump and hence gives also no non-constant solution of (4.3). Thus only the second type discontinuity of F'' can generate a non-constant solution. However we are not able to prove it in general.

Definition 4.2. *Let F be a potential that loses its smoothness at its local minimizer(s). We say that it satisfies the $\alpha-$*condition *at its local minimizer ν if there exist constants $\beta_1, \beta_2 > 0$ and $\alpha \in (1, 2)$ such that*

$$\beta_1 |s - \nu|^\alpha \leq F(s) - F(\nu) \leq \beta_2 |s - \nu|^\alpha, \quad \forall s \in (\nu - \zeta, \nu + \zeta). \qquad (\alpha)$$

Usually ζ is chosen such that F is convex in $(\nu - \zeta, \nu + \zeta)$. The possibility of such choice is guaranteed by the definition of potential (Definition 2.23). We show that for a potential F that has lost the C^2-smoothness at its minimizer ν and satisfies $\alpha-$*condition* there, the initial value problem (4.3) has also a non-constant solution. We will follow the concept of proving similar property for model (4.1) introduced by Drábek, Manásevich and Takáč in [DMT11].

We have to do some preparation first. Let us introduce the function $U_+ : (-\vartheta_+, \vartheta_+) \to [\nu, \nu + \zeta)$ by the implicit formula

$$\frac{\varepsilon}{\sqrt{2}} \int_{\nu}^{U_+(y)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}} = |y|, \quad y \in (-\vartheta_+, \vartheta_+), \qquad (4.7)$$

where a finite positive ϑ_+ is given by

$$\vartheta_+ \stackrel{def}{=} \frac{\varepsilon}{\sqrt{2}} \int_{\nu}^{\nu+\zeta} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}},$$

where ζ is from (α).
Similarly, we define $U_- : (-\vartheta_-, \vartheta_-) \to (\nu - \zeta, \nu]$ by the formula

$$\frac{\varepsilon}{\sqrt{2}} \int_{U_-(y)}^{\nu} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}} = |y|, \quad y \in (-\vartheta_-, \vartheta_-), \qquad (4.8)$$

where again a finite positive ϑ_- is given by

$$\vartheta_- \stackrel{def}{=} \frac{\varepsilon}{\sqrt{2}} \int_{\nu-\zeta}^{\nu} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}}.$$

See the sketch of functions U_+ and U_- in Figure 4.1.

Lemma 4.3. *Functions U_+ and U_- are classical non-constant solutions to initial value problem (4.3).*

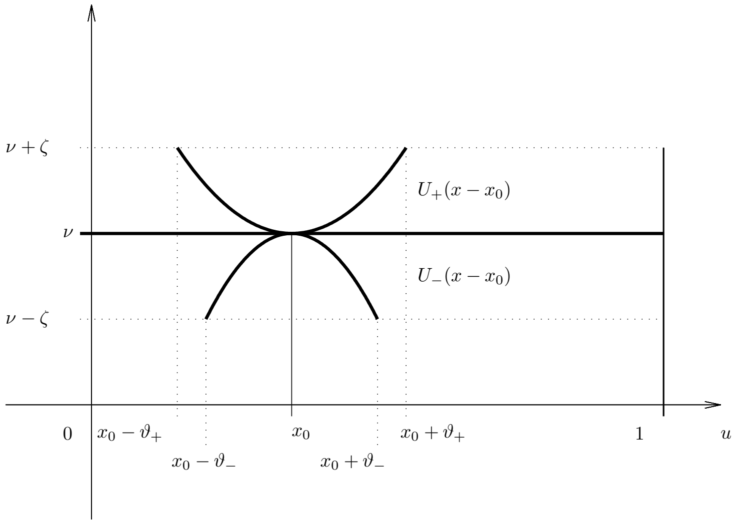

Figure 4.1: Functions U_+ and U_- defined by (4.7), (4.8), respectively.

Proof. The proof is straightforward. We get a solution to (4.3) using the separation-of-variables method twice. We use a common trick using the chain rule to handle the second order equation,

$$u'' = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{\mathrm{d}u'}{\mathrm{d}x} = \frac{\mathrm{d}u'}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = u'\frac{\mathrm{d}u'}{\mathrm{d}u}.$$

Therefore the equation $\varepsilon^2 u'' = F'(u)$ can be equivalently rewritten as

$$u'\frac{\mathrm{d}u'}{\mathrm{d}u} = \frac{1}{\varepsilon^2}F'(u).$$

Applying the separation-of-variables method and considering the initial conditions $u(0) = \nu$ and $u'(0) = 0$ we get

$$\frac{1}{2}(u')^2 = \frac{1}{\varepsilon^2}(F(u) - F(\nu)),$$

which is the first integral (1.5). Applying the separation for the second time one gets

$$\operatorname{sgn}(u'(x)) \int_{\nu}^{u(x)} \frac{\varepsilon}{\sqrt{2}} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}} = \int_0^x \mathrm{d}y = x.$$

Using the separation of variables we got an implicit formula for the solution of (4.3) in some neighbourhood of 0. Let us discuss the ambiguity of the signs. If we expect $u(x) \geq \nu$ for all $x \in (-\vartheta_+, \vartheta_+)$, we have $xu'(x) \geq 0$ and hence we can write

$$\frac{\varepsilon}{\sqrt{2}} \int\limits_{\nu}^{u(x)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}} = |x|,$$

which is a definition of U_+ (4.7). If we demand $u(x) \leq \nu$ for $x \in (-\vartheta_-, \vartheta_+)$, then $xu'(x) \leq 0$ and

$$\int\limits_{\nu}^{u(x)} \frac{\varepsilon}{\sqrt{2}} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}} = -|x|,$$

or equivalently,

$$\frac{\varepsilon}{\sqrt{2}} \int\limits_{u(x)}^{\nu} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}} = |x|,$$

which is a definition of U_- in (4.8).

Fulfilling the condition (α) plays the clue role in here. It is a sufficient condition that ensures the convergence (finiteness) of the singular integral in the definition of U_+, U_-.

□

The following lemma (as a variation of [DMT11, Lemma 3.4.]) suggests that all non-constant solutions of (4.3) are equal either to U_+ or U_-.

Lemma 4.4. *Let the potential F have a local minimum at $\nu \in \mathbb{R}$ and let (α) holds. Then there exists δ, $0 < \delta < \min\{\vartheta_+, \vartheta_-\}$ with the following property: If u is a solution of*

$$\begin{aligned} -\varepsilon^2 u'' + F'(u) &= 0, \\ u(x_0) &= \nu, \\ u'(x_0) &= 0, \end{aligned} \tag{4.9}$$

in the interval $[x_0, x_0 + \delta)$ such that it is non-constant in every subinterval $[x_0, x_0 + \delta'], 0 < \delta' < \delta$, then either $u(x) = U_+(x - x_0)$ or $u(x) = U_-(x - x_0)$ in $[x_0, x_0 + \delta)$.

An analogous result holds also for $(x_0 - \delta, x_0]$.

Proof. From Lemma 4.3 we get that U_+, U_- solve the initial value problem (4.3). Due to the equation being autonomous, $U_+(x-x_0)$ and $U_-(x-x_0)$ solve the problem (4.9).[4] The "uniqueness"[5] remains to be shown. We follow the proof of [DMT11, Lemma 3.2.].
We introduce the function $\varrho : (\nu-\zeta, \nu+\zeta) \to \mathbb{R}$ with

$$\varrho(s) \overset{def}{=} \frac{\varepsilon}{\sqrt{2}} \int_{\nu}^{s} \frac{\mathrm{d}r}{\sqrt{F(r)-F(\nu)}}, \quad s \in (\nu-\zeta, \nu+\zeta). \tag{4.10}$$

This is a continuous strictly increasing function, moreover it is C^2 in both $(\nu-\zeta, \nu)$ and $(\nu, \nu+\zeta)$. Its inverse $\sigma : (-\vartheta_-, \vartheta_+) \to (\nu-\zeta, \nu+\zeta)$ is continuous, strictly increasing and continuously differentiable in $(-\vartheta_-, \vartheta_+)$ with $\sigma'(0) = 0$.

The derivative can be expressed as follows

$$\sigma'(z) = \frac{1}{\varrho'(\sigma(z))} = \frac{1}{\varepsilon}\sqrt{2(F(\sigma(z) - F(\nu)))} > 0, \quad z \in (-\vartheta_-, \vartheta_+) \setminus \{0\}. \tag{4.11}$$

Let u be a solution of (4.9). Then in $(x_0-\delta, x_0+\delta)$, where $\delta < \min\{\vartheta_-, \vartheta_+\}$ the first integral relation holds,

$$\frac{\varepsilon^2}{2}(u'(x))^2 = F(u(x)) - F(\nu). \tag{4.12}$$

Combining (4.11) and 4.12 we come to

$$(u'(x))^2 = \frac{2}{\varepsilon^2}(F(u(x)) - F(\nu)) = (\varrho'(u(x)))^{-2},$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(\varrho(u(x))) = \pm 1, \quad \forall x \in (x_0-\delta, x_0+\delta).$$

If $u' > 0$ for some $(0, \delta')$, where $0 < \delta' < \delta$, then the composition $\varrho \circ u$ is an identity. Using this when substituting $u(x)$ into (4.10), one easily gets that $u(x) = U_+(x-x_0)$. Other cases (left neighbourhood and/or decreasing u) can be treated analogously. □

[4] The proof of this *translation property* is proven in [Ver85, Lemma 2.1.]

[5] Notice that it is not uniqueness indeed. We show that there are only two non-constant solutions (one strictly increasing and the other strictly decreasing) in the left neighbourhood of the point x_0. And, symmetrically, two in the right neighbourhood of x_0.

Hence the two previous lemmas guarantee the existence of a non-trivial solution of the initial value problem emanating from a local minimizer satisfying (α). However, our goal is to classify the stationary points of functional J_ε. They correspond to soltuions of the boundary value problem

$$\varepsilon^2 u''(x) - F'(u(x)) = 0, \tag{4.13}$$

accompanied with homogeneous Neumann boundary conditions. We focus on it in the next section.

To ease our further work, we establish what we mean by the non-smooth potential.

Definition 4.5 (Non-smooth Potential)**.** *Let $F \in C^1(\mathbb{R})$ be a* potential*, i.e. let satisfy Definition 2.23. Let F have a continuous second derivative F'' defined at all points except the local minimizers $\nu_i, i \in \{1, 2, \dots, n\}$. Let F satisfy (α) for $\nu = \nu_i$, for every $i \in \{1, 2, \dots, n\}$. Then we call F a non-smooth potential.*

4.2 New Types of Solution for Non-Smooth Potential

Losing the uniqueness of the solution at the local minimizers may bring us new types of stationary points of J_ε that did not occur for twice continuously differentiable F. Let ν be a local minimizer of non-smooth potential F and $u : [x_1, x_1 + \delta) \to \mathbb{R}$ a strictly increasing solution of

$$\begin{aligned} -\varepsilon^2 u'' + F'(u) &= 0, \\ u(x_1) &= \nu, \\ u'(x_1) &= 0. \end{aligned} \tag{4.14}$$

Its existence and uniqueness has been proven by Lemmas 4.3 and 4.4. However, the existence is only local. We want to continue $u(x)$ to some compact interval $J = [x_1, x_2]$, such that $u(x)$ is strictly monotone in J and $u'(x)$ vanish at both boundary points of the interval. The first integral suggests that $u(x_2)$ must fulfil $F(u(x_2)) = F(\nu)$.

For x_2 to be uniquely determined we add the demand on J to be the minimal interval with the property, i.e. in J there is no internal point with vanishing derivative, i.e.

$$u'(x) > 0 \quad \forall x \in (x_1, x_2) = \text{int } J. \tag{4.15}$$

This ensures that $u(x_2)$ is the nearest bigger value the same level as ν. Similar assertion as for the mapping φ in Section 2.2 holds.

Proposition 4.6. *Let $x_1 < x_2$ be such that there exists $u(x)$ a solution of (4.14) satisfying $u'(x_2) = 0$ and (4.15). Then it holds*

1. $u(x_1) < u(x_2)$,
2. $F(u(x_2)) = F(u(x_1))$,
3. $F(y) > F(u(x_1)), \quad \forall y \in (u(x_1), u(x_2))$,
4. $F'(u(x_2)) \leq 0$.

Proof. The proof is omitted as it follows the proof of Proposition 2.19. □

Let us focus on the last property in Proposition 4.6. The derivative $F'(u(x_2))$ is either *zero* or *negative*. In both cases we get a *kink* solution, however it is reasonable to distinguish these two cases. We will focus on them in Sections 4.2.1 and 4.2.2, respectively.

4.2.1 Transition Between Equal Minima

Let u satisfy the assumptions of Proposition 4.6 and let $F'(u(x_2)) = 0$. Thanks to the properties guaranteed by said Proposition, $u(x_2)$ is a local minimizer of F. The strictly increasing function u therefore ensures a smooth transition between two neighbouring minima. Its existence (and existence of the opposite transition) is proposed by the following claim, which is a special case of [DMT11, Proposition 4.3.].

Proposition 4.7. *Let F be a* non-smooth potential, *with two local minima $\nu_1 < \nu_2$ such that $F(\nu_1) = F(\nu_2) < F(y)$ for all $y \in (\nu_1, \nu_2)$. Then there exist functions $u : J \to (\nu_1, \nu_2)$ and $v : J \to (\nu_1, \nu_2)$ in bounded interval $J = (x_1, x_2)$ with the following properties: $u = u(x)$ and $v = v(x)$ satisfy 4.13 in J and*

1. $\lim_{x \to x_1^+} u(x) = \nu_1$ *and* $\lim_{x \to x_2^-} u(x) = \nu_2$,
2. $u' > 0$ *in* J,
3. $\lim_{x \to x_1^+} v(x) = \nu_2$ *and* $\lim_{x \to x_2^-} v(x) = \nu_1$,
4. $v' < 0$ *in* J.

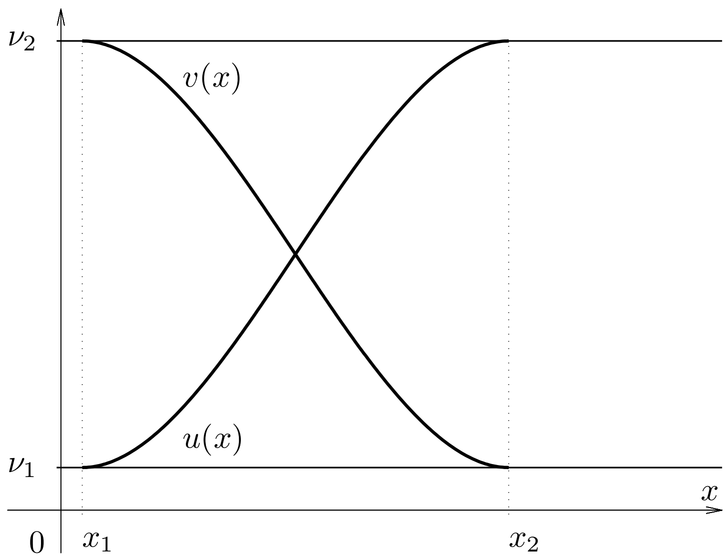

Figure 4.2: Sketch of functions u and v from Proposition 4.7.

The functions u, v are unique up to the shift of the interval J.

See the sketch of functions u, v in Figure 4.2. We follow the proof by Drábek, Manásevich and Takáč.

Proof. For x_1 fixed there exists a unique solution in $(x_1, x_1 + \delta)$ satisfying $u' > 0$. Such u satisfies

$$\frac{\varepsilon}{\sqrt{2}} \int_{\nu_1}^{u(x)} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu_1)}} = x - x_1.$$

It can be continued to a unique solution of the differential equation,

$$u'(x) = \frac{1}{\varepsilon}\sqrt{2(F(u(x)) - F(\nu_1))},$$

satisfying $F(u) > F(\nu_1)$ for all $x \in (x_1, x_2)$, where we take x_2 as a maximal value having this property. If we define

$$\tilde{\nu_2} \overset{def}{=} \lim_{x \to x_2^-} u(x),$$

then $\nu_1 < \tilde{\nu_2} \leq \nu_2$ and

$$\frac{\varepsilon}{\sqrt{2}} \int_{\nu_1}^{\nu_2} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu_1)}} = x_2 - x_1.$$

Thanks to our assumptions on the non-smooth potential F, is the integral finite in both cases ($\tilde{\nu_2} < \nu_2$ or $\tilde{\nu_2} = \nu_2$), so $x_2 \in \mathbb{R}$. Let the strict inequality hold. Then $F(\tilde{\nu_2}) > F(\nu_1)$ and we continue the solution beyond x_2 to some $(x_1, x_2 + \delta'), \delta' > 0$, which contradicts the maximality of x_2. So $\tilde{\nu_2} = \nu_2$ and the proof is finished. The existence of v can be proven analogously. $\square$

As a bonus we get a formula for the length of the interval J,

$$l(J) = \frac{\varepsilon}{\sqrt{2}} \int_{\nu_1}^{\nu_2} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu_1)}}. \tag{4.16}$$

This type of solution can be observed for example at the non-smooth double-well potential, let say $F = |1 - u^2|^\alpha$, $1 < \alpha < 2$, which is discussed in detail in [DR11]. The smooth transitions between ± 1 together with constant solution provide the existence of a whole continua of solutions for the boundary value problem (2.6). We will focus on that in Section 4.3.

4.2.2 Solution from an Elevated Minimum

In this section we introduce a solution that is, in many features, similar to the one shown in the previous section. However, some differences in its behaviour (that will be discussed later) suggest to distinguish those two types. It is again a solution emanating from a local minimum of a non-smooth potential, but it does not arrive into another minimum, but to a point where the derivative of F is negative (for an increasing solution, or positive for a decreasing solution). Therefore the minimum we start at, must be elevated to higher level than the corresponding neighbouring minimum. The existence of such solution is given by the following proposition.

Proposition 4.8. *Let F be a* non-smooth potential *with a local minimum ν.*

1. *Let there exist $a > \nu$ such that $F(y) > F(\nu) = F(a)$ for all $y \in (\nu, a)$ and $F'(a) \neq 0$. Then there exists a function $u : J' \to (\nu, a)$ for some interval $J' = (x_1, x_3)$ with the following properties:*
   - *(a) $u(x)$ satisfies (4.13),*
   - *(b) $\lim_{x \to x_1^+} u(x) = \lim_{x \to x_3^-} u(x) = \nu$,*
   - *(c) $u'(x) > 0$ in (x_1, x_2), $u'(x) < 0$ in (x_2, x_3), where $x_2 = \frac{x_3 + x_1}{2}$,*
   - *(d) $u(x_2) = a$,*

(e) $u(x) = u(2x_2 - x)$ *for* $x \in (x_2, x_3)$.

2. *Similarly, let* $b < \nu$ *such that* $F(y) > F(\nu) = F(b)$ *for all* $y \in (b, \nu)$ *and* $F'(b) \neq 0$. *Then there exists a function* $v : \tilde{J} \to (b, \nu)$ *for some interval* $\tilde{J} = (\tilde{x_1}, \tilde{x_3})$ *with the following properties:*

   (a) $v(x)$ *satisfies (4.13),*

   (b) $\lim_{x \to \tilde{x_1}^+} v(x) = \lim_{x \to \tilde{x_3}^-} v(x) = \nu$,

   (c) $v'(x) < 0$ *in* $(\tilde{x_1}, \tilde{x_2})$, $v'(x) > 0$ *in* $(\tilde{x_2}, \tilde{x_3})$, *where* $\tilde{x_2} = \frac{\tilde{x_3} + \tilde{x_1}}{2}$,

   (d) $v(\tilde{x_2}) = b$,

   (e) $v(x) = v(2\tilde{x_2} - x)$ *for* $x \in (\tilde{x_2}, \tilde{x_3})$.

See the illustration of Propostion 4.8 in Figure 4.3.

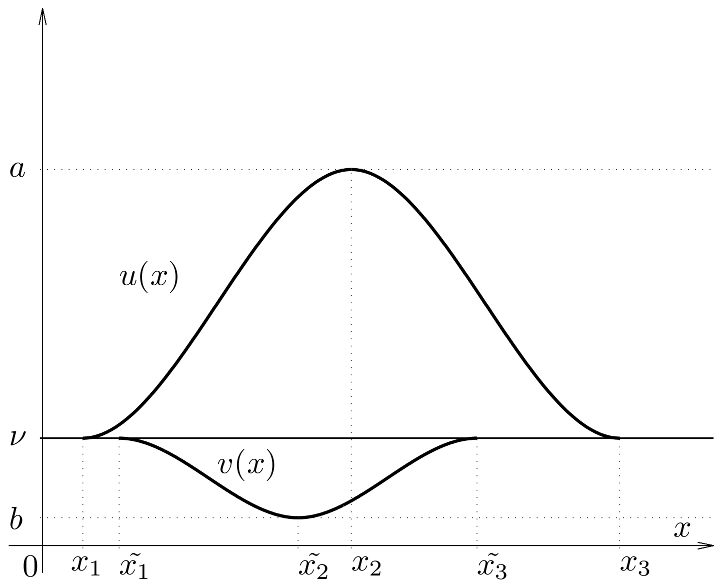

Figure 4.3: Sketch of the functions u and v from Proposition 4.8.

Proof. We proceed again the proof for the first case only. For proving the existence of the solution in (x_1, x_2) we can follow the proof of Proposition 4.7. We come into $u(x_2) = a, u'(x_2) = 0$. But as there is a unique solution to the initial value problem (4.13), $u(x_2) = a, u'(x_2) = 0$, we can continue $u(x)$. As $F'(a) < 0$, the solution is decreasing in $(x_2, x_2 + \delta)$ and fulfils

$$\frac{\varepsilon}{\sqrt{2}} \int_{u(x)}^{a} \frac{\mathrm{d}s}{\sqrt{F(s) - F(a)}} = x - x_2.$$

We can assume that $\delta = x_3 - x_2$. The symmetry of the solution (proven in the same way as Proof of Proposition 2.10) gives all the claimed properties. □

A simple example for this case is the non-smooth triple-well m-potential,

$$F = |1 - u^2|^{\alpha_1}(|u|^\alpha + m),\ \alpha \in (1, 2), m \in (0, \frac{1}{2}), \alpha_1 \in (1, 2), \tag{4.17}$$

sketched in Figure 4.4. Notice that for the existence of the solution u the potential F can be twice continuously differentiable in the neighbourhood of the global minimizers, i.e. we can admit $\alpha_1 = 2$. Such F no longer satisfies the definition of a non-smooth potential (compare with Definition 4.5).

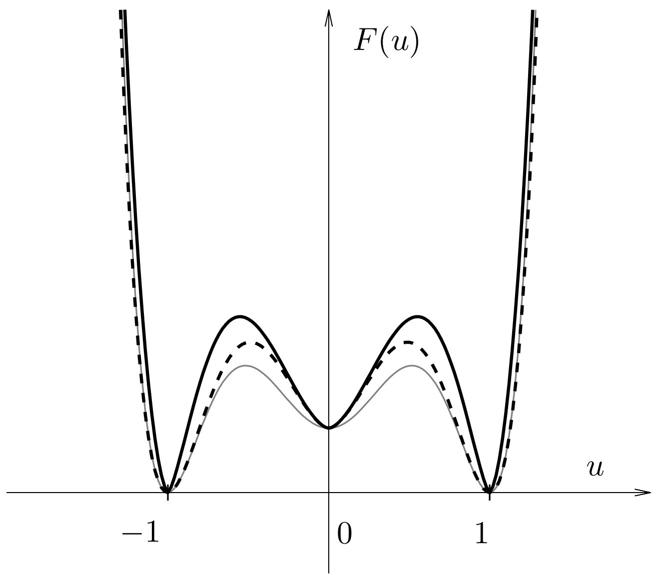

Figure 4.4: Non-smooth triple-well m-potential given by (4.17) with parameters $\alpha_1 = 1.5$ and $\alpha = \frac{5}{3}$ (solid) compared with "semi-non-smooth" triple-well m-potential ($\alpha_1 = 1.5, \alpha = 2$, dashed) and standard triple-well m-potential ($\alpha_1 = 2, \alpha = 2$, grey).

Remark 4.9. *Definition 4.5 of the non-smooth potential is too strict, as we usually do not need all minima to satisfy the α-condition.*

4.3 Continua of Solutions Between Equal Minima

Both types of new solutions to the initial value problem are satisfying the equation (4.13) equipped with homogeneous Neumann boundary conditions in some particular interval. Similarly as in the smooth case we can concatenate the solutions.

However, there is a crucial difference between a non-constant solution starting at local minimum ν and the solution starting from Dom φ (or Im φ) introduced in Section 2.2. As there exists a constant solution $u(x) \equiv \nu$ the concatenation is no longer strictly determined. The freedom of the concatenation is illustrated by Example 4.10.

Example 4.10. *Let us take $F = |1 - u^2|^\alpha$. It satisfies the assumptions of Proposition 4.7, so there are both strictly increasing and strictly decreasing kink solutions $u(x), v(x)$ of the initial value problem in the interval J. Let $l(J) < \frac{1}{2}$ be the length of interval J. Then*

$$y(x) = \begin{cases} -1 & \textit{for } [0, x_1), \\ u(x - x_1) & \textit{for } [x_1, x_2], \\ 1 & \textit{for } (x_2, x_3), \\ v(x - x_3) & \textit{for } [x_3, x_4], \\ -1 & \textit{for } (x_4, 1], \end{cases}$$

is a solution of the boundary value problem (2.6). It holds that

$$x_2 - x_1 = l(J) = x_4 - x_3,$$

where $l(J)$ is given by (4.16). Hence there are four parameters with two (mutually independent) conditions. If we denote $t_1 := x_1, t_2 := x_3 - x_2$ and $t_3 := 1 - x_4$, we get a triplet of parameters, for which it holds that

$$\sum_{i=1}^{3} t_i = 1 - 2 \cdot l(J), \quad t_i \geq 0.$$

The above example should suggest the following. Besides the kinks of fixed length $l(J)$ we can complete the solution to the basic interval $[0, 1]$ with constant solutions of arbitrary lengths. The only restriction is given by the total length of the interval $(0, 1)$. After this rather illustrative motivation let us describe the problem more rigorously.

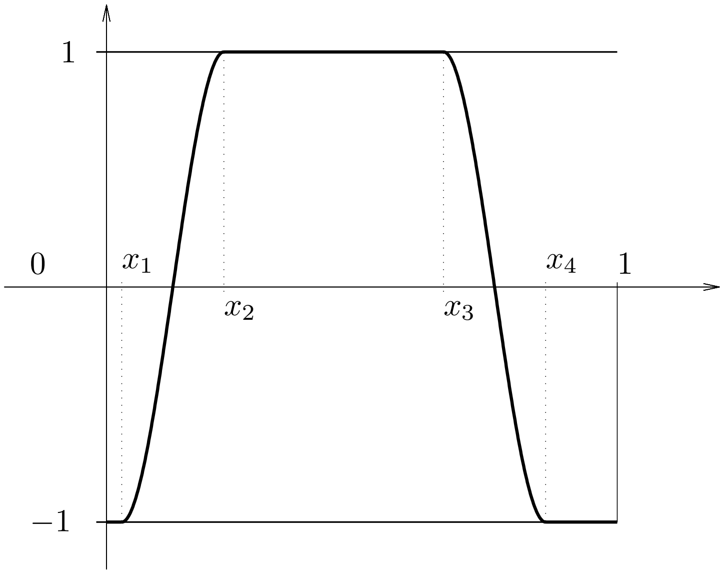

Figure 4.5: Sketch of solution $y(x)$ from Example 4.10.

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4.3.1 Manifolds of Solutions for the Double-Well Potential

In this section we describe the continua of stationary points of J_ε with *non-smooth double-well potential* F. We did not define the non-smooth double-well potential. For our purpose it suffices that F is a potential, that has two equal neighbouring minima (and not more!), satisfying α-condition. A variation of this assertion can be found in [DMT11].

Theorem 4.11. *Let F satisfy the assumptions stated in Proposition 4.7 and let $k = \left\lfloor \frac{1}{l(J)} \right\rfloor$. Then there exist $N = 2k$ manifolds $\mathcal{C}_d^\pm \subset W^{1,2}(0,1)$, where $d \in \{1, 2, \dots, k\}$ such that every $y \in \mathcal{C}_d^+$ is a solution of the boundary value problem (2.6) satisfying $y(0) = \nu_2$, consisting of d kinks. (Similarly $y \in \mathcal{C}_d^-$ has $y(0) = \nu_1$). Moreover, there exist diffeomorphisms $H_d^+(H_d^-) : \mathcal{C}_d^+(\mathcal{C}_d^-) \to \Delta^d$ for $d = \{1, 2, \dots, k\}$ that ensure that each manifold of solutions is diffeomorphic to a d-dimensional simplex.*

Proof. Each solution is determined by d (the number of kinks it consists of), the sign (+ for starting at the higher value and − for the lower one) and the parameters $t = (t_1, t_2, \dots, t_{d+1})$. For the parameters it holds that $t_i > 0, i \in \{1, 2, \dots, d+1\}$ and

$$\sum_{i=1}^{d+1} t_i = 1 - d \cdot l(J) =: T,$$

which is non-negative. If $k \cdot l(J) = 1$, then Δ^k degenerates in a point, otherwise T is positive. Then the parameters $\frac{1}{T}t_i, i \in \{1, \ldots, d+1\}$, having the sum equal to 1 uniquely determine a point of a simplex in $\mathbb{R}^d$. □

We can distinguish the points of the simplex being either boundary and interior points. Also two qualitatively different functions in the manifold correspond to them. See the illustration in Figure 4.6.

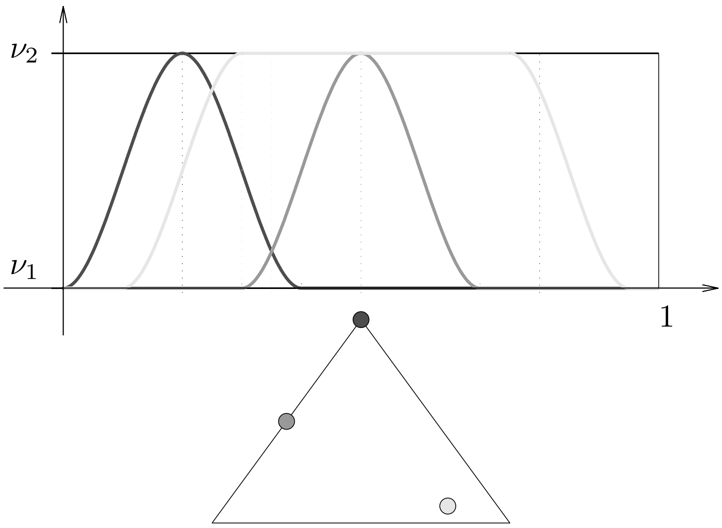

Figure 4.6: Three qualitatively different functions from the manifold $\mathcal{C}_2^-$ with their images in diffeomorphism H_2^-, a points in two-dimensional simplex (a triangle). This example has the values $l(J) = 0.2, T = 0.6$ and the triplets of parameters are $(0, 0, 0.6)$ - black solution; $(0.3, 0, 0.3)$ - dark grey solution; $(0.10, 0.45, 0.05)$ - light grey solution. The first two functions are boundary points of $\mathcal{C}_2^-$, the last is an interior point.

Drábek and Robinson have shown in [DR11] that a boundary point of each simplex corresponds to a function that is a stationary point of J_ε of the saddle type, whereas the interior points correspond to functions that are local minimizers of J_ε. A degenerate manifold where the sum of parameters $T = 0$ is created by a single point that is of a saddle type.

We started with the non-smooth double-well potential, as the simplest example that enables the transition. The situation does not change rapidly for higher number of minima. However, the broader possibilities for concatenation the kinks cause significant increase of the number of continua.

4.3.2 Non-Smooth Triple-Well Potential

We show an interesting connection to a graph theory issue, when counting the number of continua of solution. We start with the definition of non-smooth triple-well potential.

Definition 4.12. *Let F be an even function satisfying the definition of non-smooth potential. Let F have three equal minimizers $\pm\nu$ and 0. Then we call F a non-smooth triple-well potential.*

An example of non-smooth triple-well potential is the function,

$$F(u) = |1 - u^2|^\alpha |u|^\alpha.$$

Note that any non-smooth triple-well potential fulfils the assumptions of Theorem 4.7, even twice. Hence for both couples of local minimizers $-1, 0$ and $0, 1$ there exist manifolds of solutions consisting of transitions between those two minima. However, we can also combine the transitions.
As F is even, the length of the transition $-1 \leftrightarrow 0$ and $0 \leftrightarrow 1$ is equal and can be expressed as $l(J)$, see (4.16).

Thanks to the that, we can restrict ourselves to a simple graph approach. The graph G_3 in Figure 4.7 represents the possible transitions between the minima of the potential F.

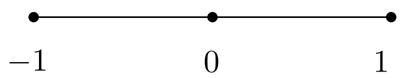

Figure 4.7: Graph G_3 representing the possibilities of the transitions between minima at the non-smooth triple-well potential.

The graph G_3 can be characterized by its adjacency matrix

$$_3A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.18}$$

As the kinks ensure transition between values at which also a constant solutions exist, we can join the kinks freely (as far as we still fit the basic interval $(0, 1)$).

Theorem 4.13. *Let the even function F be a non-smooth triple-well potential. Let $l(J) \leq 1$ be the length of a transition between neighbouring minima. Then J_ε with potential F has continua of critical points. Denote*

$$k = \left\lfloor \frac{1}{l(J)} \right\rfloor .$$

*Then each continuum of the dimension $d \leq k$ is uniquely represented by a d-*walk[6] *in the graph G_3 (Figure 4.7). The number of continua is given by*

$$\sum_{d=1}^{k} |{}_3A^d|,$$

where we define $|M|$ for a square matrix $M = [m_{i,j}], i, j \in \{1, 2, \ldots n\}$ as

$$|M| := \sum_{i=1}^{n} \sum_{j=1}^{n} m_{i,j}. \tag{4.19}$$

On the other hand, each d-walk w represents a continuum and the length of the walk d gives its dimension.

Proof. The first part of the theorem is a simple corollary of the Theorem 4.11 It is a basic knowledge from the graph theory that the number of d-walks in a graph from vertex i to vertex j is represented by the item on position (i, j) in the d-th power of the adjacency matrix. As we are concerned about all the d-walks no matter where they start or finish, it suffices to sum all the items in ${}_3A^d$. □

Computing some of the powers,

$${}_3A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad {}_3A^3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix},$$

we notice that

$${}_3A^3 = 2 \cdot {}_3A.$$

And as $|{}_3A| = 4$ and $|{}_3A^2| = 8$, we come easily to the following conclusion.

Theorem 4.14. *The number N of continua of solution of the even non-smooth triple-well potential F (given by the number of walks of the length not greater than k) can be expressed as*

[6] This is a common abbreviation for *a walk of the length d.*

$$N = 12 \sum_{i=1}^{l} 2^{i-1},$$

for $k = 2l, l \in \mathbb{N}$ *and*

$$N = 12 \sum_{i=1}^{l-1} 2^{i-1} + 2^{l+1},$$

for $k = 2l - 1, l \in \mathbb{N}$.

4.3.3 Special Case: Periodic Non-Smooth Multi-Well Potential

The graph approach for the non-smooth triple-well potential suggests a further generalization. We can compute the number of continua for the case that is represented by the graph G_n in Figure 4.8 and its $(n \times n)$ adjacency matrix ${}_nA$,

$${}_nA = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

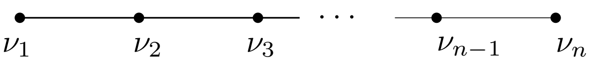

Figure 4.8: Graph G_n representing the possibilities of the transition between minima at the locally periodic non-smooth multi-well potential.

The assumption, that the kinks between the minima are equally long, was quite natural in the triple-well case. However, it is becoming way too restrictive for the multi-well case, with n equal global minimum points. First, there exists a constant

$$d_\nu := \nu_{i+1} - \nu_i, \quad i \in \{1, 2, \ldots, n-1\}, \tag{4.20}$$

independent of i. Secondly, it requires that

$$\int_{\nu_1}^{\nu_2} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}} = \int_{\nu_i}^{\nu_{i+1}} \frac{\mathrm{d}s}{\sqrt{F(s) - F(\nu)}}, \quad i \in \{1, 2, \ldots n-1\}, \tag{4.21}$$

where $F(\nu)$ abbreviates $F(\nu_j)$ for any $j \in \{1, 2, \dots n\}$, as they are equal. It does not require F being *locally periodic*[7], however it is the simplest way to ensure fulfilling both the conditions (4.20) and (4.21).

Theorem 4.15. *For a locally periodic non-smooth multi-well potential we can compute $l(J)$ from (4.16) and define $k = \left\lfloor \frac{1}{l(J)} \right\rfloor$. Then the number of continua of solution is given by*

$$\sum_{d=1}^{k} |_n A^d|,$$

where $|\ .\ |$ is defined in (4.19).

Remark 4.16. *As there is no connection between k and n, there is no simple formula*[8] *that would immediately give number of continua of dimension up to k. However, we believe there is some simpler answer than the one in Theorem 4.15 to the question of the number of d-walks in $(n-1)$-path graph.*

For a general non-smooth multi-well potential, despite having all local minima equal, it is difficult to establish the number of continua. Now the lengths of the kinks between two neighbouring minima come into account. We may denote $\omega_{i,i+1}$ the length of the transition[9] between ν_i and ν_{i+1}. Transformed to the graph problem, the number of continua corresponds to the number of walks in $(n-1)$-path graph with *weighted length less than* 1. That is a number that the graph theory does not give us an efficient method to establish.

[7]By local periodicity we mean that for any $x \in (\nu_1, \nu_n)$ it holds that

$$(x + k \cdot d_\nu) \in [\nu_1, \nu_n], k \in \mathbb{Z} \Rightarrow F(x + k \cdot d_\nu) = F(x),$$

where d_ν is defined by (4.20).

[8]We find interesting that it is possible to determine the number of *closed k-walks* in any graph G as the $tr(A_G)$ (the trace of k-th power of its adjacency matrix A_G). This number is equal to $\sum_{i=1}^{n} \lambda_i^k$, where λ_i are eigenvalues of A_G. Lovász suggests in [Lov93, Problem 29., p.20 and p.184] that $\lambda_i = \cos \frac{i\pi}{n+1}, k \in \{1, 2, \dots, n\}$ for our graph G_n. However, the closed walks represent only those continua for which the solutions fulfil $u(0) = u(1)$, while we are interested also in those continua where $u(0) \neq u(1)$, so the suggested method is not helpful.

[9]A simple corollary of Proposition 4.7 is that $\omega_{i,i+1} = \omega_{i+1,i}$.

4.4 Continua of Solutions from an Elevated Minimum

The whole matter is a bit less complicated for the solutions emanating from a non-smooth elevated minimum. We have two options that give qualitatively different results.

Theorem 4.17. *Let F fulfil the assumptions of Proposition 4.8, part 1. and $F(y) > F(\nu), \forall y < \nu$ and let $k = \left\lfloor \frac{1}{l(J')} \right\rfloor$, where $l(J')$ is the length of the interval J'*

or

let F fulfil the assumptions of Prop. 4.8, part 2. and $F(y) > F(\nu), \forall y > \nu$ and let $k = \left\lfloor \frac{1}{l(\tilde{J})} \right\rfloor$, where $l(\tilde{J})$ is the length of the interval $\tilde{J}$.

Then there exists k manifolds $\mathcal{C}_d \subset W^{1,2}(0,1), d \in \{1, 2, \ldots, k\}$ such that for each there exists a diffeomorphism H_d between $\mathcal{C}_d$ and a d-dimensional simplex Δ^d.

An example of potentials that meet the assumptions of Proposition 4.17 is given in Figure 4.9.

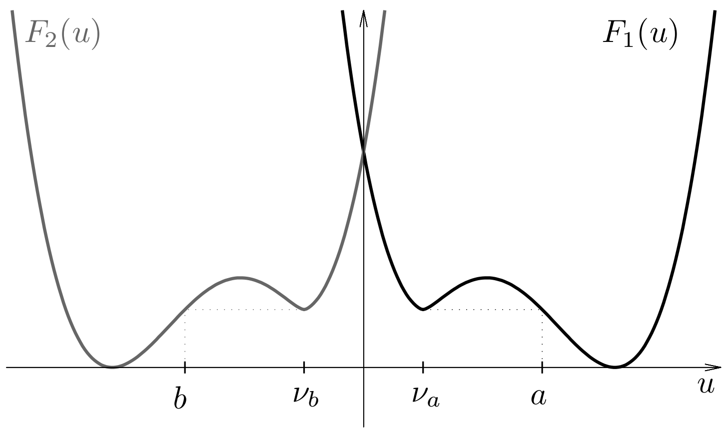

Figure 4.9: Potentials F_1 and F_2 that fulfil the assumptions of Proposition 4.17, part first and second, respectively.

Proof. If $k \geq 1$, the solution starting at ν can be composed of maximally k arcs (see Figure 4.3) with its extreme in a (or b, depends on the assumptions)

and constant parts equal to ν between these arcs. Any manifold is uniquely determined by the $d \leq k'$ the number of arcs, therefore the number of continua is k.
Similarly as in the previous case, denoting the constant parts of a solution with $t_i, i \in \{1, 2, \ldots, d+1\}$, we come into

$$\sum_{i=1}^{d+1} t_i = 1 - d \cdot l(J) = T'.$$

then a vector $(t_1, t_2, \ldots, t_{d+1})$ uniquely determines a point in a d-dimensional simplex, as $t_i \geq 0$ and $\sum \frac{1}{T} t_i = 1$.
(Here $l(J)$ stands for either $l(J')$ or $l(\tilde{J})$ with dependence on the assumptions.) □

In the situation, where both cases 1. and 2. of Proposition 4.8 are fulfilled, becomes more complicated, as two type of arcs can be concatenated. The lengths of both types of arcs are not equal in general. The following proposition covers therefore only the special case, when they are equal.

Theorem 4.18. *Let F fulfil assumptions of both parts of Proposition 4.8, $l(J') = l(\tilde{J})$ and let us denote $k = \left\lfloor \frac{1}{l(J')} \right\rfloor$. Then there exist $2^{k+1} - 2$ manifolds of solution, diffeomorphic to simplex.*

This case occurs for example for the non-smooth triple-well m-potential (see Figure 4.4).

Proof. We focus on the number of continua denoting it with N. When concatenating the arcs of two types, we can always choose out of two options (either upwards or downwards). Thus, we have 2^d ways how to concatenate d arcs. As $d \in \{1, 2, \ldots, k\}$, to get N, we sum

$$N = \sum_{i=1}^{k} 2^d = -1 + \sum_{i=0}^{k} 2^d = -1 + 2^{k+1} - 1 = 2^{k+1} - 2.$$

□

Again, for $l(J') \neq l(\tilde{J})$ the situation is becoming more complicated and the number of continua is not easy to be determined.

Chapter 5

Conclusion

We offer a short summary of the thesis in which we managed to achieve some original results.

For a general smooth potential we succeeded in describing the stationary points of its Lyapunov functional J_ε. The non-constant ones are kinks and their concatenations. The clue point of the research is employing the function φ (Definition 2.19), assigning maximum of the kink to its initial point. Solution diagram as a tool of effective presentation of the critical points was depicted already in [DR11]. We managed to prove some of its properties, mainly the necessary condition of its monotonicity (Theorem 3.11). Then we succeeded also in proving continuity and limit properties for a general smooth potential (Proposition 3.15).
Chapter 4, dedicated to non-smooth potentials, presents assumptions under which the manifolds of solution occur for general non-smooth potentials. The graph theory approach to the problem is introduced and used to determine the number of continua for many common situations. See Theorems 4.13, 4.14, 4.15, 4.17 and 4.18.

The thesis is naturally not exhaustive. There are some interesting problems remaining opened. For example, we would like to have more properties of the branches of solution diagram. We do not know about the derivative of the branches at the distinctive points, not even for the double-well potential. Moreover, we have not proven the conditions of monotonicity of the branches for the general case.
Another opened problem arising from Theorem 4.15 belongs to the graph theory and is discussed in Remark 4.16. Is there a formula determining the number of k-walks in n-path?

We will try not to leave this questions unanswered and focus on them in further research.

Appendix A

Counterexample: Sufficient Condition in Theorem 3.11 is not Necessary

In this section we show that there exists a standard double-well potential, that does not fulfil the assumptions given in Theorem 3.11 and still its solution diagram is monotone in $(-1, 0)$ and $(0, 1)$. Such potential is a counterexample showing that the condition given in Theorem 3.11 is not necessary. Consider the function f,

$$f(x) = \begin{cases} -x(x+1) + 0.02(\cos(4\pi x) - 1) & \text{for } x \leq 0, \\ x(x-1) - 0.02(\cos(4\pi x) - 1) & \text{for } x > 0. \end{cases} \tag{A.1}$$

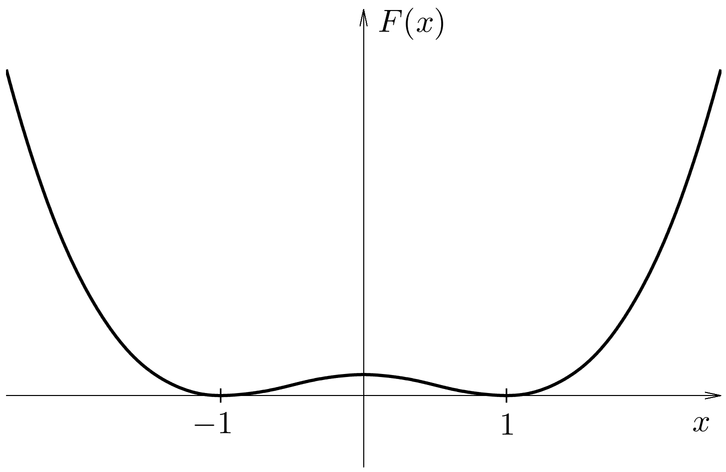

Figure A.1: The standard double-well potential F given by (A.3).

We claim that there is a double-well potential $F \in C^2(\mathbb{R})$ such that

$F' = f$, for which the branches ε_n of its solution diagram are monotonic even though there are inflection points of f in $(-\xi, 0)$. In this example it is important, that the concavity of F' in $(-1, 0)$ is not "violated rapidly" (see Figure A.2). First we show that $F \in C^2(\mathbb{R})$ or equivalently $f \in C^1(\mathbb{R})$. The

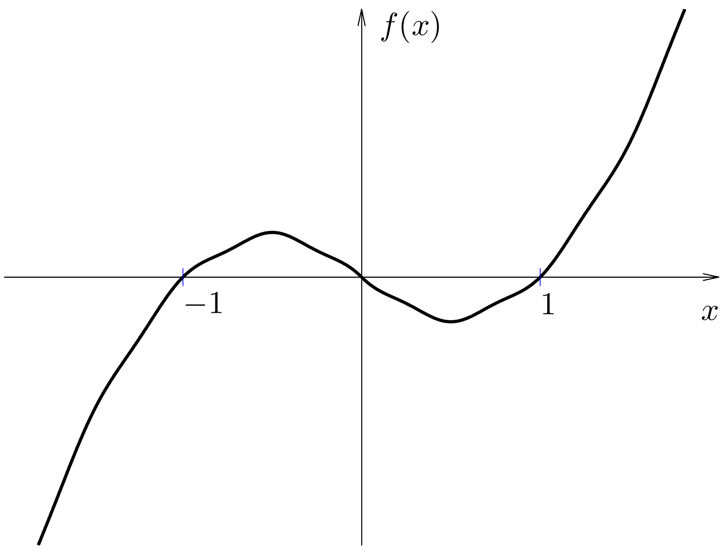

Figure A.2: The derivative $f = F'$ (see (3.24)) of a standard double-well potential F.

continuity of f is trivial, as f is a sum of a parabola and cosine function and $f(0+) = f(0-) = f(0) = 0$.
The derivative,

$$f'(x) = \begin{cases} -2x - 1 - 0.08\pi \sin(4\pi x) & \text{for } x < 0, \\ 2x - 1 + 0.08\pi \sin(4\pi x) & \text{for } x > 0, \\ -1 & \text{for } x = 0, \end{cases} \tag{A.2}$$

is continuous in $\mathbb{R}$ as at the trouble point 0 it holds $f'(0+) = f'(0-) = -1$. The potential itself is obtained by integrating f,

$$F(x) = \operatorname{sgn}(x) \left(\frac{x^3}{3} + \frac{x}{50} - \frac{\sin 4\pi x}{200\pi} \right) - \frac{x^2}{2} + \frac{11}{75}, \tag{A.3}$$

where sgn is the sign function. The integration constant is chosen such that $F(\pm 1) = 0$.
Potential F is even thanks to f being odd and the roots of f (± 1 and 0) are extremal points of F. The inflection points of F are equal to roots of $f'(x)$,

which are $x = \pm\frac{1}{2}$, hence $\xi = \frac{1}{2}$. Therefore F is a double-well potential.
It is not difficult to establish that there are inflection points of f in $(-\frac{1}{2}, 0)$,

$$f''(x) = -\frac{8}{25}\pi^2 \cos(4\pi x) - 2.$$

So in $(-\frac{1}{2}, 0)$ which is a basic period of the scaled cosine function we get two roots r_1, r_2. Their approximate values are $r_1 = -0.180, r_2 = -0.320$.

Hence we have a double-well potential that does not satisfy the conditions given in Theorem 3.11 but still its solution diagram branches are monotone as we claim that

$$f(x) > xf'(x) \quad \text{for } x \in (-\frac{1}{2}, 0). \tag{A.4}$$

It has been shown in the Proof of Theorem 3.11 that this condition is sufficient for the branches of the solution diagram to be monotone.
Proving the equation (A.4) requires solving a transcendent equation. It can be shown using numerical methods that it holds. A geometrical interpretation can be seen in Figure A.3 while the numerical calculation is illustrated in Figure A.4.

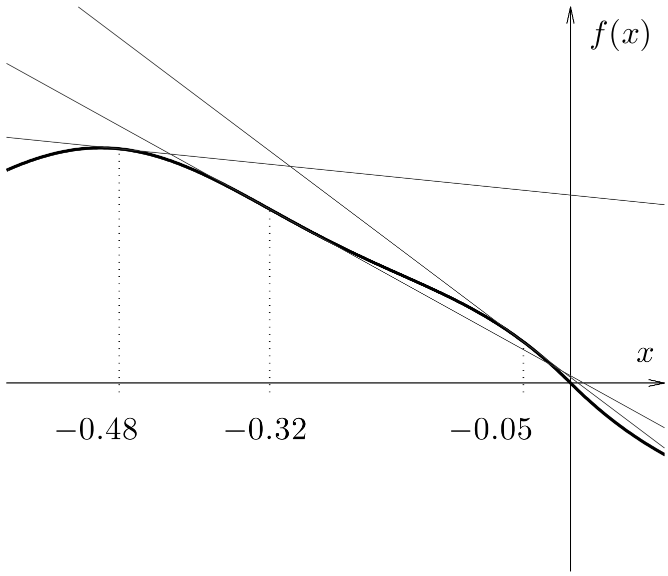

Figure A.3: Geometrical interpretation of inequality (A.4).

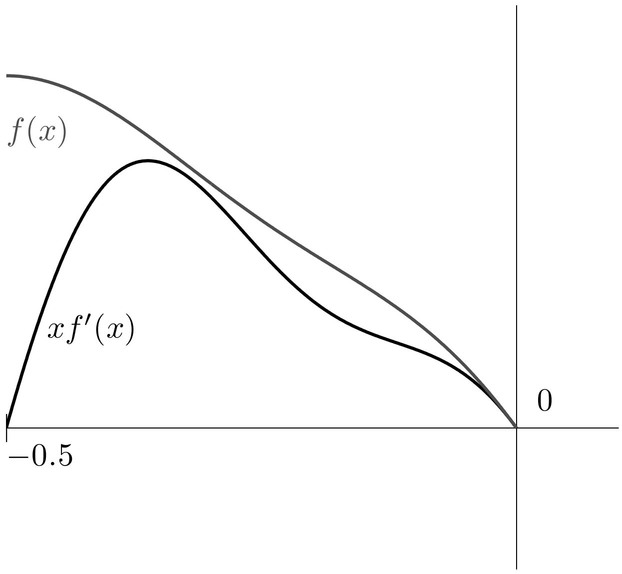

Figure A.4: Numerical evalution of inequality (A.4).

Appendix B

Proof of Properties of φ

This chapter focuses on properties of the function φ that is defined for smooth potential[1] F and assigns every starting point of a finite kink solution to the boundary problem (2.6) its maximum.

First, we introduce the notation. For the whole chapter we expect F being a smooth potential (Definition 2.23) and φ function from Definition 2.19. Step by step we unravel the properties of φ, summed up in Lemma 3.13.

As potential F has a finite number of extremes, both Dom φ and Im φ can be written as a finite union of intervals. See Figure 2.9, Dom φ consists of intervals labelled with "2a", those with "3a" build Im φ.

Definition B.1. *Let D_i, E_j for $i = \{1, 2, \dots, \bar{i}\}, j = \{1, 2, \dots, \bar{j}\}$ be pairwise disjoint intervals such that $\bigcup_i D_i = \mathrm{Dom}\ \varphi$ and $\bigcup_j E_j = \mathrm{Im}\ \varphi$. We assume that D_i and E_j are maximal having this property.*

Lemma B.2. *D_i is open set for every $i \in \{1, 2, \dots, \bar{i}\}$.*

Proof. Let us take some $\overline{x} \in D_i : \varphi(x) =: \overline{y}$. From Proposition 2.20 we know that $\forall z \in (\overline{x}, \overline{y}) : F(z) > F(\overline{x})$ and $F'(\overline{x}) > 0, F'(\overline{y}) < 0$. Let us focus on the boundary points of D_i. We denote

$$\begin{aligned} x_1 :=&\ \max\{x \in \mathbb{R} : x < \overline{x} \quad \text{and } F'(x) = 0\}, \\ x_2 :=&\ \max\{x \in \mathbb{R} : x < \overline{x} \quad \text{and } x \in \mathrm{Dom}\ \tilde{\varphi} \setminus \mathrm{Dom}\ \varphi\}. \end{aligned}$$

Notice that those are the only options that can bound D_i from bottom. Similarly, from above,

$$\begin{aligned} x_3 :=&\ \min\{x \in \mathbb{R} : x > \overline{x} \quad \text{and } F'(x) = 0\}, \\ x_4 :=&\ \min\{x \in \mathbb{R} : x > \overline{x} \quad \text{and } x \in \mathrm{Dom}\ \tilde{\varphi} \setminus \mathrm{Dom}\ \varphi\}. \end{aligned}$$

[1] However, φ can be defined also for non-smooth potentials.

In other words, $F'(\tilde{\varphi}(x_2)) = 0, F'(\tilde{\varphi}(x_4)) = 0$. Denoting $x_a := \max\{x_1, x_2\}$ and $x_b := \min\{x_3, x_4\}$, one can write $D_i = (x_a, x_b)$. Interval D_i does not contain its boundary points, hence D_i is an open set for every index i. $\square$

Lemma B.3. *Function φ is continuous in D_i, for every index i.*

Proof. Let $x_0 \in D_i$ be arbitrary but fixed. Let us take E_j that contains $y_0 := \varphi(x_0)$. Definition of φ together with Proposition 2.20 give $F'(x_0) > 0$ and $F'(y_0) < 0$. Thus there exist some small neighbourhoods $U(x_0, \delta)$ and $U(y_0, \delta')$ such that

$$\forall x \in U(x_0, \delta) \exists! y \in U(y_0, \delta'), \text{ such that } \ \varphi(x) = y,$$

and therefore φ is continuous in some $U(x_0)$.
We show the continuity in D_i via contradiction. Let $x_d \in D_i$ be a discontinuity point of φ in D_i. Let us take such x_d, that $|x_d - x_0|$ is minimal. We can exclude that the discontinuity is of second type, because then $x_d \notin \text{Dom } \varphi$, a contradiction. Therefore only jump discontinuity can occur. There are two possibilities:

- $\varphi(x_d) < \lim_{x \to x_d} \varphi(x)$.
  According to the definition of φ and properties of F, it follows that there is a local minimizer of F between D_i and E_j. We denote it with x_m. Thanks to the minimality of x_d it must hold that $\varphi(x_d) = x_m$. But $F'(x_m) = F'(\varphi(x_d)) = 0$, a contradiction with properties of φ.

- $\varphi(x_d) > \lim_{x \to x_d} \varphi(x)$.
  The continuity of F suggest that there exists some x_m, a local minimizer of F such that $\varphi(x_d) > x_m$ and $F(x_d) < F(x_m)$. But then there must exist some $x_u \in (x_d, x_0)$ such that $\tilde{\varphi}(x_u) = x_m$. Again, $F'(x_m) = 0$, which implies $x_u \notin \text{Dom } \varphi$ and therefore $x_d \notin D_i$, a contradiction.

Thus, φ is continuous in D_i, for every i. $\square$

Lemma B.4. *Function φ is strictly decreasing in every D_i.*

Proof. It is a simple corollary of continuity and injectivity (given by Proposition 2.25) that $\varphi|_{D_i}$ is strictly monotone. As $F' > 0$ in D_i and $F' < 0$ in $\varphi(D_i)$, we get that φ is strictly decreasing. $\square$

Lemma B.5. *There exists bijection b between indices $i = \{1, 2, \ldots, \bar{i}\}$ and $j = \{1, 2, \ldots, \bar{j}\}$ such that*

$$\varphi(D_i) = E_{b(i)}. \tag{B.1}$$

A simple corollary is that $\bar{i} = \bar{j}$.

Proof. Let us choose some D_i. For arbitrary $x_0 \in D_i$, its image $\varphi(x_0)$ belongs to some E_j. The restricted function $\varphi|_{D_i}$ is continuous, so $\varphi(D_i)$ is an interval. As E_j is maximal connected subset of Im φ, we get that

$$\varphi(D_i) \subseteq E_j.$$

We show the equality via contradiction. We expect points $y_1, y_2 \in E_j \subseteq \text{Im } \varphi$ for which exist $x_1 \in D_i, x_2 \in D_k, i \neq k$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y_2$. The sets $\varphi(D_i)$ and $\varphi(D_k)$ are disjoint, otherwise we would come immediately to the contradiction with the definition of φ, similarly as in the proof of Proposition 2.25. The images of the sets D_i, D_k are also open sets (thanks to strict monotonicity of restrictions of φ). Hence there must exist a point $y \in E_j$ between $\varphi(D_i)$ and $\varphi(D_k)$ such that $y \notin \text{Im } \varphi$, a contradiction.

The bijection b is given by the image of any point in D_i. The continuity of $\varphi|_{D_i}$ ensures that it is well defined. Hence, also every $\varphi|_{D_i}$ is a bijection. □

Corollary B.6. *For any index j, E_j is open set.*

Proof. Thanks to Lemma (B.5), for every E_j there exists some D_i such that $\varphi(D_i) = E_j$. As $\varphi|_{D_i}$ is a strictly monotone bijection between D_i and E_j and D_i is open set, we get that E_j is open set, for any j. □

Bibliography

[Bar74] Hans Jochen Bartsch. *Handbook of mathematical formulas.* Academic Press, 1974.

[CL55] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations.* International series in pure and applied mathematics. McGraw-Hill, 1955.

[DM05] Pavel Drábek and Jaroslav Milota. *Basic Nonlinear Methods.* Západočeská univerzita v Plzni, 2005.

[DM13] Pavel Drábek and Jaroslav Milota. *Methods of Nonlinear Analysis.* Birkhäuser, 2013.

[DMT11] Pavel Drábek, Raúl F. Manásevich, and Peter Takáč. Manifolds of critical points in a quasilinear model for phase transition. *Contemporary Mathematics*, 2011.

[DR11] Pavel Drábek and Stephen B. Robinson. Continua of local minimizers in a non-smooth model of phase transition. *Zeitsrchrift für angewandte Mathematik und Physik ZAMP*, 2011.

[Drá11] Pavel Drábek. New-type solutions of the modified fisher-kolmogorov equation. *Abstract and Applied Analysis*, 2011.

[Eva98] Lawrence C. Evans. *Partial Differential Equations, Graduate Studies in Mathematics 19.* American Mathemaical Society, 1998.

[Fus90] Giorgio Fusco. A geometric approach to the dynamics of $u_t = \varepsilon^2 u_{xx} + f(u)$ for small ε. *Lecture Notes in Physics*, 1990.

[Lov93] Lászlo Lovász. *Combinatorial problems and exercises.* AMS/Chelsea publication. AMS Chelsea Publishing, 1993.

[Ott07] Josef Otta. Analytical and numerical analysis of the quasilinear bistable equation. Master's thesis, Faculty of Applied Sciences, University of West Bohemia, 2007.

[PT01] Lambertus A. Peletier and William C. Troy. *Spatial Patterns, Higher Order Models in Physics and Mechanics*. Birkhäuser, 2001.

[Roc97] R.Tyller Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics and Physics Series. PRINCETON University Press, 1997.

[SW81] J. Smoller and A. Wassermann. Global bifurcation of steady-state solutions. *Journal of Differential Equations*, 1981.

[Ver85] Ferdinand Verhulst. *Nonlinear Differential Equations and Dynamical Systems*. Springer-Verlag, 1985.

[ZC04a] Vladimir A. Zorich and Roger Cooke. *Mathematical Analysis I*. Mathematical Analysis. Springer, 2004.

[ZC04b] Vladimir A. Zorich and Roger Cooke. *Mathematical Analysis II*. Number sv. 2 in Mathematical Analysis II. Springer, 2004.